CSP DICHOTOMY FOR SPECIAL POLYADS

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Abstract. For a digraph $\mathbb{H}$, the Constraint Satisfaction Problem with template $\mathbb{H}$, or CSP($\mathbb{H}$), is the problem of deciding whether a given input digraph $\mathbb{G}$ admits a homomorphism to $\mathbb{H}$. The CSP dichotomy conjecture of Feder and Vardi states that for any digraph $\mathbb{H}$, CSP($\mathbb{H}$) is either in P or NP-complete. Barto, Kozik, Maróti and Niven (Proc. Amer. Math. Soc, 2009) confirmed the conjecture for a class of oriented trees called special triads. We generalize this result, establishing the dichotomy for a class of oriented trees which we call special polyads. We prove that every tractable special polyad has bounded width and provide the description of special polyads of width 1. We also construct a tractable special polyad which neither has width 1 nor admits any near-unaninimity polymorphism.

1. Introduction

Let $\mathbb{H}$ be a fixed finite digraph. The Constraint Satisfaction Problem with template $\mathbb{H}$, or CSP($\mathbb{H}$) for short, is the following decision problem:

INPUT: A finite digraph $\mathbb{G}$.
QUESTION: Is there a homomorphism from $\mathbb{G}$ to $\mathbb{H}$?

In graph theory, CSP($\mathbb{H}$) is also called $\mathbb{H}$-coloring problem. This class of problems has recently recieved a lot of attention, mainly because of the work of Feder and Vardi [7] from 1999. In this article the authors conjectured a large natural class of NP decision problems avoiding the complexity classes between P and NP-complete (assuming that P$\neq$NP). Many natural decision problems, such as k-SAT, graph k-colorability or solving systems of linear equations over finite fields belong to this class. In the same article they proved that each such problem can be expressed as CSP($\mathbb{G}$) for some digraph $\mathbb{G}$. Therefore their dichotomy conjecture can be formulated as follows:

Conjecture (The Dichotomy Conjecture)**.** *For every digraph* $\mathbb{H}$*,* CSP($\mathbb{H}$) *is either tractable or NP-complete.*

For brevity, we sometimes say that a digraph $\mathbb{H}$ is tractable if CSP($\mathbb{H}$) is tractable and NP-complete if CSP($\mathbb{H}$) is NP-complete.

The dichotomy was established for a number of special cases, including oriented paths (which are all tractable) [8], oriented cycles [6], undirected

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graphs [10] and many others. The work of Jeavons, Cohen and Gyssens [11], refined by Bulatov, Jeavons and Krokhin [4], has shown a strong connection between the constraint satisfaction problem and universal algebra. This "algebraic approach" led to a rapid development of the subject and is essential to our paper. For more information on the algebraic approach to CSP, see the survey of Krokhin, Bulatov and Jeavons [12]. Using the algebraic approach (in particular, a result of Maróti and McKenzie [14, Theorem 1.1]), Barto, Kozik and Niven [3] established the CSP dichotomy for digraphs without sources or sinks (i.e., digraphs such that each vertex has an incoming and an outgoing edge).

In the class of all digraphs, oriented trees are in some sense very far from digraphs without sources or sinks. Except the oriented paths, the simplest class of oriented trees are the *triads* (i.e., oriented trees with one vertex of degree 3 and all other vertices of degree 1 or 2). Though the dichotomy conjecture for triads remains open, it was confirmed by Barto, Kozik, Maróti and Niven [2] for the so-called *special triads*, a certain class of triads possessing enough structure to provide a structural description of the tractable and NP-complete cases. Our paper generalizes their result to the *special polyads* (which will be defined later). A *polyad* is an oriented tree with at most one vertex of degree greater than 2. Special polyads are a straightforward generalization of special triads.

A digraph $\mathbb{G}$ is said to have *bounded width* if CSP($\mathbb{G}$) can be solved in polynomial time by local consistency methods (see [7]). It was proved earlier that if $\mathbb{G}$ has a compatible *majority operation* [7] or compatible *totally symmetric idempotent operations* of all arities [5], then it has bounded width (and thus CSP($\mathbb{G}$) is tractable). In [13], Larose and Zádori conjectured a full characterization of digraphs with bounded width. This conjecture was confirmed by Barto and Kozik [1]. Our paper relies on their result that digraphs with compatible *weak near-unanimity operations* of almost all arities have bounded width (see Theorem 2.4).

In [2], the authors proved that every special triad is either NP-complete or it has a compatible majority operation or compatible totally symmetric idempotent operations of all arities. We concentrated on the special polyads for several reasons. Though the special polyads do possess the same kind of structure as the special triads, allowing us to apply some of the techniques used in [2], it was not obvious whether the results from [2] can be extended to them.

We were also interested in the following question: Will every tractable special polyad be tractable for a "simple" reason, by which we mean satisfying some strong conditions ensuring tractability (e.g., possessing a compatible majority operation, near-unanimity operation or totally symmetric idempotent operations of all arities)? We were not able to find such a strong condition for every tractable special polyad, therefore we need the result from [1] in its full strength. Using our techniques we constructed a special polyad which admits neither totally symmetric idempotent operations of all arities nor any near-unanimity operation, but still is tractable (see Section 5). Moreover, we wanted to determine whether there exist tractable special polyads without bounded width. The answer to this question is negative.

We believe that the techniques developed in this article can be applied to a far broader class of oriented trees.

2. Preliminaries

2.1. **Digraphs.** A *digraph* $\mathbb{G} = (G, E)$ is a set of *vertices* G together with a binary relation $E \subseteq G^2$, the *edge* relation. For $\langle a, b\rangle \in E$ we write $a \xrightarrow{\mathbb{G}} b$ or simply $a \to b$ when there is no danger of confusion. The degree of a vertex is the number $\deg(v) = |\{\langle a, b\rangle \in E : a = v \text{ or } b = v\}|$.

A digraph $\mathbb{G}'$ is a *subgraph* of $\mathbb{G}$ (we write $\mathbb{G}' \subseteq \mathbb{G}$), if $G' \subseteq G$ and $E' \subseteq E$. If $E' = E \cap {G'}^2$, then $\mathbb{G}'$ is an *induced subgraph* of $\mathbb{G}$ (or a subgraph *induced by G'*), denoted by $\mathbb{G}[G']$.

Let $\mathbb{G}$ and $\mathbb{H}$ be digraphs. A mapping $f : G \to H$ is a *homomorphism* from $\mathbb{G}$ to $\mathbb{H}$, if it preserves the edges, i.e., for all $a, b \in \mathbb{G}$ such that $a \xrightarrow{\mathbb{G}} b$ we have $f(a) \xrightarrow{\mathbb{H}} f(b)$. We say that $\mathbb{G}$ is *homomorphic* to $\mathbb{H}$ and write $\mathbb{G} \to \mathbb{H}$, if there exists a homomorphism from $\mathbb{G}$ to $\mathbb{H}$. A digraph $\mathbb{G}$ is a *core*, if every homomorphism $\mathbb{G} \to \mathbb{G}$ is bijective.

For each digraph $\mathbb{H}$ there exists a unique (up to isomorphism) core digraph $\mathbb{H}'$ such that $\mathbb{H} \leftrightarrow \mathbb{H}'$, it is called the *core of* $\mathbb{H}$ and denoted $\mathrm{core}(\mathbb{H})$. For any digraph $\mathbb{G}$, $\mathbb{G} \to \mathbb{H}$ if and only if $\mathbb{G} \to \mathbb{H}'$.

Let $\mathbb{G}_1, \dots, \mathbb{G}_n$ be digraphs. The *product* of $\mathbb{G}_1, \dots, \mathbb{G}_n$ is the digraph $\prod_{i=1}^n \mathbb{G}_i = (G_1 \times \cdots \times G_n, E)$ where $\langle \bar{a}, \bar{b}\rangle \in E$ iff $\langle a_i, b_i\rangle \in E_i$ for each $i = 1, \dots, n$. The product of n copies of $\mathbb{G}$ is called the *n-th power* of $\mathbb{G}$ and denoted $\mathbb{G}^n$.

An *oriented path* of length n is a digraph $\mathbb{P} = (P, E)$ with pairwise distinct vertices $P = \{v_0, v_1, \dots, v_n\}$ and edges $E = \{e_0, e_1, \dots, e_{n-1}\}$ such that $e_i \in \{\langle v_i, v_{i+1}\rangle, \langle v_{i+1}, v_i\rangle\}$ for each i. The vertex v_0 is called the *initial vertex*, denoted by $\mathrm{init}(\mathbb{P})$, and v_n is called the *terminal vertex*, denoted by $\mathrm{term}(\mathbb{P})$.

Let $\mathbb{G} = (G, E)$ be a digraph and $a, b \in G$. We say that a is *connected* to b in $\mathbb{G}$ via a path $\mathbb{P}$ if $\mathbb{P} \subseteq \mathbb{G}$, $a = \mathrm{init}(\mathbb{P})$ and $b = \mathrm{term}(\mathbb{P})$. By the *distance* of two connected vertices a, b (denoted $\mathrm{dist}_{\mathbb{G}}(a, b)$) we mean the minimal length of an oriented path connecting a to b. The relation of connectedness is an equivalence relation on G. Its classes are called *components of connectivity.* $\mathbb{G}$ is *connected* if each two vertices $a, b \in G$ are connected.

2.2. **Oriented trees.** A digraph $\mathbb{T} = (T, E)$ is called an *oriented tree* if for each $a, b \in T$ there exists precisely one path connecting a to b. (Alternatively, an oriented tree is a digraph which can be obtained from an undirected tree, i.e., connected undirected graph without cycles, by orienting its edges.) There exists a unique mapping $\mathrm{lvl} : T \to \mathbb{N} \cup \{0\}$ satisfying the following conditions:

(i) If $a \to b$, then $\mathrm{lvl}(b) = \mathrm{lvl}(a) + 1$.
(ii) There exists a vertex $a \in T$ with $\mathrm{lvl}(a) = 0$.

For $a \in T$, $\mathrm{lvl}(a)$ is called the *level* of a. The *height* of $\mathbb{T}$, denoted by $\mathrm{hgt}(\mathbb{T})$, is the highest level of a vertex in $\mathbb{T}$. For any $i \geq 0$ we define the set

$$\mathrm{Level}_{\mathbb{T}}(i) = \{a \in T : \mathrm{lvl}(a) = i\}$$

(dropping the index when $\mathbb{T}$ is known from the context).

An oriented path $\mathbb{P}$ is *minimal* if $\mathrm{lvl}(\mathrm{init}(\mathbb{P})) = 0$, $\mathrm{lvl}(\mathrm{term}(\mathbb{P})) = \mathrm{hgt}(\mathbb{P})$ and $0 < \mathrm{lvl}(v) < \mathrm{hgt}(\mathbb{P})$ for all $v \in P \setminus \{\mathrm{init}(\mathbb{P}), \mathrm{term}(\mathbb{P})\}$. Below is an example of a minimal path.

