Minimal Taylor Algebras

as a Common Framework for the Three Algebraic Approaches to the **CSP**

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CONSTRAINT SATISFACTION PROBLEMS

$\mathrm{CSP}(A;\ R_1, R_2, \ldots, R_k)$

A finite domain
$R_i \subseteq A^{r_i}$

Examples

- 3-SAT
- 3-COLORING
- solving system of equations over...

INPUT: conjunction of constraints
e.g. $R_1(x,y,z) \wedge R_2(z,x) \wedge R_1(u,u,y)$

QUESTION: satisfiable?

Theorem Computational complexity depends only on $\mathcal{C}$ – the clone of polymorphisms [Jeavons et al 90s]

- contains projections $(x_1, \ldots, x_n) \mapsto x_i$
- closed under composition

operations $A^n \to A$ compatible with each R_i

[Bulatov et al 00s]

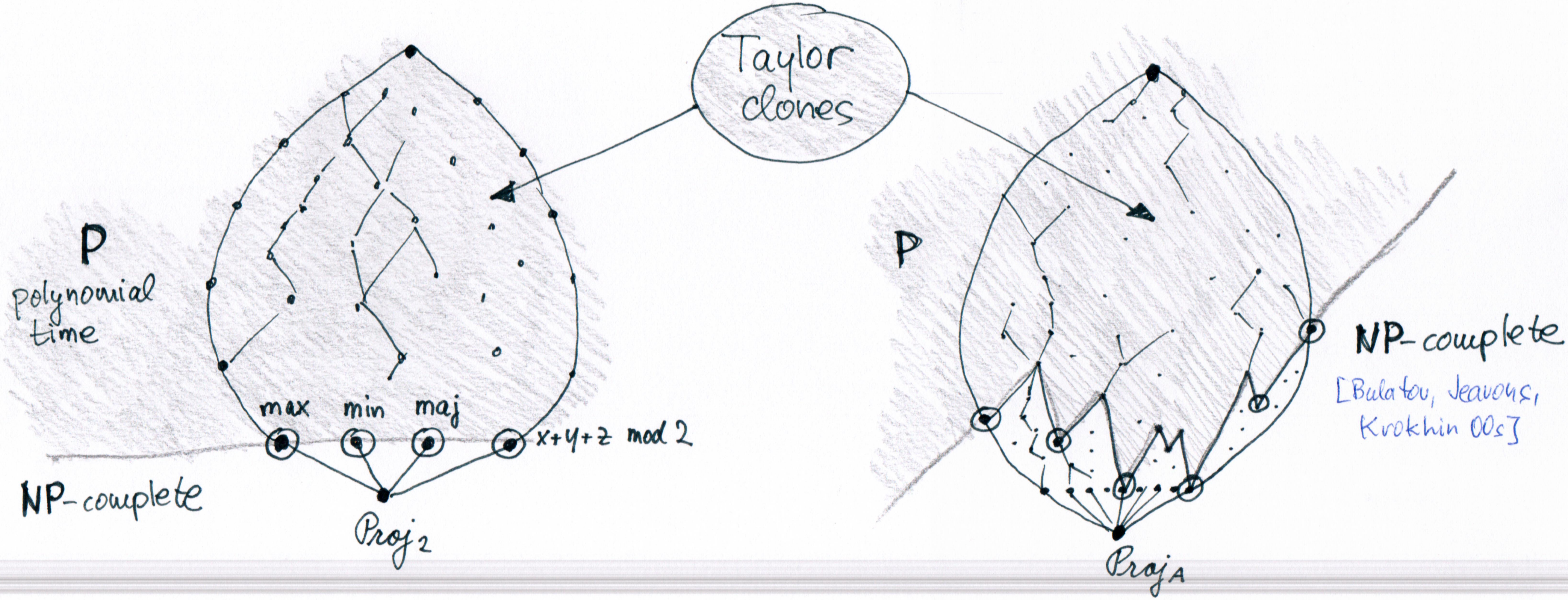
Theorem Can assume $\mathcal{C}$ idempotent

$\forall f \in \mathcal{C}\ \forall a \in A\ f(a,a,\ldots,a) = a$

DICHOTOMY THEOREM

$A = \{0,1\}$ [Schaefer 70s]

$|A| > 2$ [Bulatov 17, Zhuk 17]

