NMAG 405 - Universal Algebra 1 - fall semester 2020/21

Homework 4

Deadline 17.12.2020, 9:00

1. (10 points) Let $\mathbf{R}$ be a fixed ring (with 1), and let $\mathcal{V}$ be the variety of (left)-modules over $\mathbf{R}$. Prove that the free $\mathbf{R}$-module is isomorphic to $R^{(X)} = \{(u_x)_{x\in X} \in R^X \mid \text{only finitely many } u_x \text{ are not } 0\}$. Use the universal algebraic characterization of free algebras as "terms modulo identities", do not use the categorical/module-theoretical definition of freeness.

2. (10 points) Let $\mathcal{V}$ be the variety of algebras $(A, \cdot, l, r)$ of type $(2, 1, 1)$ that satisfy the identities
$$l(x \cdot y) \approx x, \quad r(x \cdot y) \approx y, \quad l(x) \cdot r(x) \approx x.$$
   (a) Show that every non-trivial member of $\mathcal{V}$ is infinite.
   (b) Prove that, if $\mathbf{A} \in \mathcal{V}$ is generated by $\{a_1, a_2, \dots, a_n\}$, then it is already generated by $\{(a_1 \cdot a_2), a_3, \dots, a_n\}$
   (c) Prove that $\mathbf{F}_\mathcal{V}(n) = \mathbf{F}_\mathcal{V}(m)$ for all positive integers n, m.

3. (10 points) Let $\mathbf{A}$ be the semigroup given by the following multiplication table:

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

   Prove that the variety generated by $\mathbf{A}$ is exactly the variety of commutative semigroups satisfying $x^3 \approx x^4$.