# The complete classification for quantified equality constraints

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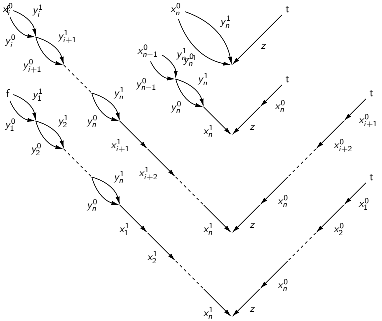


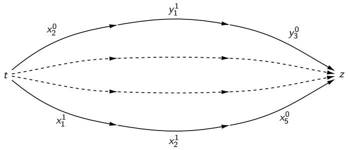

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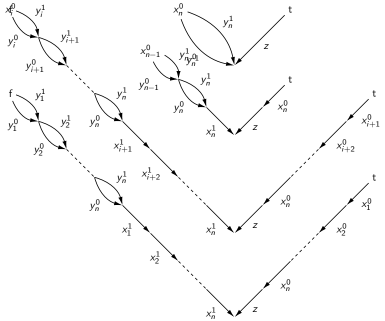

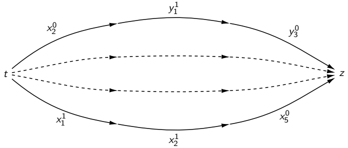


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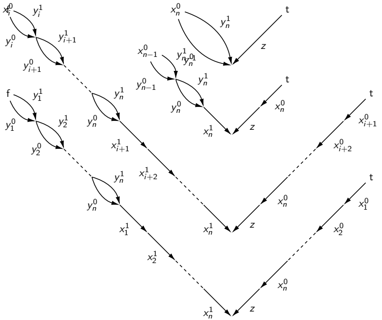

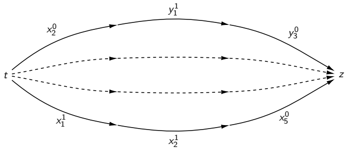


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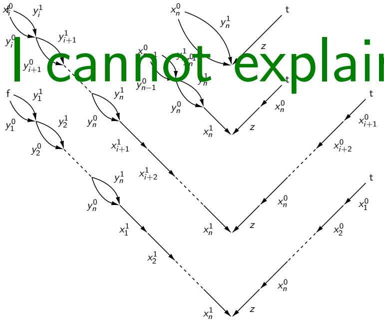
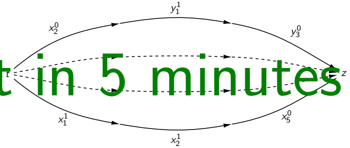
I cannot explain it in 5 minutes






One suspects that
a winning strategy for the EP is to
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To build a reduction
we start a new sequence of $\rightarrow$
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# Quantified Equality Constraints

What is the complexity of QCSP($\mathbb{N}; x = y \rightarrow y = z$)?

- QCSP($\mathbb{N}; x = y \rightarrow y = z$) is coNP-hard [Bodirsky, Chen, 2010].

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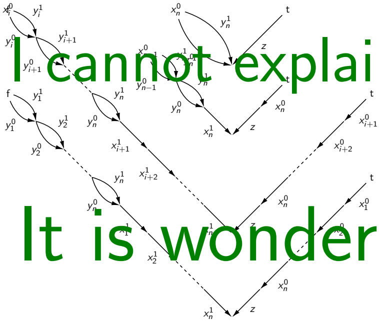

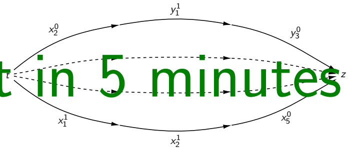


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## Classification of the complexity

### Theorem

Let $\Gamma$ be an equality constraint language.

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$(x_1 = x_2 \vee x_3 \neq x_4)$, $(x_1 \neq x_2 \vee x_2 = x_3 \vee x_3 \neq x_4)$

# Bounded alternation

## Bounded alternation

**$\Pi_k$-QCSP$(A;\Gamma)$**

Given a $\Pi_k$-sentence

$$\forall x_{1,1} \ldots \forall x_{1,n_1} \exists x_{2,1} \ldots \exists x_{2,n_2} \ldots \exists x_{k,1} \ldots \exists x_{k,n_k}$$

$$(R_1(\ldots) \wedge \cdots \wedge R_s(\ldots))$$

where $R_1, \ldots, R_s \in \Gamma$.

Decide whether it holds.

## Bounded alternation

### $\Pi_k$-**QCSP**$(A;\Gamma)$

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Decide whether it holds.