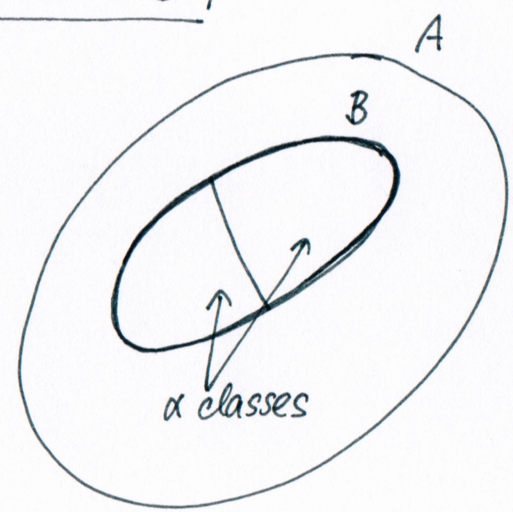


⊙ minimal Taylor clones $\leftrightarrow$ "hardest" CSPs in P

TAYLOR CLONES

Taylor clone no factor is Proj_2 (here finite, idempotent)

$$\boxed{\begin{array}{l} \text{factor of } \mathcal{C} = \text{clone } \mathcal{C}{\restriction}B/\alpha \text{ on } B/\alpha \\ \qquad B \subseteq A \text{ invariant subset} \\ \qquad \alpha \text{ invariant equivalence on } B \end{array}}$$



Minimal Taylor clone

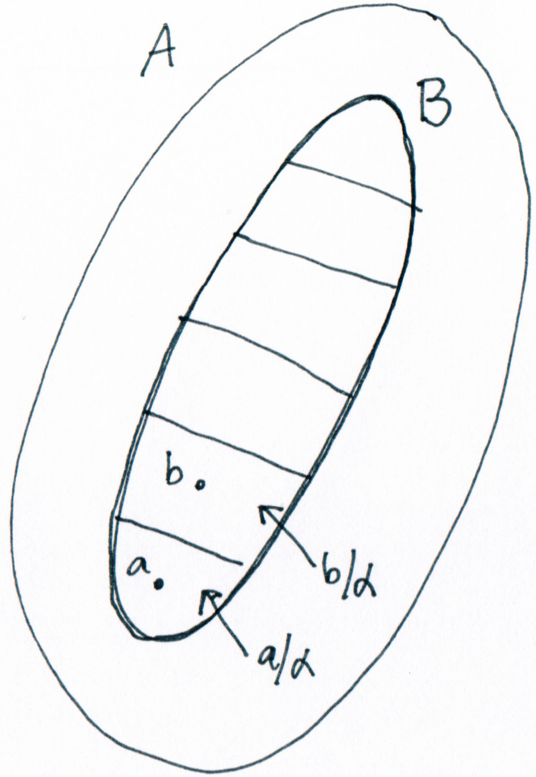
minimal (wrt $\subseteq$) among
Taylor clones on A

Tools for Taylor clones:

classic
Bulatov's theory
Zhuk's theory
absorption theory

BULATOV'S THEORY

$\mathcal{C}$ clone on A $\longmapsto$ directed graph on A



$a \longrightarrow b$ if $\exists$ factor $\mathcal{C}{\restriction_B}/\alpha = \mathcal{D}$ such that $a, b \in B$, $a/\alpha \neq b/\alpha$ and

$a \xrightarrow{\text{semilattice}} b$ $\exists$ binary $s \in \mathcal{D}$ such that s on $\{a/\alpha, b/\alpha\}$ is max on $\{0,1\}$

$a \xrightarrow{\text{majority}} b$ $\exists$ ternary $m \in \mathcal{D}$ such that m on $\{a/\alpha, b/\alpha\}$ is majority on $\{0,1\}$

$a \xrightarrow{\text{abelian}} b$ $\mathcal{D}$ is abelian ... "essentially a module" (for Taylor)