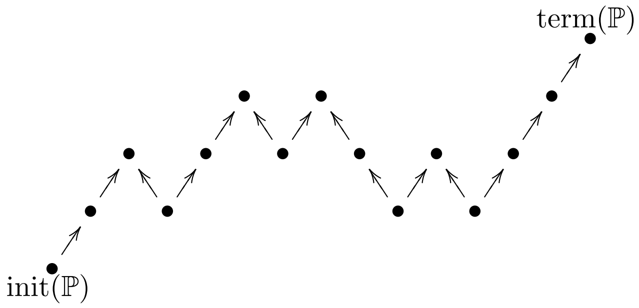


FIGURE 1. A minimal path of height 4.

Let $\mathbb{P}_1, \ldots, \mathbb{P}_n$ be minimal paths of the same height l. It is known that there exists a minimal path $\mathbb{Q}$ of height l homomorphic to all the paths $\mathbb{P}_1, \ldots, \mathbb{P}_n$ (see for example [9]).

2.3. **The Constraint Satisfaction Problem.** Let $\mathbb{H}$ be a digraph. The Constraint Satisfaction Problem with template $\mathbb{H}$ (or $\mathrm{CSP}(\mathbb{H})$ for short, also known as the $\mathbb{H}$-coloring roblem) is the following decision problem:

INPUT: A digraph $\mathbb{G}$.
QUESTION: Is there a homomorphism from $\mathbb{G}$ to $\mathbb{H}$?

The CSP dichotomy conjecture of Feder and Vardi from [7] can be stated as follows:

Conjecture (The Dichotomy Conjecture)**.** *For every digraph* $\mathbb{H}$*,* $\mathrm{CSP}(\mathbb{H})$ *is either tractable or NP-complete.*

A digraph $\mathbb{H}$ is said to have *bounded width*, if $\mathrm{CSP}(\mathbb{H})$ can be solved in polynomial time by local consistency methods (see [7]), and *width 1*, if it can be solved by $(1, k)$-consistency algorithm for some fixed k (see [5]).

It is easily seen that $\mathrm{CSP}(\mathbb{H}) = \mathrm{CSP}(\mathrm{core}(\mathbb{H}))$. Thus we can restrict ourselves to digraphs which are cores.

2.4. **CSP and compatible operations.** In this subsection we introduce certain special types of operations and their connection to the complexity of $\mathrm{CSP}(\mathbb{H})$. First, we will define the notion of compatible operation, a generalization of endomorphism. Recall that by an *r-ary operation* on a set A we mean a mapping $A^r \to A$. Note that the r-ary operations compatible with $\mathbb{H}$ are precisely the homomorphisms from $\mathbb{H}^r$ to $\mathbb{H}$.

Definition 2.1. Let $\mathbb{H} = (H, E)$ be a digraph and let f be an r-ary operation on H. We say that f is *compatible with* $\mathbb{H}$ (or is a *polymorphism* of $\mathbb{H}$), if it satisfies the following condition: if $a_i, b_i \in H$ and $a_i \xrightarrow{\mathbb{H}} b_i$ for $i = 1, \ldots, r$, then $f(a_1, \ldots, a_r) \xrightarrow{\mathbb{H}} f(b_1, \ldots, b_r)$.

The compatible operations play a key role in the algebraic approach to CSP (see [12] for more details). In the rest of this subsection we introduce three theorems connecting the computational complexity of CSP($\mathbb{H}$) with existence of certain "nice" compatible operations. We will later use these algebraic tools to prove tractability or NP-completeness of CSP for special polyads, which we will introduce in the next section.

Definition 2.2. An r-ary operation f on a set A is *idempotent*, if it satisfies $f(a, a, \dots, a) = a$ for all $a \in A$.

(i) Let $r \geq 2$. An r-ary operation ω on A is called a *weak near-unanimity* operation (or a *weak-NU*), if it is idempotent and satisfies

$$\omega(a, \dots, a, b) = \omega(a, \dots, a, b, a) = \dots = \omega(b, a, \dots, a)$$

for all $a, b \in A$. We define the binary operation $\circ_\omega$ by setting

$$a \circ_\omega b = \omega(a, \dots, a, b).$$

(ii) A weak-NU ν of arity ≥ 3 is called a *near-unanimity* operation (*NU*), if $a \circ_\nu b = a$ for all $a, b \in A$. A ternary NU is called a *majority* operation.

(iii) An r-ary operation τ is *totally symmetric idempotent* (*TSI*), if it is idempotent and satisfies

$$\tau(a_1, a_2, \dots, a_r) = \tau(a'_1, a'_2, \dots, a'_r)$$

whenever $\{a_1, a_2, \dots, a_r\} = \{a'_1, a'_2, \dots, a'_r\}$. (Note that a totally symmetric idempotent operation is a weak-NU.)

Remark. It can be easily seen that an operation obtained by composing operations compatible with $\mathbb{H}$ is also compatible with $\mathbb{H}$. In particular, if ω is a weak-NU operation compatible with $\mathbb{H}$, then $\circ_\omega$ is also compatible with $\mathbb{H}$, as we can obtain it by composing ω with the *projection operations* (i.e., the operations $p^i_r(x_1, \dots, x_r) = x_i$, which are indeed compatible with $\mathbb{H}$).

Our tool to prove NP-completeness is the following theorem, a combination of a result of Bulatov, Jeavons and Krokhin from [4] and a result of Maróti and McKenzie [14].

Theorem 2.3. *Let $\mathbb{H}$ be a digraph. If* core($\mathbb{H}$) *admits no compatible weak-NU operation, then* CSP($\mathbb{H}$) *is NP-complete.*

The algebraic dichotomy conjecture states that the converse is also true. It can be formulated as follows:

Conjecture (The Algebraic Dichotomy Conjecture)**.** *Let $\mathbb{H}$ be a core digraph. If $\mathbb{H}$ admits a compatible weak-NU operation, then* CSP($\mathbb{H}$) *is tractable, otherwise it is NP-complete.*

The algebraic dichotomy conjecture is a strengthening of the conjecture of Feder and Vardi. In Theorem 3.2 we prove that this conjecture holds for special polyads. To prove tractability, we apply the following theorem by Barto and Kozik [1]:

Theorem 2.4. *Let $\mathbb{H}$ be a core digraph. The following conditions are equivalent:*

(i) *$\mathbb{H}$ has bounded width.*

(ii) $\mathbb{H}$ *admits compatible weak-NU operations of almost all arities (i.e., there exists* r_0 *such that for all* $r \geq r_0$ $\mathbb{H}$ *admits a compatible* r*-ary weak-NU).*

The following characterization of digraphs of width 1 is due to Dalmau and Pearson [5]:

Theorem 2.5. *Let* $\mathbb{H}$ *be a core digraph. The following conditions are equivalent:*

(i) $\mathbb{H}$ *has width 1.*
(ii) $\mathbb{H}$ *admits compatible totally symmetric idempotent operations of all arities.*

3. Special polyads

3.1. **The definition.** In this subsection we define the *special polyads*, a certain class of oriented trees generalizing the special triads treated in [2]. An oriented tree is called a *polyad* if at most one of its vertices has degree greater than 2.

Definition 3.1. (i) By a *half-branch* we mean a minimal path, the *root* of the half-branch $\mathbb{P}$ is its initial vertex.
(ii) Let $\mathbb{P}$ and $\mathbb{P}'$ be two disjoint minimal paths of the same height. The *branch* $\langle \mathbb{P}, \mathbb{P}' \rangle$ is the oriented tree obtained by identifying the terminal vertices of $\mathbb{P}$ and $\mathbb{P}'$ into a single vertex. The *root* of the branch $\langle \mathbb{P}, \mathbb{P}' \rangle$ is the initial vertex of $\mathbb{P}$.
(iii) Let n, k be nonnegative integers, $n + k > 0$ and let $\langle \mathbb{P}_i, \mathbb{P}'_i \rangle$ $(1 \leq i \leq n)$ and $\mathbb{P}_{n+i}$ $(1 \leq i \leq k)$ be n branches and k half-branches of the same height (pairwise disjoint). The *special polyad* given by $\langle \mathbb{P}_1, \mathbb{P}'_1 \rangle, \dots, \mathbb{P}_{n+k}$ is the oriented tree $\mathbb{T}$ obtained by identifying the roots of $\langle \mathbb{P}_1, \mathbb{P}'_1 \rangle, \dots, \mathbb{P}_{n+k}$ into a single vertex, the *root*.

In the following, we will denote the root of $\mathbb{T}$ by 0, the initial vertex of $\mathbb{P}'_i$ by i and the top-level vertex of $\langle \mathbb{P}_i, \mathbb{P}'_i \rangle$ or $\mathbb{P}_i$ by $\widehat{i}$ (see the figure below, arrows indicate "direction" of paths). Let us also define

$$\mathrm{Base}_{\mathbb{T}} = \mathrm{Level}_{\mathbb{T}}(0) = \{0, 1, \dots, n\},$$
$$\mathrm{Top}_{\mathbb{T}} = \mathrm{Level}_{\mathbb{T}}(\mathrm{hgt}(\mathbb{T})) = \{\widehat{1}, \dots, \widehat{n+k}\}$$
$$\mathrm{Half}_{\mathbb{T}} = \{\widehat{n+1}, \dots, \widehat{n+k}\}$$

and

$$\mathrm{Paths}_{\mathbb{T}} = \{\mathbb{P}_1, \mathbb{P}_2, \dots, \mathbb{P}_{n+k}, \mathbb{P}'_1, \mathbb{P}'_2, \dots, \mathbb{P}'_n\}$$

(we will usually drop the index $\mathbb{T}$).

In our terminology, a special triad from [2] is a special polyad with 3 branches and no half-branches.

3.2. **The main result.** The following theorem is the main result of our paper.

Theorem 3.2. *For every special polyad* $\mathbb{T}$*,* CSP($\mathbb{T}$) *is either NP-complete or tractable. More specifically,*

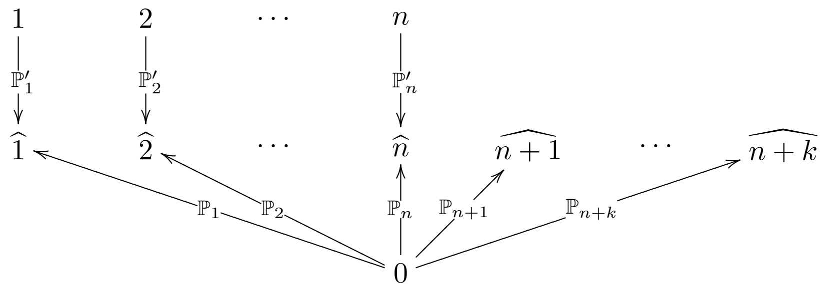


FIGURE 2. A special polyad.

(i) core($\mathbb{T}$) *has bounded width, if and only if* core($\mathbb{T}$) *admits a compatible weak near-unanimity operation, otherwise* $\mathbb{T}$ *is NP-complete.*
(ii) $\mathbb{T}$ *has width* 1*, if and only if* $\mathbb{T}$ *admits a compatible binary weak-NU (i.e., a binary idempotent commutative operation).*

Corollary 3.3. *The CSP dichotomy conjecture holds for special polyads.*

We will prove Theorem 3.2 in the next section.