**Question**

What is the complexity of $\Sigma_k$-QCSP$(A;\Gamma)$ and $\Pi_k$-QCSP$(A;\Gamma)$ for each equality constraint language $\Gamma$ and each $k$?

## Bounded alternation

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where $R_1, \dots, R_s \in \Gamma$.
Decide whether it holds.

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What is the complexity of $\Sigma_k$-QCSP$(A;\Gamma)$ and $\Pi_k$-QCSP$(A;\Gamma)$ for each equality constraint language $\Gamma$ and each $k$?

- $\Pi_{2k}$-QCSP$(A;\Gamma)$ and $\Pi_{2k+1}$-QCSP$(A;\Gamma)$ are polynomially equivalent for $k \geq 2$.

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- $\Sigma_{2k+1}$-QCSP$(A;\Gamma)$ and $\Sigma_{2k+2}$-QCSP$(A;\Gamma)$ are polynomially equivalent for $k \geq 2$

## Classification for bounded alternation

**Theorem**

Let $\Gamma$ be an equality constraint language, $k \geq 2$ be even.

# Classification for bounded alternation

**Theorem**

Let $\Gamma$ be an equality constraint language, $k \geq 2$ be even.

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## Classification for bounded alternation

**Theorem**

Let $\Gamma$ be an equality constraint language, $k \geq 2$ be even.

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A predicate is Horn if it has a CNF definition in which each clause contains at most one positive literal.

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## Classification for bounded alternation

### Theorem

Let $\Gamma$ be an equality constraint language, $k \geq 3$ be odd.

- If $\Gamma$ is negative, then $\Sigma_k$-QCSP$(\mathbb{N}; \Gamma)$ is in Logspace.

A predicate is negative if it has a CNF definition in which all of the clauses are either equalities, or are disjunctions of negative literals

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# **Predicate** $x = y \to y = z$

# Predicate $x = y \to y = z$

- QCSP($\mathbb{N}; x = y \to y = z$) is PSpace-complete.

# Predicate $x = y \rightarrow y = z$

- QCSP($\mathbb{N}; x = y \rightarrow y = z$) is PSpace-complete.
- $\Pi_k$-QCSP($\mathbb{N}; x = y \rightarrow y = z$) is coNP-complete for every $k \geq 2$.

# Predicate $x = y \to y = z$

- QCSP($\mathbb{N}; x = y \to y = z$) is PSpace-complete.
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- $\Sigma_k$-QCSP($\mathbb{N}; x = y \to y = z$) is coNP-complete for every $k \geq 3$.

# Predicate $x = y \rightarrow y = z$

- $\mathrm{QCSP}(\mathbb{N}; x = y \rightarrow y = z)$ is PSpace-complete.
- $\Pi_k$-$\mathrm{QCSP}(\mathbb{N}; x = y \rightarrow y = z)$ is coNP-complete for every $k \geq 2$.
- $\Sigma_k$-$\mathrm{QCSP}(\mathbb{N}; x = y \rightarrow y = z)$ is coNP-complete for every $k \geq 3$.
- $\Sigma_1$-$\mathrm{QCSP}(\mathbb{N}; x = y \rightarrow y = z)$,
  $\Sigma_2$-$\mathrm{QCSP}(\mathbb{N}; x = y \rightarrow y = z)$, and
  $\Pi_1$-$\mathrm{QCSP}(\mathbb{N}; x = y \rightarrow y = z)$ are in Logspace.

# Next step (Temporal QCSP)

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Suppose $\Gamma$ is a set of predicates on $\mathbb{Q}$ definable as boolean combinations of $x < y$ and $x = y$.

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What is the complexity of $\mathrm{QCSP}(\mathbb{Q}; \Gamma)$ for every $\Gamma$?

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- A complete classification for positive languages (W. Charatonik, M. Wrona, 2008)
- A dichotomy for self-dual languages (M. Wrona, 2014)

## Next step (Temporal QCSP)

Suppose $\Gamma$ is a set of predicates on $\mathbb{Q}$ definable as boolean combinations of $x < y$ and $x = y$.

**Open question**

What is the complexity of $\text{QCSP}(\mathbb{Q}; \Gamma)$ for every $\Gamma$?

- Complexity of $\text{CSP}(\mathbb{Q}; \Gamma)$ was classified (Bodirsky, Kára, 2010)
- A complete classification for positive languages (W. Charatonik, M. Wrona, 2008)
- A dichotomy for self-dual languages (M. Wrona, 2014)
- $\text{QCSP}(\mathbb{Q}; x = y \to y \leq z)$ is in P (J. Rydval, 2022)

## Next step (Temporal QCSP)

Suppose $\Gamma$ is a set of predicates on $\mathbb{Q}$ definable as boolean combinations of $x < y$ and $x = y$.

