CSP lecture 21/22 winter semester – Problem Set 5

A ternary operation $m : A^3 \to A$ is called a *majority operation* if $m(a,a,b) = m(a,b,a) = m(b,a,a) = a$ for each $a, b \in A$ (note that for $|A| \leq 2$ there is a unique majority operation on A, otherwise there are more of them).

Problem 1 Let $R \subseteq A^n$ be a relation compatible with a majority operation on A. Denote $\pi_{i,j}(R)$ the projection of R onto the coordinates i, j $(1 \leq i, j \leq n)$, that is,

$$\pi_{i,j}(R) = \{(a_i, a_j) : (a_1, \ldots, a_n) \in R\} \ .$$

Prove that R is determined by these binary projections, that is,

$$(a_1, \ldots, a_n) \in R \text{ if and only if } (\forall i, j,\ 1 \leq i, j \leq n)\ (a_i, a_j) \in \pi_{i,j}(R)$$

(Hint: start with $n = 3$)

Problem 2 Let $\mathbb{A} = (A; \ldots)$ be a relational structure with a majority polymorphism. Show that there exists a relational structure $\mathbb{B} = (A; \ldots)$ which contains only binary relations such that $\mathbb{A}$ is pp-definable from $\mathbb{B}$ and $\mathbb{B}$ is pp-definable from $\mathbb{A}$. For $A = \{0,1\}$, conclude that CSP($\mathbb{A}$) $\leq_P$ 2–SAT (and thus CSP($\mathbb{A}$) is solvable in polynomial time).

Problem 2' Let $\mathbb{A} = (\mathbb{Z}; R_1, \ldots, R_k)$, where all relations $R_1, \ldots, R_k$ admit a quantifier-free definition over the relations $y < x + c$ and $y = x + c$, where $c \in \mathbb{Z}$. E.g. R can be the 4-ary relation that holds on (x, y, z, t) iff $(x > y + 1 \vee x > z - 6) \wedge (x = z \Rightarrow t = y + 1)$ holds. Suppose that the ternary median operation is a polymorphism of $\mathbb{A}$. Show that CSP($\mathbb{A}$) is solvable in polynomial time.

Problem 3 Let $\mathbb{A} = (\{0,1\}; \ldots)$ be a relational structure with polymorphism min (from Problem Set 4). Show that each n-ary relation of $\mathbb{A}$ is an affine subspace of $\mathbb{Z}_2^n$. Conclude that CSP($\mathbb{A}$) is solvable in polynomial time.

Problem 4 Let $\mathbb{A} = (\{0,1\}; C_0, C_1, H)$ be as in Problem Set 1 (the corresponding CSP is HORN-3-SAT). Let $S_{a_1,\ldots,a_k}$ denote the k-ary relation $\{0,1\}^k \setminus \{(a_1, \ldots, a_k)\}$, e.g., $H = S_{110}$.

Prove that a relation $R \subseteq \{0,1\}^n$ is compatible with the binary minimum operation if and only if R is pp–definable from $\mathbb{A}$.

Below is a strategy to prove the harder implication $\Rightarrow$. It uses the following notation:

$$[n] = \{1, 2, \ldots, n\}$$
$$\chi_{\mathbf{a}} = \{i \in [n] : a_i = 1\} \text{ for a tuple } \mathbf{a} = (a_1, \ldots, a_n) \in \{0,1\}^n$$
$$\chi_R = \{\chi_{\mathbf{a}} : \mathbf{a} \in R\}$$

In words, $\chi_{\mathbf{a}}$ is the set of coordinates where the tuple has 1. In this way, a tuple from $\{0,1\}^n$ corresponds to a subset of $[n]$. The set χ_R describes the whole relation as a family of subsets of $[n]$.

- Prove that $S_{11\ldots10}$ and $S_{11\ldots1}$ are pp–definable from $\mathbb{A}$ (for any arity). It may be helpful to recall that $S_{110}(x, y, z)$ iff $(x \wedge y) \to z$.
- Assume first that $[n] \in \chi_R$ (that is, $(1, 1, \ldots, 1) \in R$). For a subset $X \subseteq [n]$ denote

$$Cl(X) := \bigcap_{Y : X \subseteq Y \in \chi_R} Y$$

  Prove that $X = Cl(X)$ if and only if $X \in \chi_R$ (use that $\wedge$ is a polymorphism of R).

- Prove that

$$R(\mathbf{a}) \text{ if and only if } \bigwedge_{X:X=\{i_1,\ldots,i_k\}\subseteq[n]} \quad \bigwedge_{j:j\in Cl(X)} S_{11\ldots 10}(a_{i_1}, a_{i_2}, \ldots, a_{i_k}, a_j)$$

- Using the above, give a pp–definition in the other case (when $[n] \notin \chi_R$).

Problem 5 Prove that for each relational structure $\mathbb{A} = (A; \ldots)$ with $A = \{0, 1\}$, either CSP($\mathbb{A}$) is solvable in polynomial time or CSP($\mathbb{A}$) is NP–complete (this is *Schaefer's dichotomy theorem* (1978)). Describe the two cases in terms of polymorphisms.