4. Proof of Theorem 3.2

For a positive integer n, let $[n] = \{1, \ldots, n\}$.

4.1. **Preliminary results.** First, we will reduce the problem to core special polyads. In the next two easy lemmata we prove that the core of a special polyad is still a special polyad and inherits its "nice" polymorphisms.

Lemma 4.1. *Let* $\mathbb{T}$ *be a special polyad with* n *branches and* k *half-branches. Then* core($\mathbb{T}$) *is a special polyad with* n' *branches and* k' *half-branches where* $0 < n' + k' \leq n + k$.

Proof. It is easily seen that a homomorphism from a minimal path of height l to an oriented tree of height l maps the initial vertex to a vertex of level 0 and the terminal vertex to a vertex of level l. The rest follows directly from this fact. □

Lemma 4.2. *Let* $\mathbb{H}$ *be a digraph. If* $\mathbb{H}$ *has a compatible* r*-ary weak-NU* ω*, then there exists an* r*-ary weak-NU* ω' *compatible with* core($\mathbb{H}$) *such that if* ω *is a NU, then* ω' *is also a NU and if* ω *is TSI, then* ω' *is also TSI.*

Proof. Let $f : \mathbb{H} \to \text{core}(\mathbb{H})$ and $g : \text{core}(\mathbb{H}) \to \mathbb{H}$ be homomorphisms. Then the homomorphism $f \circ g : \text{core}(\mathbb{H}) \to \text{core}(\mathbb{H})$ is bijective and since core(H) is finite, there exists $k > 0$ such that $(f \circ g)^k = \text{id}_{\text{core}(\mathbb{H})}$. For $\bar{x} \in \text{core}(H)^r$ we define $\omega'(\bar{x}) = (f \circ (g \circ f)^{k-1})(\omega(g(x_1), \ldots, g(x_r)))$. The rest is easy. □

In the rest of this subsection we show that if an oriented tree $\mathbb{T}$ has a compatible partial weak-NU or NU defined for the tuples of vertices of the same level, it can be easily extended to a full weak-NU or NU. Similar fact is true for having partial TSI operations of all arities.

Let A be any set and $K \subseteq A^r$. By a *partial* r*-ary operation* on a set A with *domain* K we mean a mapping $f : K \to A$. We define *partial* *weak-NU*, *partial NU* and *partial TSI* in an obvious fashion, restricting the conditions required in Definition 2.2 to tuples from the domain. The notion of compatibility generalizes to partial operations similarly:

Definition 4.3. Let $\mathbb{H} = (H, E)$ be a digraph and let f be a partial r-ary operation on H with domain K. We say that f is *compatible with* $\mathbb{H}$, if it satisfies the following condition: if $\bar{a}, \bar{b} \in K$ and $a_i \xrightarrow{\mathbb{H}} b_i$ for $i = 1, \dots, r$, then $f(\bar{a}) \xrightarrow{\mathbb{H}} f(\bar{b})$.

Lemma 4.4. *Let $\mathbb{T}$ be an oriented tree.*

(i) *Each partial weak-NU compatible with $\mathbb{T}$ with domain $\bigcup_{k=0}^{\operatorname{hgt}\mathbb{T}} \operatorname{Level}(k)^r$ (i.e., tuples of vertices of the same level) can be extended to a weak-NU $\omega' \supseteq \omega$ compatible with $\mathbb{T}$ in such a way that if ω is a partial NU, then ω' is a NU.*

(ii) *Each partial TSI τ_r compatible with $\mathbb{T}$ with domain $\bigcup_{k=0}^{\operatorname{hgt}\mathbb{T}} \operatorname{Level}(k)^r$ can be extended to a TSI operation $\tau_r' \supseteq \tau_r$ compatible with $\mathbb{T}$.*

Proof. To prove (i), we define ω' as follows (let $\bar{a} \in T^r$):

(1) If all the vertices a_i have the same level, then we put $\omega'(\bar{a}) = \omega(\bar{a})$.

(2) If there exists $i \in [r]$ such that $\operatorname{lvl}(a_j) = k$ for all $j \neq i$ and $\operatorname{lvl}(a_i) \neq k$, then

  (2a) if $r = 2$, we define $\omega'(a_1, a_2) = a_1$ if $\operatorname{lvl}(a_1) < \operatorname{lvl}(a_2)$ and $\omega'(a_1, a_2) = a_2$ else,

  (2b) if $r \geq 3$, we define $\omega'(\bar{a}) = a_2$ if $i = 1$ and $\omega'(\bar{a}) = a_1$ else.

(3) In all other cases we put $\omega'(\bar{a}) = a_1$.

First, we will prove that ω' is a weak-NU. Let $a, b \in T$ be arbitrary. We want to prove that $\omega'(a, \dots, a, b) = \omega'(a, \dots, a, b, a) = \dots = \omega'(b, a, \dots, a)$. Clearly, for all of these tuples the same case of the definition applies. In case (1) the equalities hold because ω is a weak-NU, while in case (2) the result is independent on the coordinate at which the 'b' occurs. Moreover, $a \circ_{\omega'} b = a$ in case (2b); and so ω' is a NU whenever ω is a partial NU.

To prove compatibility, choose $\bar{a}, \bar{b} \in T^r$ such that $a_i \to b_i$ for each i. The same case of the definition applies for both $\omega'(\bar{a})$ and $\omega'(\bar{b})$. From the compatibility of ω' (case (1)) and the fact that $a_i \to b_i$ (cases (2) and (3)) it follows that $\omega'(\bar{a}) \to \omega'(\bar{b})$ and (i) is proved.

In order to prove (ii), for $\bar{a} \in T^r$ let $a_{i_1}, \dots, a_{i_k}$ $(i_1 < \dots < i_k)$ be the vertices of minimal level among $\{a_1, \dots, a_r\}$. We define

$$\tau_r'(\bar{a}) = \tau_r(a_{i_1}, \dots, a_{i_k}, \underbrace{a_{i_k}, \dots, a_{i_k}}_{(r-k)\text{-times}}).$$

It is easy to check that τ_r' is TSI. The compatibility of τ_r' follows immediately from the compatibility of τ_r. □

4.2. **Reduction to $\mathcal{A}(\mathbb{T})$.** Let $\mathbb{T}$ be a special polyad. In this subection we translate the question if $\mathbb{T}$ has a compatible r-ary weak-NU, NU or TSI operations of all arities into a question whether there exists a weak-NU, NU or TSI operations of all arities compatible with a certain family $\mathcal{A}(\mathbb{T})$ of digraphs on the set $\operatorname{Base} \cup \operatorname{Top}$. This translation significantly simplifies the proof of Theorem 3.2 and also allows us to construct special polyads with some desired properties such as the one in Section 5.

Definition 4.5.

(i) Let $\mathcal{I} \subseteq \mathrm{Paths}$ be nonempty. We define $\bigotimes_{\mathbb{S}\in\mathcal{I}} \mathbb{S}$ to be the component of connectivity of the digraph $\prod_{\mathbb{S}\in\mathcal{I}} \mathbb{S}$ containing the tuple $\langle \mathrm{init}(\mathbb{S}) : \mathbb{S} \in \mathcal{I}\rangle$. (Note that $\bigotimes$ is, up to isomorphism, associative and commutative.)

(ii) Let us denote by $\mathcal{R}$ the mapping from the set $\mathcal{P}(\mathrm{Paths})$ (the *power set* of Paths) to itself defined by

$$\mathcal{R}(\mathcal{I}) = \{\mathbb{P} \in \mathrm{Paths} : \bigotimes_{\mathbb{S}\in\mathcal{I}} \mathbb{S} \to \mathbb{P}\}$$

for $\mathcal{I} \neq \emptyset$; we put $\mathcal{R}(\emptyset) = \emptyset$.

We will need the following easy lemma.

Lemma 4.6. *Let $\mathcal{I} = \{\mathbb{S}_1, \dots, \mathbb{S}_r\} \subseteq \mathrm{Paths}$ be nonempty. Then the tuple of terminal vertices $\langle \mathrm{term}(\mathbb{S}_1), \dots, \mathrm{term}(\mathbb{S}_r)\rangle$ belongs to $\bigotimes_{i=1}^r \mathbb{S}_i$ and any homomorphism $\psi : \bigotimes_{i=1}^r \mathbb{S}_i \to \mathbb{T}$ maps the tuple $\langle \mathrm{init}(\mathbb{S}_1), \dots, \mathrm{init}(\mathbb{S}_r)\rangle$ to a vertex of level* 0 *and $\langle \mathrm{term}(\mathbb{S}_1), \dots, \mathrm{term}(\mathbb{S}_r)\rangle$ to a vertex of level* $\mathrm{hgt}(\mathbb{T})$*; the image of $\bigotimes_{i=1}^r \mathbb{S}_i$ under ψ is a minimal path from* Paths.

Proof. Let $\mathbb{Q}$ be a minimal path (of height $\mathrm{hgt}(\mathbb{T})$) homomorphic to all the paths $\mathbb{S}_1, \dots, \mathbb{S}_r$ via $\varphi_1, \dots, \varphi_r$, respectively. Consider the natural homomorphism $\varphi : \mathbb{Q} \to \prod_{i=1}^r \mathbb{S}_i$ defined by $\varphi(\bar{x}) = \langle \varphi_1(x_1), \dots, \varphi_r(x_r)\rangle$. Since $\mathbb{Q}$ is connected, it follows that $\varphi : \mathbb{Q} \to \bigotimes_{i=1}^r \mathbb{S}_i$; and thus $\varphi(\mathrm{term}(\mathbb{Q})) = \langle \mathrm{term}(\mathbb{S}_1), \dots, \mathrm{term}(\mathbb{S}_r)\rangle \in \bigotimes_{i=1}^r \mathbb{S}_i$. The homomorphism $\psi \circ \varphi : \mathbb{Q} \to \mathbb{T}$ maps $\mathbb{Q}$ onto a minimal path $\mathbb{P} \in \mathrm{Paths}$. Therefore $\psi(\mathrm{init}(\mathbb{S}_1), \dots, \mathrm{init}(\mathbb{S}_r)) = (\psi \circ \varphi)(\mathrm{init}(\mathbb{Q})) = \mathrm{init}\,\mathbb{P}$ has level 0 and $\psi(\mathrm{term}(\mathbb{S}_1), \dots, \mathrm{term}(\mathbb{S}_r))$ has level $\mathrm{hgt}(\mathbb{T})$. The rest is obvious. $\square$

In the following lemma we prove that $\mathcal{R}$ is a *closure operator* on the set Paths.

