CSP lecture 16/17 winter semester – Problem Set 5

A ternary operation $m : A^3 \to A$ is called a *majority operation* if $m(a,a,b) = m(a,b,a) = m(b,a,a) = a$ for each $a, b \in A$ (note that for $|A| \leq 2$ there is a unique majority operation on A, otherwise there is more of them).

Problem 2. Let $R \subseteq A^n$ be a relation compatible with a majority operation on A. Denote $\pi_{i,j}(R)$ the projection of R onto the coordinates i, j ($1 \leq i, j \leq n$), that is,

$$\pi_{i,j}(R) = \{(a_i, a_j) : (a_1, \dots, a_n) \in R\} \ .$$

Prove that R is determined by these binary projections, that is,

$$(a_1, \dots, a_n) \in R \text{ if and only if } (\forall i, j,\ 1 \leq i, j \leq n)\ (a_i, a_j) \in \pi_{i,j}(R)$$

(Hint: start with $n = 3$)

Problem 3. Let $\mathbb{A} = (A; \dots)$ be a relational structure with a majority polymorphisms. Show that there exists a relational structure $\mathbb{B} = (A; \dots)$ which contains only binary relations such that $\mathbb{A}$ is pp-definable from $\mathbb{B}$ and $\mathbb{B}$ is pp-definable from $\mathbb{A}$. For $A = \{0,1\}$ conclude that CSP($\mathbb{A}$) $\leq_P$ 2–SAT (and thus CSP($\mathbb{A}$) is solvable in polynomial time).

Problem 4. Let $\mathbb{A} = (\{0,1\}; \dots)$ be a relational structure with polymorphism min (from Problem Set 4). Prove that CSP($\mathbb{A}$) is solvable in polynomial time. (Hint: show that each n-ary relation is an affine subspace of $\mathbb{Z}_2^n$.)

Problem 5. Let $\mathbb{A} = (\{0,1\}; C_0, C_1, H)$ be as in Problem Set 1 (the corresponding CSP is HORN-3-SAT). Let $S_{a_1,\dots,a_k}$ denote the k-ary relation $\{0,1\}^k \setminus \{(a_1, \dots, a_k)\}$, eg. $H = S_{110}$.

Prove that a relation $R \subseteq \{0,1\}^n$ is compatible with the binary minimum operation if and only if R is pp–definable from $\mathbb{A}$.

Below is a strategy to prove the harder implication $\Rightarrow$. It uses the following notation:

$$[n] = \{1, 2, \dots, n\}$$
$$\chi_{\mathbf{a}} = \{i \in [n] : a_i = 1\} \text{ for a tuple } \mathbf{a} = (a_1, \dots, a_n) \in \{0,1\}^n$$
$$\chi_R = \{\chi_{\mathbf{a}} : \mathbf{a} \in R\}$$

In words, $\chi_{\mathbf{a}}$ is the set of coordinates, where the tuple has 1. In this way, a tuple from $\{0,1\}^n$ corresponds to as a subset of $[n]$. The set χ_R describes the whole relation as a family of subsets of $[n]$.

- Prove that $S_{11\dots10}$ and $S_{11\dots1}$ are pp–definable from $\mathbb{A}$ (for any arity). It may be helpful to recall that $S_{110}(x,y,z)$ iff $(x \wedge y) \to z$.
- Assume first that $[n] \in \chi_R$ (that is, $(1,1,\dots,1) \in R$). For a subset $X \subseteq [n]$ denote

  $$Cl(X) := \bigcap_{Y : X \subseteq Y \in \chi_R} Y$$

  Prove that $X = Cl(X)$ if and only if $X \in \chi_R$ (use that $\wedge$ is a polymorphism of R).
- Prove that

  $$R(\mathbf{a}) \text{ if and only if } \bigwedge_{X : X = \{i_1, \dots, i_k\} \subseteq [n]} \ \bigwedge_{j : j \in Cl(X)} S_{11\dots10}(a_{i_1}, a_{i_2}, \dots, a_{i_k}, a_j)$$

- Using the above, give a pp–definition in the other case (when $[n] \notin \chi_R$).

Problem 6. Prove that for each relational structure $\mathbb{A} = (A; \dots)$ with $A = \{0,1\}$ either CSP($\mathbb{A}$) is solvable in polynomial time or CSP($\mathbb{A}$) is NP–complete. Describe the two cases in terms of polymorphisms.