NMAG405 - Universal Algebra 1 - winter term 2023/24

Homework 1

Deadline 03.11.23 12:20

1. (10 points) A *latin square* $(A, *)$ is an algebra of type (2), such that for each $a, b \in A$ there exists a unique $x \in A$ with $x * a = b$ and a unique $y \in A$ with $a * y = b$; we then denote x by b/a and y by $a\backslash b$. (For finite A each row and each column of the multiplication table of $*$ contains every element of A exactly once, hence the name.)
   A *quasigroup* is an algebra $(A, *, \backslash, /)$ of type $(2, 2, 2)$, which satisfies the identities:

   $$y \approx x * (x\backslash y) \approx x\backslash(x * y) \approx (y/x) * x \approx (y * x)/x.$$

   Let A be a fixed set. Prove that the map Φ that assigns to every latin square $(A, *)$ the algebra $(A, *, \backslash, /)$ as above, and the map Ψ that forgets the operations $\backslash, /$ are mutually inverse bijections between the set of latin squares and the quasigroups (with universe A).

2. (10 points) A set $C \subseteq \mathbb{R}^n$ is called *convex* if $\mathbf{x}, \mathbf{y} \in C$ implies $\theta\mathbf{x} + (1 - \theta)\mathbf{y} \in C$, for all $\theta \in [0, 1]$. For fixed dimension n, let $\mathrm{Cvx}(\mathbb{R}^n)$ be the set of all convex subsets of $\mathbb{R}^n$. Show that $(\mathrm{Cvx}(\mathbb{R}^n), \subseteq)$ is a complete algebraic lattice.

3. (10 points) A map $f\colon L_1 \to L_2$ between two lattices is called *monotone* if $x \leq y$ implies $f(x) \leq f(y)$. Let L be a complete lattice, and $f\colon L \to L$ an monotone map. Prove that there is a fixpoint a of f, i.e. a point $a \in L$ such that $f(a) = a$.