Lemma 4.7. *The following statements hold:*

(i) $\mathcal{I} \subseteq \mathcal{R}(\mathcal{I})$ *for any* $\mathcal{I} \subseteq \mathrm{Paths}$. *(extensivity)*

(ii) *If* $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathrm{Paths}$, *then* $\mathcal{R}(\mathcal{I}) \subseteq \mathcal{R}(\mathcal{J})$. *(monotonicity)*

(iii) $\mathcal{R}(\mathcal{R}(\mathcal{I})) = \mathcal{R}(\mathcal{I})$ *for all* $\mathcal{I} \subseteq \mathrm{Paths}$. *(idempotency)*

Proof. In the following, let $\mathcal{I} = \{\mathbb{S}_1, \dots, \mathbb{S}_r\}$. The projection homomorphisms $\pi_j(\bar{x}) = x_j$ witness $\bigotimes_{i=1}^r \mathbb{S}_i \to \mathbb{S}_j$ for all j and (i) is proved.

To prove (ii), let $\mathbb{P} \in \mathcal{R}(\mathcal{I})$, $\varphi : \bigotimes_{i=1}^r \mathbb{S}_i \to \mathbb{P}$. By (i), for each i there exists a (projection) homomorphism $\pi_{\mathbb{S}_i} : \bigotimes_{\mathbb{S}\in\mathcal{J}} \mathbb{S} \to \mathbb{S}_i$. The mapping $\psi : \bigotimes_{\mathbb{S}\in\mathcal{J}} \mathbb{S} \to \mathbb{P}$ defined by $\psi(\bar{x}) = \varphi(\pi_{\mathbb{S}_1}(\bar{x}), \dots, \pi_{\mathbb{S}_r}(\bar{x}))$ is a homomorphism witnessing $\mathbb{P} \in \mathcal{R}(\mathcal{J})$.

It remains to prove (iii). The inclusion $\mathcal{R}(\mathcal{R}(\mathcal{I})) \supseteq \mathcal{R}(\mathcal{I})$ follows from (i). Let $\mathbb{P} \in \mathcal{R}(\mathcal{R}(\mathcal{I}))$ and let $\varphi : \bigotimes_{\mathbb{S}\in\mathcal{R}(\mathcal{I})} \mathbb{S} \to \mathbb{P}$. For each $\mathbb{S} \in \mathcal{R}(\mathcal{I})$ there exists a homomorphism $\varphi_{\mathbb{S}} : \bigotimes_{i=1}^r \mathbb{S}_i \to \mathbb{S}$. Similarly as before the composition $\psi(\bar{x}) = \varphi(\langle \varphi_{\mathbb{S}}(\bar{x}) : \mathbb{S} \in \mathcal{R}(\mathcal{I})\rangle)$ is a homomorphism from $\bigotimes_{i=1}^r \mathbb{S}_i$ to $\mathbb{P}$, and the proof is finished. $\square$

Now we are ready to define the family $\mathcal{A}(\mathbb{T})$.

Definition 4.8. For any $\mathcal{I} \subseteq$ Paths, let $\mathbb{T}(\mathcal{I})$ be the digraph on the set Base $\cup$ Top defined by the following condition:

$$a \xrightarrow{\mathbb{T}(\mathcal{I})} b \text{ iff } a \text{ is connected to } b \text{ via } \mathbb{P} \text{ for some } \mathbb{P} \in \mathcal{R}(\mathcal{I}).$$

Let us denote by $\mathcal{A}(\mathbb{T})$ the family of digraphs $\mathcal{A}(\mathbb{T}) = \{\mathbb{T}(\mathcal{I}) : \mathcal{I} \subseteq \text{Paths}\}$. We say that an operation on the set Base $\cup$ Top is *compatible with* $\mathcal{A}(\mathbb{T})$, if it is compatible with all the digraphs $\mathbb{T}(\mathcal{I}) \in \mathcal{A}(\mathbb{T})$.

Below is a figure of the digraph $\mathbb{T}$(Paths). From Lemma 4.7 it follows that all digraphs from $\mathcal{A}(\mathbb{T})$ are subgraphs of this digraph.

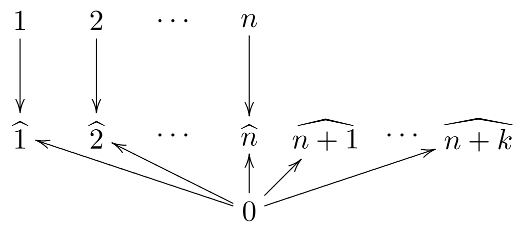


FIGURE 3. The digraph $\mathbb{T}$(Paths).

The following immediate corollary summarizes the connection between $\mathcal{R}$ and compatible operations of $\mathbb{T}$.

Corollary 4.9. *Let f be an r-ary operation compatible with $\mathbb{T}$ and $\mathcal{I} \subseteq$* Paths. *If $a_i \xrightarrow{\mathbb{T}(\mathcal{I})} b_i$ for all $i = 1, \dots, r$, then*

$$f(\bar{a}) \xrightarrow{\mathbb{T}(\mathcal{I})} f(\bar{b}).$$

Finally, we conclude this section with the "reduction" lemma, which allows us to look for compatible weak-NUs on $\mathcal{A}(\mathbb{T})$, a family of quite simple digraphs, instead of $\mathbb{T}$.

Lemma 4.10. *Let $\mathbb{T}$ be a special polyad. The following statements hold:*

(i) $\mathbb{T}$ *admits an r-ary compatible weak-NU, if and only if $\mathcal{A}(\mathbb{T})$ admits an r-ary compatible weak-NU.*
(ii) $\mathbb{T}$ *admits an r-ary compatible NU, if and only if $\mathcal{A}(\mathbb{T})$ admits an r-ary compatible NU.*
(iii) $\mathbb{T}$ *admits an r-ary compatible TSI, if and only if $\mathcal{A}(\mathbb{T})$ admits an r-ary compatible TSI.*

Proof. For an r-ary operation f compatible with $\mathbb{T}$, let f' be the restriction of f to the domain $\text{Base}^r \cup \text{Top}^r$. Choose arbitrary $\mathcal{I} \subseteq$ Paths, $\bar{a} \in \text{Base}^r$ and $\bar{b} \in \text{Top}^r$ such that $a_i \xrightarrow{\mathbb{T}(\mathcal{I})} b_i \, (1 \leq i \leq r)$. From the previous corollary it follows that the partial operation f' is compatible with $\mathcal{A}(\mathbb{T})$. The first implications now follow from Lemma 4.4 (which can be easily generalized to compatibility with a family of oriented trees on a set), as the properties of being weak-NU, NU or TSI are preserved by restriction.

It remains to prove the converse implications. For each $\mathcal{I} \subseteq$ Paths we fix an arbitrary $\mathbb{S}_{\mathcal{I}} \in \mathcal{I}$ and whenever $\bigotimes_{\mathbb{S} \in \mathcal{I}} \mathbb{S}$ is homomorphic to $\mathbb{P} \in$ Paths, we fix a homomorphism $\varphi_{\mathcal{I},\mathbb{P}} : \bigotimes_{\mathbb{S} \in \mathcal{I}} \mathbb{S} \to \mathbb{P}$ in such a way that if $\mathbb{P} \in \mathcal{I}$, then $\varphi_{\mathcal{I},\mathbb{P}}$ is the projection homomorphism.

To prove the converse implications of (i) and (ii), let ω' be an r-ary weak-NU compatible with $\mathcal{A}(\mathbb{T})$. We will define a partial operation ω on $\mathbb{T}$ with domain $\bigcup_{k=0}^{\mathrm{hgt}\,\mathbb{T}} \mathrm{Level}(k)^r$. Let $\bar{a} \in \mathrm{Level}(k)^r$. For $k \notin \{0, \mathrm{hgt}(\mathbb{T})\}$, let $\mathbb{S}_i \in \mathrm{Paths}$ be such that $a_i \in \mathbb{S}_i$ and denote the set $\{\mathbb{S}_1, \ldots, \mathbb{S}_r\}$ by $\mathcal{I}$. For each i let a'_i be the vertex from $\{a_1, \ldots, a_r\} \cap \mathbb{S}_i$ *second closest* to $\mathrm{init}(\mathbb{S}_i)$. (To be precise, if $\{a_1, \ldots, a_r\} \cap \mathbb{S}_i = \{a_i\}$, then $a'_i = a_i$, else if a_j is the vertex from $\{a_1, \ldots, a_r\} \cap \mathbb{S}_i$ with minimal distance from $\mathrm{init}(\mathbb{S}_i)$, then we define a'_i to be the vertex from $\{a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_r\} \cap \mathbb{S}_i$ with minimal distance from $\mathrm{init}(\mathbb{S}_i)$. This is needed to ensure the NU property, i.e., that $a \circ_\omega b = a$, in the case that $a, b \in \mathbb{P}$ for some $\mathbb{P} \in \mathrm{Paths}$ and b is closer to init(Paths) than a.)

(1) If $k = 0$ or $k = \mathrm{hgt}(\mathbb{T})$, we put $\omega(\bar{a}) = \omega'(\bar{a})$.
(2) Else, if $\bar{a} \in \bigotimes_{i=1}^r \mathbb{S}_i$, let $\mathbb{P} \in \mathrm{Paths}$ be the minimal path connecting $\omega'(\langle \mathrm{init}(\mathbb{S}_i) : 1 \leq i \leq r \rangle)$ to $\omega'(\langle \mathrm{term}(\mathbb{S}_i) : 1 \leq i \leq r \rangle)$. We put $\omega(\bar{a}) = \varphi_{\mathcal{I},\mathbb{P}}(\langle a'_i : \mathbb{S}_i \in \mathcal{I} \rangle)$.
(3) If $\bar{a} \notin \bigotimes_{i=1}^r \mathbb{S}_i$, then
   (3a) if $r \geq 3$ and there exist $i, j \in [r]$ such that $\{a_l : l \neq j\} \subseteq \mathbb{S}_i$, we put $\omega(\bar{a}) = a'_i$.
   (3b) if $r = 2$, we put $\omega(a_1, a_2) = a'_1$ if $\mathbb{S}_{\mathcal{I}} = \mathbb{S}_1$ and $\omega(a_1, a_2) = a'_2$ else.
   (3c) In all other cases we define $\omega(\bar{a}) = a_1$.

