Constraint Satisfaction Problems of Bounded Width

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Relational structures and homomorphisms

Definition

Type is finite sequence of natural numbers

Relational structure of type $r_1, \ldots, r_n$ is a tuple $(A, R_1, \ldots, R_n)$, where A is a finite set, R_i is a relation of arity r_i, i.e. $R_i \subseteq A^{r_i}$

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Let $\mathbb{X} = (X, S_1, \ldots, S_n)$ and $\mathbb{A} = (A, R_1, \ldots, R_n)$ be relational structures of the same type $r_1, \ldots, r_n$.
A homomorphism $f : \mathbb{X} \to \mathbb{A}$ is a mapping $f : X \to A$ preserving all the relations, i.e. $(f(x_1), \ldots, f(x_{r_i})) \in R_i$ for any $(x_1, \ldots, x_{r_i}) \in S_i$.

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Example

Relational structures of type 2 are directed graphs. Homomorphism = edge-preserving mapping

Constraint Satisfaction Problem (CSP)

Definition (CSP with fixed template)

Let $\mathbb{A} = (A, R_1, \dots, R_n)$ *be a relational structure (template).*
CSP$(\mathbb{A})$ *is the following decision problem*

INPUT *A rel. structure* $\mathbb{X} = (X, S_1, \dots, S_n)$ *of the same type as* $\mathbb{A}$
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Question

For a fixed $\mathbb{A}$, what is the time complexity of $CSP(\mathbb{A})$? (Clearly in NP)

k-coloring problem

Fix $k \in \mathbb{N}$

$\mathbb{A} = (\{1, 2, \ldots, k\}, R)$ type 2, where
$(x, y) \in R$ iff $x \neq y$

$\mathbb{X} = \{X, E\}$.

A mapping $f : X \to \{1, 2, \ldots, k\}$ is a homomorphism iff it is a k-coloring (if $(x, y) \in E$, then $f(x) \neq f(y)$)

$CSP(\mathbb{A}) = k$-COL

Complexity:

- **P** if $k \leq 2$
- **NP-complete** if $k \geq 3$

3-SAT

$\mathbb{A} = (\{0,1\}, R_1, R_2, R_3, R_4)$ type 3, 3, 3, 3, where
$(x,y,z) \in R_1$ iff $x \vee y \vee z$
$(x,y,z) \in R_2$ iff $x \vee y \vee \neg z$
$(x,y,z) \in R_3$ iff $x \vee \neg y \vee \neg z$
$(x,y,z) \in R_4$ iff $\neg x \vee \neg y \vee \neg z$

$\mathbb{X} = (\{x_1, \dots, x_4\},\ \{(x_1,x_2,x_4),(x_2,x_3,x_3)\},\{(x_4,x_3,x_1),(x_2,x_1,x_3)\},\emptyset,\emptyset)$

Consider the formula

$(x_1 \vee x_2 \vee x_4) \wedge (x_2 \vee x_2 \vee x_3) \wedge (x_4 \vee x_3 \vee \neg x_1) \wedge (x_2 \vee x_1 \vee \neg x_3)$

A mapping $f : X \to A$ is a homomorphism, if it is an evaluation of variables $x_1, \dots, x_4$ which makes the formula above true

$CSP(\mathbb{A}) = 3 - SAT$

Complexity ... NP-complete

Systems of linear equations over $GF(p^k)$

$\mathbb{A} = \big(GF(p^k),\ R,\ R_i(i \in GF(p^k))\big)$ type $3, 1, 1, \ldots, 1$, where
$(x, y, z) \in R_1$ iff $x + y = z$
$R_i = \{i\}$

$\mathbb{X} = \big(\{x_1, \ldots, x_5\},\ S,\ S_i(i \in GF(p^k))\big)$, where
$S = \{(x_1, x_3, x_5), (x_2, x_5, x_4)\}$
$S_4 = \{x_1, x_2\}$
$S_i = \emptyset$, for $i \neq 4$

homomorphism = solution of the following system of lin. eq. over $GF(p^k)$

$x_1 + x_3 = x_5,\ x_2 + x_5 = x_4,\ x_1 = 4,\ x_2 = 4$

$CSP(\mathbb{A}) = SysLinEq(p^k)$

Complexity ... P

The dichotomy conjecture

The conjecture of Feder and Vardi 93

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Theorem (Feder, Vardi 93)

For every $\mathbb{A}$ there exists a directed graph $\mathbb{A}'$ such that $CSP(\mathbb{A})$ has the same complexity as $CSP(\mathbb{A}')$.

Towards the algebraic approach

WLOG

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Definition

Let $R \subseteq A^m$. We say that an operation $f : A^n \to A$ is compatible with R (or R is compatible with f), if for every $(a_{ij})_{i=1\dots m, j=1\dots n}$ $a_{ij} \in A$ such that columns are in R, then $(f(a_{11}, \dots a_{1n}), \dots f(a_{m1}, \dots, a_{mn})) \in R$.

We say that an operation $f : A^n \to A$ is a polymorphism of $\mathbb{A}$, if it is compatible with all the relations in $\mathbb{A}$.

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Every polymorphism is idempotent, i.e. $f(a, a, \ldots, a) = a$ for all $a \in A$.

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Example

A projection is a polymorphism of every relational structure.

Polymorphism - a better example

malcev...

Bulatov, Jeavons, Krokhin 00

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The complexity of $\mathbb{A}$ depends only on $\mathbf{A}$!

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Theorem

If $HSP(\mathbf{A})$ contains a trivial algebra (i.e. at least 2-element algebra such that every operation is a projection), then $CSP(\mathbb{A})$ is NP-complete.

Algebraic dichotomy conjecture and some results

Conjecture (BJK 00)

If $HSP(\mathbf{A})$ doesn't contain a trivial algebra, then $CSP(\mathbb{A})$ is in P. Otherwise, it is NP-complete.

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- True, if $\mathbb{A}$ contains all unary relations (Bulatov 05)
- True, if $\mathbb{A}$ is a digraph such that all vertices have an incoming and an outgoing edge (Barto, Kozik, Niven 06)

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- **"Consistency checking."** We didn't know when this algorithm gives a correct answer. Just some very special cases were known

  Bulatov, Carvalho, Dalmau, Feder, Kiss, Marković, Maróti, Valeriote, Vardi

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$\mathbb{X}, \mathbb{A}$ ... relational structures of the same type. k ... a natural number.

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- If there is a homomorphism $\mathbb{X} \to \mathbb{A}$, then there exists a nonempty k-strategy.

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A relational structure $\mathbb{A}$ has bounded width, if it has width k for some k.

Bounded width is everywhere!

Bounded width has many equivalent formulation

- Combinatorics: bounded tree width duality
- Logic: via definability in certain infinitary logic
- Programming: solvability in Datalog (fragment of Prolog)
- Pebble games
- ...

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Theorem (Barto, Kozik 08)

Yes!

Our basic tool

Theorem (No trivial algebras!)

Let $\mathbb{A}$ be a relational structure. TFAE

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$$w(b, a, \dots, a) = w(a, b, a, \dots, a) = \dots = w(a, \dots, a, b)$$

(so called Weak-NU operation)

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- *?????(BJK conjecture) $CSP(\mathbb{A})$ is in P*

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- *(BK 08) $CSP(\mathbb{A})$ has bounded width*