NMAG 405 Universal Algebra 1, fall semester 2017–2018

Homework 1

Deadline 2 Nov 2017, 10:40

1.1. (10 points) Fix a set A. A *Latin square* is a binary operation $*$ on A such that, for each $a, b \in A$, there exists a unique $x \in A$ with $a * x = b$ and a unique $y \in A$ with $y * a = b$; these elements are denoted by $x =: a\backslash b$ and $y =: b/a$. (Think of a finite A and the multiplication table of $\star$. The condition is that each row and each column contains each element of A exactly once.) A *quasigroup* $\mathbf{A} = (A; *, \backslash, /)$ is an algebra of type $(2, 2, 2)$ which satisfies the following identities:

$$y \approx x * (x\backslash y) \approx x\backslash(x * y) \approx (y/x) * x \approx (y * x)/x \ .$$

Prove that the mappings Φ that assigns to a Latin square the algebra $(A; *, \backslash, /)$ as above, and Ψ that forgets the operations $\backslash$ and $/$ are mutually inverse correctly defined bijections between the set of all Latin squares and the set of all quasigroups.

1.2. (10 points) Fix a prime p and a natural number n. Consider the algebra $\mathbf{A} = (\{0, 1, \ldots, p-1\}; m)$, where m is a ternary operation defined by $m(x, y, z) = (x - y + z) \mod p$. Prove that $R \leq \mathbf{A}^n$ if and only if R is an affine subspace of $\mathbb{Z}_p^n$, where R is a nonempty n-ary relation on A. (Recall from linear algebra that R is an affine subspace iff it is closed under all affine combinations.)

1.3. (10 points) Let $\mathbf{A}_n = (\{0, 1, \ldots, n-1\}; f_n)$ be the algebra with a single unary operation defined by $f_n(x) = (x + 1) \mod n$. For each pair $m, n \in \mathbb{N}$, find all homomorphisms $\mathbf{A}_n \to \mathbf{A}_m$.