It is straightforward to verify that ω is a weak-NU and that if ω' is a NU, then ω is also a NU. To prove compatibility, choose any $\bar{a} \in \mathrm{Level}(k)^r$ and $\bar{b} \in \mathrm{Level}(k+1)^r$ such that $a_i \xrightarrow{\mathbb{T}} b_i$, $i = 1, \ldots, r$. We can assume that $\mathrm{hgt}(\mathbb{T}) > 1$ (otherwise $\omega = \omega'$). If $\omega(\bar{a})$ is defined by (1), then $\omega(\bar{b})$ is defined by (2). It is easily seen that in this case $\bar{b} = \bar{b}'$ and $\omega(\bar{a}) = \varphi_{\mathcal{I},\mathbb{P}}(\langle a'_i : \mathbb{S}_i \in \mathcal{I} \rangle) \xrightarrow{\mathbb{T}} \varphi_{\mathcal{I},\mathbb{P}}(\langle b'_i : \mathbb{S}_i \in \mathcal{I} \rangle) = \omega(\bar{b})$ follows from the fact that $\varphi_{\mathcal{I},\mathbb{P}}$ is a homomorphism. The proof is analogous for the case when $\omega(\bar{b})$ is defined by (1). Now assume that neither $\omega(\bar{a})$ nor $\omega(\bar{b})$ are defined by (1). In this situation, both $\omega(\bar{a})$ and $\omega(\bar{b})$ fall into the same case of the definition. Observe that $a'_i \to b'_i$, $i = 1, \ldots, r$, and the set $\mathcal{I}$ is the same for both $\bar{a}$ and $\bar{b}$. Now $\omega(\bar{a}) \xrightarrow{\mathbb{T}} \omega(\bar{b})$ follows from the fact that $\varphi_{\mathcal{I},\mathbb{P}}$ (case (2)) and projections (cases (3a)-(3c)) are homomorphisms. We extend ω using Lemma 4.4 and the proof of (i) and (ii) is finished.

To prove the converse implication of (iii) we slightly modify the construction. Assume that $\mathcal{A}(\mathbb{T})$ admits r-ary compatible TSI τ'_r. Similarly as before, we will construct a partial TSI operation τ_r compatible with $\mathbb{T}$ with domain $\bigcup_{k=0}^{\mathrm{hgt}\,\mathbb{T}} \mathrm{Level}(k)^r$. Let $\bar{a} \in \mathrm{Level}(k)^r$. For $k \notin \{0, \mathrm{hgt}(\mathbb{T})\}$, let $\mathbb{S}_i \in \mathrm{Paths}$ be such that $a_i \in \mathbb{S}_i$ and denote the set $\{\mathbb{S}_1, \ldots, \mathbb{S}_r\}$ by $\mathcal{I}$. For each i let a'_i be the vertex from $\{a_1, \ldots, a_r\} \cap \mathbb{S}_i$ with minimal distance from $\mathrm{init}(\mathbb{S}_i)$.

(1) If $k = 0$ or $k = \mathrm{hgt}(\mathbb{T})$, we put $\tau_r(\bar{a}) = \tau'_r(\bar{a})$.
(2) Else, if $\bar{a} \in \bigotimes_{i=1}^r \mathbb{S}_i$, let $\mathbb{P} \in \mathrm{Paths}$ be the minimal path connecting $\tau'_r(\langle \mathrm{init}(\mathbb{S}_i) : 1 \leq i \leq r \rangle)$ to $\tau'_r(\langle \mathrm{term}(\mathbb{S}_i) : 1 \leq i \leq r \rangle)$. We put $\tau_r(\bar{a}) = \varphi_{\mathcal{I},\mathbb{P}}(\langle a'_i : \mathbb{S}_i \in \mathcal{I} \rangle)$.
(3) If $\bar{a} \notin \bigotimes_{i=1}^r \mathbb{S}_i$, then $\tau_r(\bar{a}) = a'_i$, where i is such that $\mathbb{S}_i = \mathbb{S}_{\mathcal{I}}$.

It is not hard to verify that τ_r is a TSI operation, just note that if $\{a_1, \ldots, a_r\} = \{b_1, \ldots, b_r\}$, then the set $\mathcal{I}$ and the paths $\mathbb{P}$ (case (2)) and

$\mathbb{S}_{\mathcal{I}}$ (case (3)) are the same for both $\bar{a}$ and $\bar{b}$. The argumentation to verify compatibility is similar as before. We conclude the proof by extending τ_r using Lemma 4.4. □

4.3. $\mathcal{A}(\mathbb{T})$ and compatible weak-NUs.

Lemma 4.11. *If $\mathcal{A}(\mathbb{T})$ admits a compatible binary weak-NU (i.e., a commutative idempotent operation), then $\mathcal{A}(\mathbb{T})$ admits compatible TSI operations of all arities.*

Proof. Let $\star$ be a binary weak-NU compatible with $\mathcal{A}(\mathbb{T})$. First, we will prove that the following holds:

$$(\exists z \in \mathrm{Base})(\forall a \in \mathrm{Base}, a \neq z)\ a \star 0 = 0.$$

Let $z, z' \in \mathrm{Base}$ be such that $z \star 0 \neq 0$, $z' \star 0 \neq 0$. Since $\star$ is compatible with the digraph $\mathbb{T}(\mathrm{Paths})$ in which $a \to \widehat{a}$ and $0 \to \widehat{a}$ for all $a \neq 0$, it follows that $a \star 0 \to \widehat{a} \star \widehat{a} = \widehat{a}$; and so $a \star 0 \in \{0, a\}$ for all $a \in \mathrm{Base}$. Therefore $z \star 0 = z$ and $z' \star 0 = z'$. But as $z \star 0 \to \widehat{z} \star \widehat{z'}$ and $z' \star 0 = 0 \star z' \to \widehat{z} \star \widehat{z'}$ in $\mathbb{T}(\mathrm{Paths})$, we conclude that $z = z'$.

Now fix $z \in \mathrm{Base}$ with the above property. We will define a partial order on the set $\mathrm{Base} \cup \mathrm{Top}$ and then use $\star$ to "compare the incomparable" elemets. For all $\widehat{a} \in \mathrm{Top}$, $\widehat{a} \neq \widehat{z}$ we put $z \prec \widehat{z} \prec 0 \prec \widehat{a}$ and if $\widehat{a} \notin \mathrm{Half}$, then also $\widehat{a} \prec a$. We define $\preceq$ to be the partial order generated by these relations. Let us fix an arbitrary linear order $\leq$ on the set $\mathrm{Top} \setminus \{\widehat{z}\}$. (We can assume without loss of generality that $z = 1$ and $\mathrm{Top} \setminus \{\widehat{z}\} = \{\widehat{2} < \widehat{3} < \cdots < \widehat{n+k}\}$.)

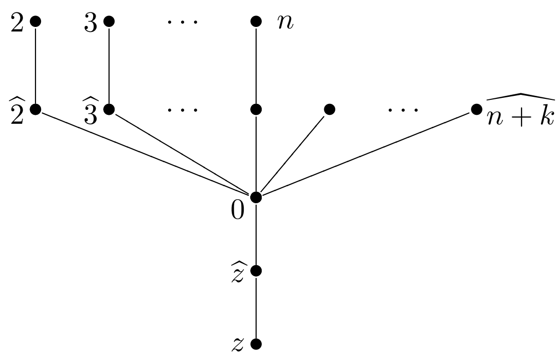


FIGURE 4. The partial order $\preceq$.

For each $i > 0$ we denote by t_i the i-ary operation defined in the following way (note that all these operations are compatible with $\mathcal{A}(\mathbb{T})$):

$$\begin{aligned} t_1(x) &= x, \\ t_2(x_1, x_2) &= x_1 \star x_2, \\ &\vdots \\ t_i(x_1, \ldots, x_i) &= t_{i-1}(x_1, \ldots, x_{i-1}) \star x_i. \end{aligned}$$

For each $\widehat{c} \in \mathrm{Top}$ we define the set $R(\widehat{c})$ as follows: we put $R(\widehat{c}) = \{\widehat{c}\}$ if $\widehat{c} \in \mathrm{Half}$ and $R(\widehat{c}) = \{\widehat{c}, c\}$ else.

Now we are ready to define the TSI operations. Again, we will use Lemma 4.4. For each $r \geq 1$ we define a partial r-ary operation τ_r in the following way: For any $\bar{a} \in \mathrm{Base}^r \cup \mathrm{Top}^r$ let $S(\bar{a})$ be the smallest subset of $\mathrm{Base} \cup \mathrm{Top}$ containing $\{a_1, \dots, a_r\}$ and closed under the operation $\star$ (i.e., $c \star c' \in S(\bar{a})$ whenever $c, c' \in S(\bar{a})$).

(1) If $S(\bar{a})$ has the least element with respect to $\preceq$, we define $\tau_r(\bar{a})$ to be that element,
(2) else let $\{\widehat{c_1} < \widehat{c_2} < \cdots < \widehat{c_m}\}$ be the set of all $\widehat{c} \in \mathrm{Top} \setminus \{\widehat{z}\}$ such that $S(\bar{a}) \cap R(\widehat{c}) \neq \emptyset$. Note that $m \geq 2$. For $i = 1, \dots, m$ we denote by a'_i the $\preceq$-least element of $S(\bar{a}) \cap R(\widehat{c_i})$. Finally, we put $\tau_r(\bar{a}) = t_m(a'_1, a'_2, \dots, a'_m)$.

It is easy to check that τ_r is totally symmetric and idempotent. To verify compatibility, choose $\mathcal{I} \subseteq \mathrm{Paths}$, $\bar{a} \in \mathrm{Base}^r$ and $\bar{b} \in \mathrm{Top}^r$ such that $a_i \xrightarrow{\mathbb{T}(\mathcal{I})} b_i$, $i = 1, \dots, r$. If $\tau_r(\bar{a})$ and $\tau_r(\bar{b})$ are defined by the same case, then it is not hard to see that $\tau_r(\bar{a}) \xrightarrow{\mathbb{T}(\mathcal{I})} \tau_r(\bar{b})$.If $\bar{a}$ falls into case (2), then so does $\tau_r(\bar{b})$. Thus it only remains to investigate the case when $\tau_r(\bar{a})$ is defined by (1) and $\tau_r(\bar{b})$ by (2). In this case, we have that $\tau(\bar{a}) = 0$ and $\tau(\bar{b}) = t_m(\widehat{c_1}, \dots, \widehat{c_m})$ for some $m \geq 2$ and $\widehat{c_i} \in \mathrm{Top} \setminus \{\widehat{z}\}$.

For each i, let $c'_i \in S(\bar{a})$ be $\preceq$-minimal such that $c'_i \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c_i}$ ($c'_i = 0$ if $\widehat{c_i} \in \mathrm{Half}$ and $c'_i \in \{0, c_i\}$ else.) Since $0 \in S(\bar{a})$, there exists j such that $c'_j = 0$. We will prove that $t_m(c'_1, \dots, c'_m) = 0$. Then the proof will be concluded, as we will have that

$$\tau_r(\bar{a}) = 0 = t_m(c'_1, \dots, c'_m) \xrightarrow{\mathbb{T}(\mathcal{I})} t_m(\widehat{c}_1, \dots, \widehat{c}_m) = \tau_r(\bar{b}).$$

Since the $\preceq$-least element of $S(\bar{a})$ is 0 and $S(\bar{a})$ is closed under $\star$, it follows that $t_{j-1}(c'_1, \dots, c'_{j-1}) \neq z$; and so $t_j(c'_1, \dots, c'_{j-1}, c'_j) = t_{j-1}(c'_1, \dots, c'_{j-1}) \star 0 = 0$. Now we have that

$$t_{j+1}(c'_1, \dots, c'_{j+1}) = t_j(c'_1, \dots, c'_j) \star c'_{j+1} = 0 \star c'_{j+1}$$

and since $c'_{j+1} \neq z$, it follows that $t_{j+1}(c'_1, \dots, c'_{j+1}) = 0$. We can proceed by induction, proving that $t_m(c'_1, \dots, c'_m) = 0$. □

The following lemma plays a key role in our proof of Theorem 3.2.