Fundamental theorem This directed graph is connected

ZHUK'S THEORY

Fundamental theorem $\mathcal{C}$ Taylor $\Rightarrow$ one of the following

• $\exists$ nontrivial 2-absorbing $B \subseteq A$ i.e. $\exists$ binary $s \in \mathcal{C}$ $s(B,A) \cup s(A,B) \subseteq B$	$\{1\}$ in $\mathrm{Clo}(\max)$ on $\{0,1\}$
• $\exists$ nontrivial 3-absorbing $B \subseteq A$ (+ extra properties) i.e. $\exists$ ternary $m \in \mathcal{C}$ $m(B,B,A) \cup m(B,A,B) \cup m(A,B,B) \subseteq B$	$\{0\}$, $\{1\}$ in $\mathrm{Clo}(\mathrm{maj})$ on $\{0,1\}$
• $\exists$ proper α such that $\mathcal{C}/\alpha$ is abelian	= in $\mathrm{Clo}(x+y+z \bmod 2)$ on $\{0,1\}$
• $\exists$ proper α such that $\mathcal{C}/\alpha$ is polynomially complete i.e. $\mathcal{C}/\alpha$ + constants generate all operations	= in $\mathrm{Clo}(\mathrm{winner})$ on $\{\mathrm{rock}, \mathrm{paper}, \mathrm{scissors}\}$

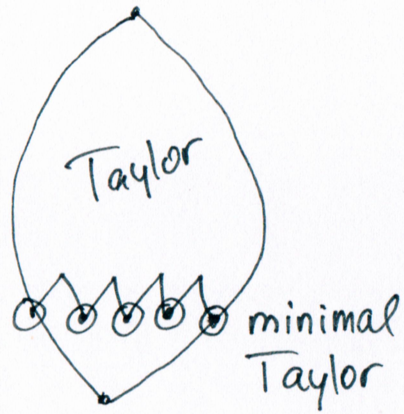
RESULTS

Taylor clones

- connection absorption $\leftrightarrow$ Zhuk
- simple theorem that implies both "Fundamental theorems"

Minimal Taylor clones

- "sufficiently general": every Taylor clone contains a minimal Taylor clone
- concepts get simpler and stronger
- surprising connections Bulatov $\leftrightarrow$ Zhuk



Follow up work

- all minimal Taylor clones on $\{0,1,2\}$ found (24 up to renaming elements) [Brady]

RESULTS: EDGES

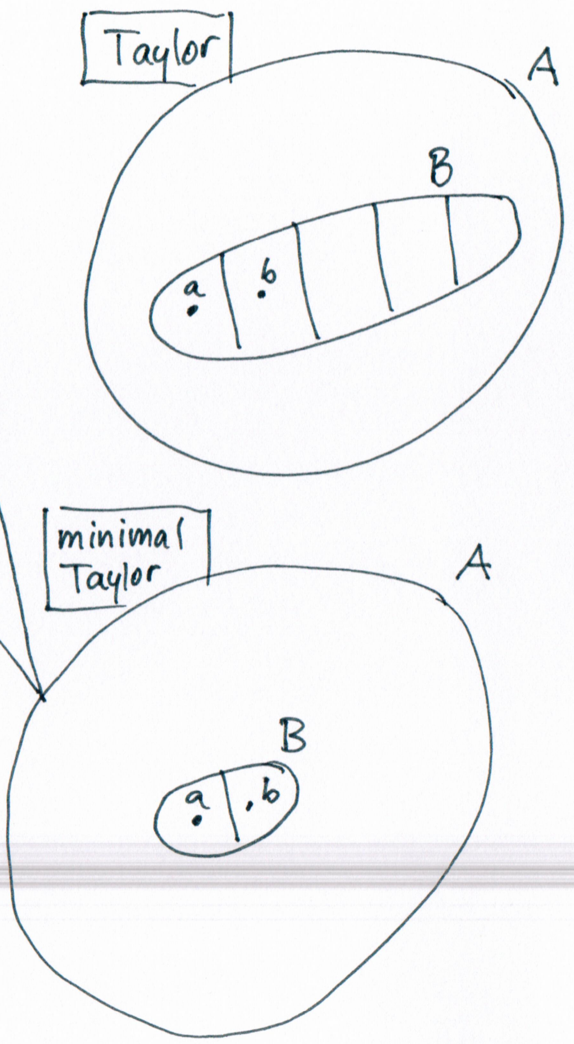
$\mathcal{C}$ minimal Taylor

Theorem Witness $\mathcal{D} = \mathcal{C}\restriction_B/\alpha$ for $a \to b$ can be chosen so that

$a \xrightarrow{\text{semilattices}} b$ $\quad$ $\mathcal{D}$ is $\mathrm{Clo}(\max)$ on $\{0,1\}$ (up to renaming elements)

