Universal Algebra Exercises - Sheet 9

Exercise 1. Given two varieties of groups $\mathcal{V}$ and $\mathcal{W}$, define

$$\mathcal{V} \cdot \mathcal{W} := \Big\{ G \text{ group} \mid \exists N \trianglelefteq G,\ N \in \mathcal{V},\ G/N \in \mathcal{W} \Big\}.$$

Show that this is a variety of groups.

Exercise 2. Let G be a group and let $\mathcal{A}$ be the variety of abelian groups. Let $\lambda_{\mathcal{A}}^{G}$ be the smallest congruence of G with an abelian quotient.

- Show that the congruence class of the identity element is the subgroup of G generated by elements of the form $[x, y] := xyx^{-1}y^{-1}$.

$$1/\lambda_{\mathcal{A}}^{G} = \mathrm{Sg}_G([x, y] \mid x, y \in G)$$

- Show that the variety $\mathcal{A} \cdot \mathcal{A}$ is axiomatized by the group laws and $[[x, y], [z, w]] = 1$.

Exercise 3. Let $\mathcal{A}_n$ be the variety of abelian groups satisfying $x^n \approx 1$. Show that

- $\mathcal{A}_3 \cdot \mathcal{A}_2 = \mathrm{Mod}\Big(\text{ group axioms } \cup \{x^6 \approx 1, \quad [x^2, y^2] \approx 1, \quad [x, y]^3 \approx 1\}\Big)$
- $\mathcal{A}_2 \cdot \mathcal{A}_2 = \mathrm{Mod}\Big(\text{ group axioms } \cup \{(x^2y^2)^2 \approx 1\}\Big)$

Exercise 4. Let cRing_n be the vareity of commutative rings satisfying $x^n \approx x$, and let $\mathbb{F}_9$ be the field of order 9. Show that $\mathrm{HSP}(\mathbb{F}_9)$ is axiomatized by the axioms of cRing_9 together with

$$x + x + x \approx 0.$$