Lemma 4.12. *If $\mathcal{A}(\mathbb{T})$ admits an r-ary weak-NU ω, then it admits an $(r+1)$-ary weak-NU ω'.*

Proof. First, let us consider the case when there exists $z \in \mathrm{Base}, z \neq 0$ such that $0 \circ_\omega z = z$. We will prove that then $\mathcal{A}(\mathbb{T})$ admits a binary idempotent commutative operation $\star$; and thus by Lemma 4.11 also an $(r+1)$-ary weak-NU (even totally symmetric) operation.

Let $\preceq$, $\leq$ and $R(\widehat{c})$, $\widehat{c} \in \mathrm{Top}$ be the same as in the proof of Lemma 4.11. We will define $\star$ for $\langle a, b \rangle \in \mathrm{Base}^2 \cup \mathrm{Top}^2$ and then extend it using Lemma 4.4.

(1) If $a \preceq b$, then we put $a \star b = b \star a = a$ and if $b \preceq a$, we put $a \star b = b \star a = b$.
(2) If a and b are $\preceq$-incomparable, then $a \in R(\widehat{c})$ and $b \in R(\widehat{d})$ for some $\widehat{c} \neq \widehat{d} \in \mathrm{Top} \setminus \{\widehat{z}\}$. We define $a \star b = b \star a = a \circ_\omega b$ if $\widehat{c} < \widehat{d}$ and $a \star b = b \star a = b \circ_\omega a$ else.

From the compatibility of $\circ_\omega$ with $\mathbb{T}(\text{Paths})$ we get that $\widehat{c} \circ_\omega \widehat{z} = \widehat{z}$ for all $\widehat{c} \in \text{Top}$. Since $c \circ_\omega 0 \to \widehat{c} \circ_\omega \widehat{z} = \widehat{z}$ and

$$0 \circ_\omega c = \omega(c, 0, \dots, 0, 0) \to \omega(\widehat{c}, \widehat{c}, \dots, \widehat{c}, \widehat{z}) = \widehat{c} \circ_\omega \widehat{z} = \widehat{z}$$

in $\mathbb{T}(\text{Paths})$, we conclude that $0 \circ_\omega c = c \circ_\omega 0 = 0$ for all $\widehat{c} \in \text{Top}, \widehat{c} \neq \widehat{z}$. Now it is not hard to prove that $\star$ is an idempotent commutative operation compatible with $\mathcal{A}(\mathbb{T})$, we leave the verification to the reader.

Second, we consider the case when ω satisfies

$$(\forall a \in \text{Base})\ 0 \circ_\omega a = 0.$$

We may assume that for all $a, b \in \text{Base} \setminus \{0\}$, if $\widehat{a} \circ_\omega \widehat{b} = \widehat{a}$, then $a \circ_\omega b = a$; otherwise we can "redefine" ω to satisfy the desired property, i.e., replace ω with the operation ω^* defined by

$$\omega^*(\bar{x}) = \begin{cases} a & \text{if } \bar{x} \in \{\langle a, \dots, a, b\rangle, \langle a, \dots, a, b, a\rangle, \dots, \langle b, a, \dots, a\rangle\} \\ & \text{for some } a, b \in \text{Base} \setminus \{0\} \text{ such that } \widehat{a} \circ_\omega \widehat{b} = \widehat{a}, \\ \omega(\bar{x}) & \text{else.} \end{cases}$$

It is easy to see that ω^* is also an r-ary weak-NU compatible with $\mathcal{A}(\mathbb{T})$ satisfying $(\forall a \in \text{Base})\ 0 \circ^*_\omega a = 0$.

Let us define the set $\text{Maj} = \{a \in \text{Base} : a \circ_\omega 0 = a\}$. We will prove the following:

$$(\forall a \in \text{Maj})(\forall b \in \text{Base})\ a \circ_\omega b = a.$$

For $a = 0$ the claim follows from the assumptions and for $b = 0$ from the definition of Maj. Let $a, b \neq 0$. Since $\circ_\omega$ is compatible with $\mathbb{T}(\text{Paths})$ and $a \circ_\omega 0 = a$, it follows that $\widehat{a} \circ_\omega \widehat{b} = \widehat{a}$. Hence $a \circ_\omega b = a$ and the claim is proved.

We will define $\omega'(\bar{a})$ for $\bar{a} = \langle a_1, \dots, a_{r+1}\rangle \in \text{Base}^{r+1} \cup \text{Top}^{r+1}$ and then apply Lemma 4.4.

(1) If $\bar{a} = \langle a, \dots, a, b\rangle$ for some $a, b \in \text{Base}$, $a \notin \text{Maj}$, we put $\omega'(\bar{a}) = a \circ_\omega b$, and if $\bar{a} = \langle \widehat{a}, \dots, \widehat{a}, \widehat{b}\rangle$ for some $\widehat{a}, \widehat{b} \in \text{Top}$, $a \notin \text{Maj}$, we put $\omega'(\bar{a}) = \widehat{a} \circ_\omega \widehat{b}$,
(2) else we define $\omega'(\bar{a}) = \omega(a_1, \dots, a_r)$.

To prove that ω' is a weak-NU, choose $a, b \in \text{Base}$. For $\widehat{a}, \widehat{b} \in \text{Top}$ we can proceed analogously. If $a \in \text{Maj}$, then case (2) applies. We have that $\omega'(b, a, \dots, a) = \dots = \omega'(a, \dots, a, b, a) = a \circ_\omega b = a$, while $\omega'(a, \dots, a, b) = \omega(a, \dots, a) = a$. Now suppose that $a \notin \text{Maj}$. In that case $\omega'(a, \dots, a, b) = a \circ_\omega b$ by (1) and $\omega'(a, \dots, a, b, a) = \dots = \omega'(b, a, \dots, a) = a \circ_\omega b$ by (2); and so the weak-NU property is verified.

To verify compatibility, choose $\mathcal{I} \subseteq \text{Paths}$, $\bar{a} \in \text{Base}^{r+1}$ and $\bar{b} \in \text{Top}^{r+1}$ such that $a_i \xrightarrow{\mathbb{T}(\mathcal{I})} b_i$, $i = 1, \dots, r+1$. If $\omega'(\bar{a})$ and $\omega'(\bar{b})$ are defined by the same case, then $\omega'(\bar{a}) \xrightarrow{\mathbb{T}(\mathcal{I})} \omega'(\bar{b})$ follows from the compatibility of $\circ_\omega$ in case (1) and ω in case (2). If $\bar{a}$ falls into case (1), then so does $\bar{b}$. The only remaining case is when $\omega'(\bar{a})$ is defined by (2) and $\omega'(\bar{b})$ by (1). In this situation we have that $\bar{b} = \langle \widehat{c}, \dots, \widehat{c}, \widehat{d}\rangle$ for some $\widehat{c}, \widehat{d} \in \text{Top}$, $c \notin \text{Maj}$ and $\omega'(\bar{b}) = \widehat{c} \circ_\omega \widehat{d}$. Since $a_i \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c}$ for $i = 1, \dots, r$, we get $\omega'(\bar{a}) = \omega(a_1, \dots, a_r) \xrightarrow{\mathbb{T}(\mathcal{I})} \omega(\widehat{c}, \dots, \widehat{c}) = \widehat{c}$; and so $\omega(a_1, \dots, a_r) \in \{0, c\}$. We also know that $0 \in \{a_1, \dots, a_r\}$, as otherwise case (1) would apply for $\bar{a}$.

First, let $\omega(a_1,\ldots,a_r)=0$. Since $0 \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c}$ and $a_{r+1} \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{d}$, from the compatibility of $\circ_\omega$ we obtain

$$\omega'(\bar{a})=\omega(a_1,\ldots,a_r)=0=0\circ_\omega a_{r+1} \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c}\circ_\omega \widehat{d}=\omega'(\bar{b}),$$

proving the compatibility condition for ω' in this case.

Second, assume that $\omega(a_1,\ldots,a_r)=c$. Notice that $c\in\{a_1,\ldots,a_r\}$ (as $\omega(0,\ldots,0)=0$), implying that $c \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c}$. We will prove that $\widehat{c}\circ_\omega \widehat{d}=\widehat{c}$. Then it will follow that

$$\omega'(\bar{a})=\omega(a_1,\ldots,a_r)=c \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c}=\widehat{c}\circ_\omega \widehat{d}=\omega'(\bar{b}),$$

which will conclude the proof. Let $j\in[r]$ be such that $a_j=0$. In the digraph $\mathbb{T}(\text{Paths})$ we have $a_j\to\widehat{d}$ and $a_i\to\widehat{c}$ for all $i=1,\ldots,r$. Therefore

$$c=\omega(a_1,\ldots,a_{j-1},a_j,a_{j+1},\ldots,a_r)\to\omega(\widehat{c},\ldots,\widehat{c},\widehat{d},\widehat{c},\ldots,\widehat{c})=\widehat{c}\circ_\omega \widehat{d}.$$

Hence $\widehat{c}\circ_\omega \widehat{d}=\widehat{c}$ and the proof is finished. □

4.4. **Q.E.D.** Finally, everything is set to prove the main result.

Proof of Theorem 3.2. Let $\mathbb{T}$ be a special polyad. By Lemma 4.1, core($\mathbb{T}$) is also a special polyad.

(i) If core($\mathbb{T}$) admits no compatible weak-NUs, then CSP($\mathbb{T}$) is NP-complete by Theorem 2.3. By Theorem 2.4 and the "reduction" Lemma 4.10, it is enough to prove that if $\mathcal{A}(\text{core}(\mathbb{T}))$ admits a weak-NU of arity r_0, then $\mathcal{A}(\text{core}(\mathbb{T}))$ admits weak-NUs of all arities $r\geq r_0$. But the latter fact follows by induction from Lemma 4.12.

(ii) By Lemma 4.11 (and Lemma 4.10), $\mathbb{T}$ admits a binary weak-NU, if and only if it admits TSI operations of all arities. The rest follows from Theorem 2.5. □

5. Constructing special polyads

In this section we will present a method of constructing special polyads with certain desired properties using $\mathcal{A}(\mathbb{T})$ and the "reduction" from Lemma 4.10. We will apply this technique to construct an interesting example: a core special polyad which is tractable, but does not have width 1 and admits no compatible near-unanimity operations.

5.1. **From $\mathcal{A}(\mathbb{T})$ back to $\mathbb{T}$.** Our aim in this subsection is to provide a characterization of families of digraphs $\mathcal{A}$ for which we can construct a special polyad $\mathbb{T}$ such that $\mathcal{A}=\mathcal{A}(\mathbb{T})$. We start with the definition of *closure system*.

Definition 5.1. By a *closure system* on a finite set A we mean a family $\mathcal{C}\subseteq\mathcal{P}(A)$ of subsets of A such that

(i) $A\in\mathcal{C}$,
(ii) if $C_1,C_2\in\mathcal{C}$, then $C_1\cap C_2\in\mathcal{C}$.

