Universal Algebra Exercises - Sheet 7

Exercise 38. Recall from the homework that, given a group G, its lattice of congruences is isomorphic to its lattice of normal subgroups. Prove an analogous statement for rings.

Exercise 39. Let p be a prime number. Show that the group $(\mathbb{Z}, +)$ is a subdirect product of the groups $\mathbb{Z}_{p^k}$.

Exercise 40. Let $\mathbb{Q}$ be the additive group of rational numbers and let p be a prime number.

(i) Let $\mathbb{Q}_p := \{a/p^k \mid a \in \mathbb{Z}, k \geq 0\}$. Show that $\mathbb{Q}_p$ is a subgroup of $\mathbb{Q}$ and that $\mathbb{Z}$ is a normal subgroup of $\mathbb{Q}_p$.

(ii) Let $\mathbb{Z}_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}$. Prove that every element of $\mathbb{Z}_{p^\infty}$ has finite order.

(iii) Let H be a subgroup of $\mathbb{Z}_{p^\infty}$ such that the order of elements in H if bounded. Show that H is a cyclic group of order p^k.

(iv) Show that $\mathbb{Z}_{p^\infty}$ is subdirectly irreducible by showing that the lattice of (normal) subgroups is a chain.

$$0 < H_1 < H_2 < \cdots < \mathbb{Z}_{p^\infty}$$

(v) Show that $\mathbb{Z}_{p^\infty}/H_k = \mathbb{Z}_{p^\infty}$ for all k.

Definition. A variety $\mathcal{V}$ is called *finitely generated* if it contains some finite algebras $A_1, \ldots, A_n$ such that $\mathcal{V} = \mathrm{HSP}(A_1, \ldots, A_n)$.

Definition. A variety $\mathcal{V}$ is called *locally finite* if every finitely generated algebra in $\mathcal{V}$ is finite.

Exercise 41. Show that every finitely generated variety is locally finite.

Hint: let $B \in \mathrm{HSP}(\mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{n})$ be finitely generated ...