**Open question**

What is the complexity of $\mathrm{QCSP}(\mathbb{Q}; \Gamma)$ for every $\Gamma$?

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- A dichotomy for self-dual languages (M. Wrona, 2014)
- $\mathrm{QCSP}(\mathbb{Q}; x = y \to y \leq z)$ is in P (J. Rydval, 2022)
- $\mathrm{QCSP}(\mathbb{Q}; x = y < z \vee x = z < y)$ is NP-complete

## Next step (Temporal QCSP)

Suppose $\Gamma$ is a set of predicates on $\mathbb{Q}$ definable as boolean combinations of $x < y$ and $x = y$.

**Open question**

What is the complexity of QCSP($\mathbb{Q}; \Gamma$) for every $\Gamma$?

- Complexity of CSP($\mathbb{Q}; \Gamma$) was classified (Bodirsky, Kára, 2010)
- A complete classification for positive languages (W. Charatonik, M. Wrona, 2008)
- A dichotomy for self-dual languages (M. Wrona, 2014)
- QCSP($\mathbb{Q}; x = y \rightarrow y \leq z$) is in P (J. Rydval, 2022)
- QCSP($\mathbb{Q}; x = y < z \vee x = z < y$) is NP-complete
- QCSP($\mathbb{Q}; x < y \vee u < v$) is PSpace-complete

## Next step (Temporal QCSP)

Suppose $\Gamma$ is a set of predicates on $\mathbb{Q}$ definable as boolean combinations of $x < y$ and $x = y$.

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- QCSP($\mathbb{Q}; x < y \vee u < v$) is PSpace-complete
- QCSP($\mathbb{Q}; x = y \rightarrow z < t$) is PSpace-complete

## Next step (Temporal QCSP)

Suppose $\Gamma$ is a set of predicates on $\mathbb{Q}$ definable as boolean combinations of $x < y$ and $x = y$.

**Open question**

What is the complexity of $\mathrm{QCSP}(\mathbb{Q}; \Gamma)$ for every $\Gamma$?

- Complexity of $\mathrm{CSP}(\mathbb{Q}; \Gamma)$ was classified (Bodirsky, Kára, 2010)
- A complete classification for positive languages (W. Charatonik, M. Wrona, 2008)
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- $\mathrm{QCSP}(\mathbb{Q}; x = y \rightarrow z < t)$ is PSpace-complete
- $\mathrm{QCSP}(\mathbb{Q}; x = y \rightarrow (x > u \vee x \geq v))$ is an open question

## Next step (Temporal QCSP)

Suppose $\Gamma$ is a set of predicates on $\mathbb{Q}$ definable as boolean combinations of $x < y$ and $x = y$.

**Open question**

What is the complexity of $\mathrm{QCSP}(\mathbb{Q};\Gamma)$ for every $\Gamma$?

- Complexity of $\mathrm{CSP}(\mathbb{Q};\Gamma)$ was classified (Bodirsky, Kára, 2010)
- A complete classification for positive languages (W. Charatonik, M. Wrona, 2008)
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- $\mathrm{QCSP}(\mathbb{Q}; x = y \to y \leq z)$ is in P (J. Rydval, 2022)
- $\mathrm{QCSP}(\mathbb{Q}; x = y < z \vee x = z < y)$ is NP-complete
- $\mathrm{QCSP}(\mathbb{Q}; x < y \vee u < v)$ is PSpace-complete
- $\mathrm{QCSP}(\mathbb{Q}; x = y \to z < t)$ is PSpace-complete
- $\mathrm{QCSP}(\mathbb{Q}; x = y \to (x > u \vee x \geq v))$ is an open question
- $\mathrm{QCSP}(\mathbb{Q}; x < y \vee y < z)$ is an open question

## Next step (Temporal QCSP)

Suppose $\Gamma$ is a set of predicates on $\mathbb{Q}$ definable as boolean combinations of $x < y$ and $x = y$.

**Open question**

What is the complexity of QCSP($\mathbb{Q}; \Gamma$) for every $\Gamma$?

- Complexity of CSP($\mathbb{Q}; \Gamma$) was classified (Bodirsky, Kára, 2010)
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- QCSP($\mathbb{Q}; x < y \vee y < z$) is an open question
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Thank you for your attention