The sets $C\in\mathcal{C}$ are called *$\mathcal{C}$-closed sets.*

Let $\mathcal{D}$ be a closure system on a finite set B. We say that $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic* if there exists a bijection $f:A\to B$ such that $\mathcal{D}=\{f[C]:C\in\mathcal{C}\}$.

Closure systems can be in a natural way identified with closure operators. The following definition is essentially just a reformulation of Definition 4.5 (ii):

Definition 5.2. Let Paths $= \{\mathbb{P}_1, \dots, \mathbb{P}_n\}$ be a finite set of minimal paths of the same height. We define the closure system $\mathcal{R}_{\bigotimes}^{\text{Paths}}$ on Paths in the following way: let the $\mathcal{R}_{\bigotimes}^{\text{Paths}}$-closed sets be precisely the empty set and the nonempty sets $\mathcal{I} \subseteq$ Paths such that

$$\mathcal{I} = \{\mathbb{P} \in \text{Paths} : \bigotimes_{\mathbb{S} \in \mathcal{I}} \mathbb{S} \to \mathbb{P}\}.$$

It is easy to check that $\mathcal{R}_{\bigotimes}^{\text{Paths}}$ is indeed a closure system. The following proposition states that each closure system on a finite set (such that the empty set is closed) is isomorphic to $\mathcal{R}_{\bigotimes}^{\text{Paths}}$ for some set of minimal paths.

Proposition 5.3. *Let $\mathcal{C}$ be a closure system on $[n]$, $\emptyset \in \mathcal{C}$. There exists a set* Paths $= \{\mathbb{P}_1, \dots, \mathbb{P}_n\}$ *of minimal paths of the same height such that for each $I \subseteq [n]$,*

$$I \in \mathcal{C} \iff \{\mathbb{P}_i : i \in I\} \in \mathcal{R}_{\bigotimes}^{\text{Paths}}.$$

Proof. Let us fix an arbitrary linear order of the nontrivial $\mathcal{C}$-closed sets (i.e., $\mathcal{C} \setminus \{\emptyset, [n]\}$), say $\mathcal{C} = \{\emptyset, C_1, \dots, C_q, [n]\}$. By an *arrow* we mean a digraph with a single edge $a \to b$ (and possibly some other discrete vertices); a *zig-zag* is a digraph with just three edges $a \to b, c \to b, c \to d$ (see the figure below).

FIGURE 5. An arrow and a zig-zag.

We say that a minimal path $\mathbb{P}$ has an *arrow at level k* if $\mathbb{P}[\text{Level}_{\mathbb{P}}(k) \cup \text{Level}_{\mathbb{P}}(k+1)]$ (the subgraph induced by vertices of level k or $k+1$) is an arrow; if it is a zig-zag, then $\mathbb{P}$ has a *zig-zag at level k*. It is an easy excercise to prove the following claim:

Claim. Let l be a positive integer and for $I \subseteq [l]$ let $\mathbb{P}_I$ denote the minimal path of height $l+2$ which has zig-zag's at levels $i \in I$ and arrows at levels $j \in \{0, \dots, l+1\} \setminus I$. For any $I_1, \dots, I_m \subseteq [l]$ the core of $\bigotimes_{i=1}^m \mathbb{P}_{I_i}$ is isomorphic to $\mathbb{P}_{I_1 \cup \dots \cup I_m}$.

The above claim is the key to our construction: For $i \in [n]$, let $\mathbb{P}_i$ be the minimal path of height $q+2$ (uniquely) determined by the following conditions:

(i) $\mathbb{P}_i$ has an arrow at level 0,
(ii) for $k = 1, \dots, q$, $\mathbb{P}_i$ has an arrow at level k if $i \in C_k$ and a zig-zag at level k else,
(iii) $\mathbb{P}_i$ has an arrow at level $q+1$.

To demonstrate the construction, consider the following example: let $n = 3$, $q = 3$, $C_1 = \{1\}$, $C_2 = \{1,2\}$, $C_3 = \{1,3\}$. The minimal paths $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ are depicted in Figure 6.

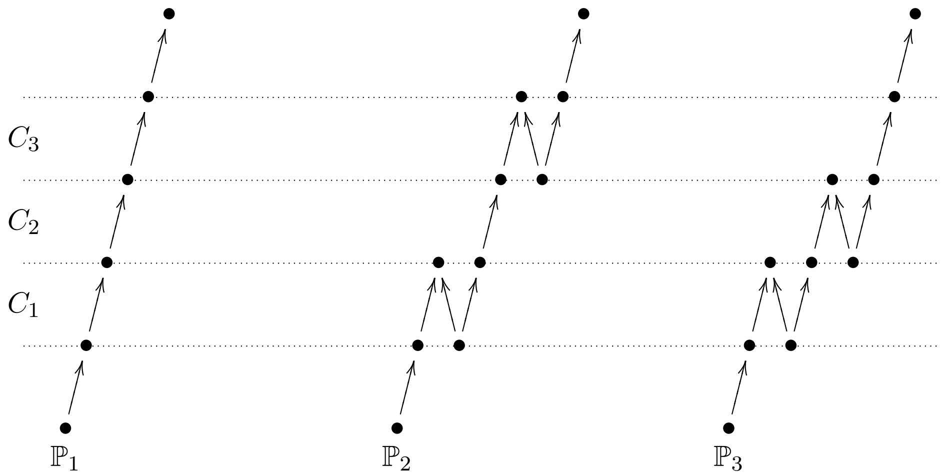


FIGURE 6. The resulting minimal paths.

The above claim implies that for all nonempty $I \subseteq [n]$ and $j \in [n]$, $\bigotimes_{i\in I} \mathbb{P}_i \to \mathbb{P}_j$, if and only if for all $C \in \mathcal{C}$ such that $j \notin C$ there exists $i \in I$ with $i \notin C$. Equivalently,

$$\bigotimes_{i\in I} \mathbb{P}_i \to \mathbb{P}_j \iff (\forall C \in \mathcal{C})\,(I \subseteq C \to j \in C).$$

Now, choose arbitrary nonempty $I \subseteq [n]$. Let $D = \bigcap\{C \in \mathcal{C} : I \subseteq C\}$ be the minimal (w.r.t. inclusion) $\mathcal{C}$-closed set containg I. From the above we get that

$$\bigotimes_{i\in I} \mathbb{P}_i \to \mathbb{P}_j \iff j \in D.$$

Thus $I \in \mathcal{C}$ (i.e., $I = D$), if and only if $\{\mathbb{P}_i : i \in I\}$ is $\mathcal{R}_{\otimes}^{\mathrm{Paths}}$-closed. □

Remark. The above construction of minimal paths was chosen for its simplicity, it is by no means optimal regarding the number of vertices of the resulting paths.

We conclude this subsection with an easy corollary of the above proposition; a key to the construction below.

Corollary 5.4. *Let $\mathcal{A}$ be a family of digraphs on the same vertex set H. The following are equivalent:*

(i) *$\mathcal{A} = \mathcal{A}(\mathbb{T})$ for some special polyad $\mathbb{T}$,*
(ii) *There exists a special polyad $\mathbb{H} = (H, E)$ of height 1 such that $(H, \emptyset) \in \mathcal{A}$ and the edge relations of members of $\mathcal{A}$ form a closure system on E.*

Moreover, if (ii) holds and $(H, \{e\}) \in \mathcal{A}$ for all $e \in E$, then $\mathbb{T}$ is a core.

Proof. (i) $\Rightarrow$ (ii): For a special polyad $\mathbb{T}$, $\mathcal{A} = \mathcal{A}(\mathbb{T})$ clearly satisfies (ii). (Note that $\mathbb{T}(\text{Paths}_{\mathbb{T}})$ is a special polyad of height 1).

(ii) $\Rightarrow$ (i): Label the edges of $\mathbb{H}$ with positive integers $1, \dots, n$ and use the previous proposition to construct the minimal paths $\mathbb{P}_i$. For $i = 1, \dots, n$, replace the edge i with the minimal path $\mathbb{P}_i$. The resulting digraph $\mathbb{T}$ is a special polyad such that $\mathcal{A} = \mathcal{A}(\mathbb{T})$.

The rest follows from the fact that if $\mathbb{T}$ is not a core, then $\mathbb{P} \to \mathbb{P}'$ for some $\mathbb{P}, \mathbb{P}' \in \text{Paths}_{\mathbb{T}}$. □

5.2. **An interesting special polyad.** In this subsection we construct a special polyad satisfying the following:

Proposition 5.5. *There exists a core special polyad $\mathbb{T}$ having the following properties:*

(i) $\text{CSP}(\mathbb{T})$ *is tractable,*
(ii) $\mathbb{T}$ *does not have width 1,*
(iii) $\mathbb{T}$ *does not admit any compatible near-unanimity operation.*

In order to construct such a special polyad, we will first introduce some notation. Let $\mathbb{H} = (H, E)$ be a special polyad of height 1 with 4 branches with the vertices and edges labeled as in the figure below:

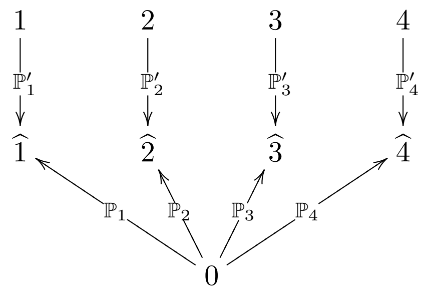


FIGURE 7. The special polyad $\mathbb{H}$ of height 1.

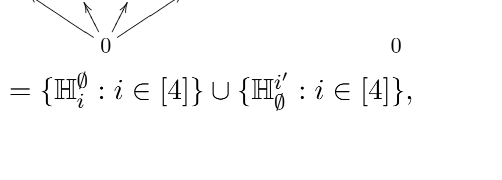
For $J \subseteq [4]$, we denote the set $\{j' : j \in J\}$ by J'. For $I, J \subseteq [4]$, we define $\mathbb{H}_I^{J'}$ to be the subgraph of $\mathbb{H}$ with vertex set H and edges $\{\mathbb{P}_i : i \in I\} \cup \{\mathbb{P}'_j : j \in J\}$.