$a \xrightarrow{\text{majority}} b$ $\quad$ $\mathcal{D}$ is $\mathrm{Clo}(\mathrm{maj})$ on $\{0,1\}$

$a \xrightarrow{\text{abelian}} b$ $\quad$ $\mathcal{D}$ is $\mathrm{Clo}(x-y+z)$ on an abelian group



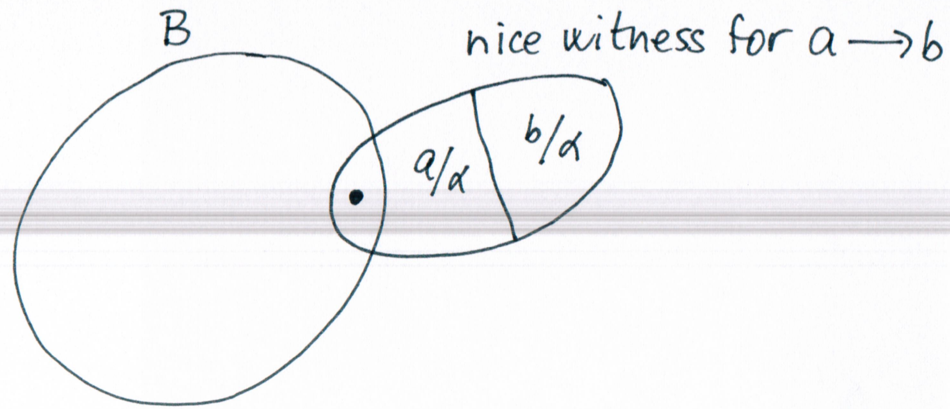
RESULTS : ABSORPTION

$\mathcal{C}$ minimal Taylor on A

Theorem TFAE for $B \subseteq A$

- B is 2-absorbing, i.e. $\exists s \in \mathcal{C}\ \ s(B,A) \cup s(A,B) \subseteq B$
- $\forall t \in \mathcal{C}$ that depends on 1st coordinate $t(B,A,A,\dots,A) \subseteq B$
- B is stable under edges

it doesn't happen that



RESULTS: WITNESSING OPERATION

$\mathcal{C}$ minimal Taylor

Theorem $\exists$ ternary $f \in \mathcal{C}$ "witnessing all edges and 2,3-absorptions"

- if $a \xrightarrow{\text{semilattice}} b$ then $f(x,y,z) = \max(x,y,z)$ on $\mathcal{D} = \mathcal{C}\restriction B/\alpha$ (nice witness)
- if $a \xrightarrow{\text{majority}} b$ then $f(x,y,z) = \mathrm{maj}(x,y,z)$ on $\mathcal{D}$
- if $a \xrightarrow{\text{abelian}} b$ then $f(x,y,z) = x - y + z$ on $\mathcal{D}$

- if B is 3-absorbing then $f(B,B,A) \cup f(B,A,B) \cup f(A,B,B) \subseteq B$
- if B is 2-absorbing then $f(B,A,A) \cup f(A,B,A) \cup f(A,A,B) \subseteq B$

Moreover, any such f generates $\mathcal{C}$.

RESULTS: OMITTING EDGE TYPES

$\mathcal{C}$ minimal Taylor

Theorem TFAE

- no abelian or semilattice edges (= only majority edges)
- $\mathcal{C}$ has a majority operation $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$
- $\mathcal{C}$ has a near unanimity operation $n(x,x,\dots x y) = n(x \dots x y x) = n(y x \dots x) = x$

Theorem TFAE

- no semilattice or majority edges
- no $\mathcal{C} \restriction B$ has a nontrivial absorbing subset
- $\mathcal{C}$ has a Mal'cev operation $p(y,x,x) = y = p(x,x,y)$

... (avoiding 1 type)

SUMMARY

Minimal Taylor clones are

- much nicer than general Taylor clones
- sufficiently general for some purposes (e.g. CSP dichotomy)

LONG TERM AIMS

- simplify proofs of CSP dichotomy theorem
- find all minimal Taylor clones
- create one coherent theory incorporating Bulatov, Zhuk, absorption + classic theories (TCT, commutator theory)

SPECIFIC QUESTIONS

- many (e.g. see the paper)
- the most embarrasing: $\mathcal{C}$ minimal Taylor on A, $a, b \in A$, $a \neq b$.
  Is always $a \rightarrow b$ or $b \rightarrow a$?