We define the family $\mathcal{A}$ of subgraphs of $\mathbb{H}$ in the following way:

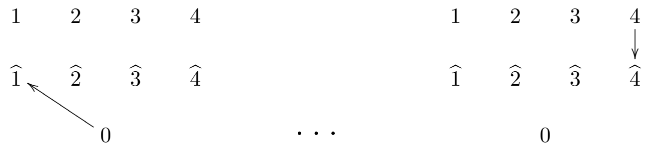
$$\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3,$$

where

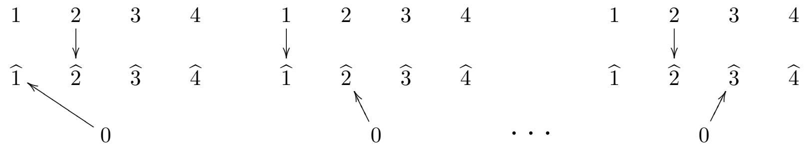
- $\mathcal{A}_0 = \{\mathbb{H}, \mathbb{H}_\emptyset^\emptyset\}$,



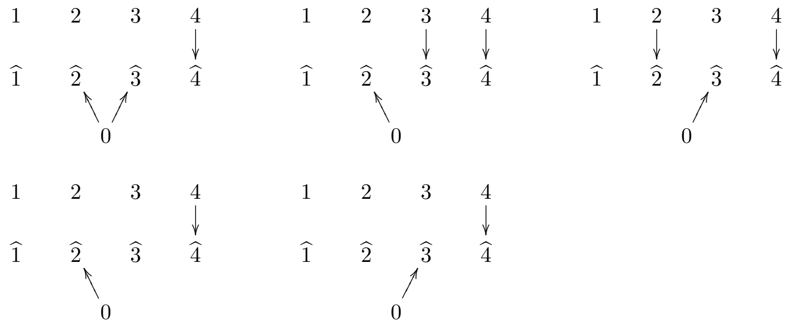
- $\mathcal{A}_1 = \{\mathbb{H}_i^\emptyset : i \in [4]\} \cup \{\mathbb{H}_\emptyset^{i'} : i \in [4]\}$,



- $\mathcal{A}_2 = \{\mathbb{H}_i^{j'} : i, j \in [3], i \neq j\}$,



- $\mathcal{A}_3 = \{\mathbb{H}_{2,3}^{4'}, \mathbb{H}_2^{3',4'}, \mathbb{H}_3^{2',4'}, \mathbb{H}_2^{4'}, \mathbb{H}_3^{4'}\}$.



It can be easily seen that the edge relations of the members of $\mathcal{A}$ form a closure system. The rest of the proof follows:

Proof of Proposition 5.5. By Corollary 5.4, there exists a core special polyad $\mathbb{T}$ such that $\mathcal{A} = \mathcal{A}(\mathbb{T})$. In the following, we use Theorem 3.2 and the "reduction" Lemma 4.10.

(i) It is enough to prove that $\mathcal{A}$ admits a compatible weak near-unanimity operation. We will define a 4-ary weak-NU ω on the set H. Let $\bar{x} \in \{0,1,2,3,4\}^4$.

(1) If $4 \notin \{x_1, x_2, x_3\}$, then
  (1.1) if $\{x_1, x_2, x_3\} = \{1,2,3\}$, we put $\omega(\bar{x}) = 1$
  (1.2) else x_1, x_2, x_3 lie on an oriented path in $\mathbb{H}$; we define $\omega(\bar{x})$ to be the middle vertex from x_1, x_2, x_3 on this path.
(2) If $4 \in \{x_1, x_2, x_3\}$, then
  (2.1) if $\bar{x} = \langle 4,4,4,4\rangle$, we put $\omega(\bar{x}) = 4$
  (2.2) else $\omega(\bar{x}) = x_i$ where i is smallest such that $x_i \neq 4$.

For $\widehat{\bar{x}} \in \widehat{[4]}^4$ we put $\omega(\widehat{\bar{x}}) = \widehat{\omega(\bar{x})}$. Finally, we extend ω using Lemma 4.4. It can be easily verified that ω is a weak-NU. (In fact, ω restricted to $H \setminus \{4, \widehat{4}\}$ is a near-unanimity.)

Compatibility with $\mathcal{A}_0$ is trivial and compatibility with $\mathcal{A}_1$ follows from the idempotency of ω. Let $\bar{x} \in (\{0\} \cup [4])^4, \bar{y} \in [4]^4$. To prove compatibility with $\mathcal{A}_2$, pick any $i, j \in [3], i \neq j$. If $\bar{x} \to \widehat{\bar{y}}$ in $\mathbb{H}_i^{j'}$, then both $\omega(\bar{x})$ and $\omega(\widehat{\bar{y}})$ are defined by (1.2) and it is easily seen that $\omega(\bar{x}) \to \omega(\widehat{\bar{y}})$ in $\mathbb{H}_i^{j'}$. As for compatibility with $\mathcal{A}_3$, let $\bar{x} \to \widehat{\bar{y}}$ in some $\mathbb{H}' \in \mathcal{A}_3$. The only interesting case is when $4 \in \bar{x}$; we see that $x_i = 4$ iff $y_i = 4$ for all $i \in [4]$. It follows that $\omega(\bar{x})$ and $\omega(\widehat{\bar{y}})$ are defined by the same case of the definition, (1.2), (2.1) or

(2.2); in all of these cases we have $\omega(\bar{x}) \to \omega(\widehat{y})$ in $\mathbb{H}'$. Thus ω is compatible with $\mathcal{A}$ and we have proved that $\mathrm{CSP}(\mathbb{T})$ is tractable.

(ii) It suffices to prove that $\mathcal{A}$ does not admit a compatible binary weak-NU (binary idempotent commutative operation). Striving for contradiction, let $\star$ be a binary weak-NU compatible with $\mathcal{A}$. In the following, a digraph above an arrow indicates that the implication was deduced from the compatibility with that digraph.

For any $i \neq j \in [3]$ we have

$$\widehat{i} \star \widehat{i} = \widehat{i} \stackrel{\mathbb{H}}{\Longrightarrow} i \star 0 \in \{i, 0\} \stackrel{\mathbb{H}_j^{i'}}{\Longrightarrow} \widehat{i} \star \widehat{j} \in \{\widehat{i}, \widehat{j}\} \stackrel{\mathbb{H}_j^{i'}, \mathbb{H}_i^{j'}}{\Longrightarrow} i \star 0 = i \text{ or } j \star 0 = j.$$

Without loss of generality we may assume that $1 \star 0 = 1$. Now

$$1 \star 0 = 1 \stackrel{\mathbb{H}_2^{1'}, \mathbb{H}_3^{1'}}{\Longrightarrow} \widehat{1} \star \widehat{2} = \widehat{1} \star \widehat{3} = \widehat{1} \stackrel{\mathbb{H}_1^{2'}, \mathbb{H}_1^{3'}}{\Longrightarrow} 2 \star 0 = 3 \star 0 = 0;$$

a contradiction.

(iii) Again, it is enough to prove that $\mathcal{A}$ admits no compatible near-unanimity operation. Suppose for contradiction that there exists an r-ary NU operation ν compatible with $\mathcal{A}$. We will prove the following claim: For all $i \in [r-2]$,

$$\nu(\underbrace{4, \dots, 4}_{i\text{-times}}, 0, 0, \dots, 0) = 0 \implies \nu(\underbrace{4, \dots, 4, 4}_{(i+1)\text{-times}}, 0, \dots, 0) = 0.$$

This claim contradicts the fact that $\nu(4, 0, \dots, 0) = 0$ and $\nu(4, \dots, 4, 0) = 4$.

Fix $i \in [r-2]$ and let

$$t(x, y, z) = \nu(\underbrace{x, \dots, x}_{i\text{-times}}, y, z, \dots, z).$$

As t is also compatible with $\mathcal{A}$, we have that

$$t(4, 0, 0) = 0 \stackrel{\mathbb{H}_{2,3}^{4'}}{\Longrightarrow} t(\widehat{4}, \widehat{2}, \widehat{3}) \in \{\widehat{2}, \widehat{3}\}.$$

If $t(\widehat{4}, \widehat{2}, \widehat{3}) = \widehat{2}$, then by compatibility with $\mathbb{H}_3^{2', 4'}$ we have $t(4, 2, 0) = 2$ and from $\mathbb{H}$ we get $t(\widehat{4}, \widehat{2}, \widehat{4}) = \widehat{2}$; a contradiction with the NU property of ν. Therefore $t(\widehat{4}, \widehat{2}, \widehat{3}) = \widehat{3}$. But

$$t(\widehat{4}, \widehat{2}, \widehat{3}) = \widehat{3} \stackrel{\mathbb{H}_2^{3', 4'}}{\Longrightarrow} t(4, 0, 3) = 3 \stackrel{\mathbb{H}}{\Longrightarrow} t(\widehat{4}, \widehat{4}, \widehat{3}) = \widehat{3}) \stackrel{\mathbb{H}_3^{2', 4'}}{\Longrightarrow} t(4, 4, 0) = 0;$$

and the claim is proved. $\square$

References

[1] L. Barto and M. Kozik. *Constraint Satisfaction Problems of Bounded Width.* Proceedings of the 50th Annual IEEE Symposium on Foundations of Computer Science, FOCS'09 (2009), 595-603.

[2] L. Barto, M. Kozik, M. Maróti and T. Niven. *CSP dichotomy for special triads.* Proc. Amer. Math. Soc. 137/9 (2009), 2921-2934.

[3] L. Barto, M. Kozik and T. Niven. *The CSP dichotomy holds for digraphs with no sources and no sinks (a positive answer to a conjecture of Bang-Jensen and Hell).* SIAM Journal on Computing 38/5 (2009), 1782-1802.

[4] A. Bulatov, P. Jeavons and A. Krokhin. *Classifying the complexity of constraints using finite algebras.* SIAM J. Comput., 34(3):720-742 (electronic), 2005.

[5] V. Dalmau and J. Pearson. *Closure functions and width 1 problems.* Proceedings of the 5th International Conference on Constraint Programming (CP99), Lecture Notes in Comput. Sci. 1713, Springer-Verlag, Berlin, 1999, pp. 159-173.
[6] T. Feder. *Classification of homomorphisms to oriented cycles and of k-partite satisfiability.* SIAM J. Discrete Math., 14(4):471-480 (electronic), 2001.
[7] T. Feder and M. Vardi. *The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory.* SIAM J. Comput., 28(1):57-104 (electronic), 1999.
[8] W. Gutjahr, E. Welzl and G. Woeginger. *Polynomial graph-colorings.* Discrete Appl. Math., 35(1): 29-45, 1992.
[9] R. Häggkvist, P. Hell, D. J. Miller and V. Neumann Lara. *On multiplicative graphs and the product conjecture.* Combinatorica 8 (1988), no. 1, 63-74.
[10] P. Hell and J. Nešetřil. *On the complexity of H-coloring.* J. Combin. Theory Ser. B, 48(1): 92-110, 1990.
[11] P. Jeavons, D. Cohen and M. Gyssens. *Closure properties of constraints.* J. ACM, 44(4):527-548, 1997.
[12] A. Krokhin, A. Bulatov and P. Jeavons. *The complexity of constraint satisfaction: An algebraic approach.* Structural Theory of Automata, Semigroups and Universal Algebra (Montreal, 2003), NATO Sci. Ser. II: Math., Phys., Chem., volume 207, 181-213, 2005.
[13] B. Larose and L. Zádori. *Bounded width problems and algebras.* Algebra Universalis, 56(3-4):439-466, 2007.
[14] M. Maróti and R. McKenzie. *Existence theorems for weakly symmetric operations.* Algebra Universalis 59 (2008), no. 3-4, 463-489.

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