On the complexity of symmetric Promise CSP

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joint work with Libor Barto

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What is a CSP?

Fix $\mathbb{A} = (A; R, S, \ldots)$ a finite relational structure on the domain A.

Definition ($\mathrm{CSP}(\mathbb{A})$, Decision version)

Input: a pp-sentence ϕ, e.g. $(\exists x_1 \exists x_2 \ldots) R(x_1) \wedge S(x_1, x_1, x_2) \wedge \ldots$
Answer Yes: ϕ is satisfied in $\mathbb{A}$
Answer No: ϕ is not satisfied in $\mathbb{A}$

Search Version: Find a satisfying assignment.
(Search version is as hard as Decision version)

Examples of CSP

- $\mathbb{K}_3 = (\{1,2,3\}; N)$ where $N = \{1,2,3\}^2 \setminus \{(1,1),(2,2),(3,3)\}$
  $\mathrm{CSP}(\mathbb{K}_3)$ is the 3-coloring problem for graphs

- $\mathbb{NAE} = (\{0,1\}; NAE)$ where $NAE = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$
  $\mathrm{CSP}(\mathbb{NAE})$: given a 3-uniform hypergraph,
  find a 2-coloring such that no hyperedge is monochromatic

- $1\text{-}\mathbb{IN}\text{-}3 = (\{0,1\}; 1\text{-}in\text{-}3)$ where $1\text{-}in\text{-}3 = \{(0,0,1),(0,1,0),(1,0,0)\}$
  $\mathrm{CSP}(1\text{-}\mathbb{IN}\text{-}3)$: given a 3-uniform hypergraph,
  find a 2-coloring in which exactly one vertex in each hyperedge receives 1

These are all well known NP-hard problems

Polymorphisms

Polymorphism of $\mathbb{A}$: a map $f : A^n \longrightarrow A$
compatible with the relations of $\mathbb{A}$

f compatible with R: f applied component-wise to tuples in R
is a tuple in R

$$\begin{pmatrix} f & (a_{1,1} & a_{1,2} & \dots & a_{1,n}) \\ f & (a_{2,1} & a_{2,2} & \dots & a_{2,n}) \\ & \vdots & \vdots & & \vdots \\ f & (a_{m,1} & a_{m,2} & \dots & a_{m,n}) \end{pmatrix} \in R$$

$$\begin{matrix} \Cap & \Cap & & \Cap \\ R & R & & R \end{matrix}$$

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What is a Promise CSP (PCSP)?

Fix two similar relational structures:

- $\mathbb{A} = (A; R^{\mathbb{A}}, S^{\mathbb{A}}, \dots)$
- $\mathbb{B} = (B; R^{\mathbb{B}}, S^{\mathbb{B}}, \dots)$
- there is a homomorphism $\mathbb{A} \longrightarrow \mathbb{B}$

Definition ($\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$, Decision version)

Input: a pp-sentence ϕ
Answer Yes: ϕ is satisfied in $\mathbb{A}$
Answer No: ϕ is not satisfied in $\mathbb{B}$

Search Version: given an input which is satisfiable in $\mathbb{A}$
find a satisfying assignment in $\mathbb{B}$.

Examples of PCSP

- $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_4)$: given a 3-colorable graph, find a 4-coloring such that no edge is monochromatic (**it is NP-hard** [Brakensiek, Guruswami '16])

- $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{NAE})$: given a 3-uniform hypergraph which admits a 2-coloring in which exactly one vertex per hyperedge is colored with the color 1, find a 2-coloring such that no hyperedge is monochromatic (**it is in P** [Brakensiek, Guruswami '18])

Polymorphisms

Polymorphism of $(\mathbb{A}, \mathbb{B})$: a map $f : A^n \longrightarrow B$
compatible with any relation pair $(R^{\mathbb{A}}, R^{\mathbb{B}})$

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Some theory

$\mathrm{Pol}(\mathbb{A})$ and $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ determine the complexity of $\mathrm{CSP}(\mathbb{A})$ and $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$, respectively.

Theorem (For CSP - Jeavons'98)

If $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$ *then* $\mathrm{CSP}(\mathbb{B})$ *is not harder than* $\mathrm{CSP}(\mathbb{A})$

Theorem (For PCSP - Brakensiek, Guruswami '16)

If $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) \subseteq \mathrm{Pol}(\mathbb{A}', \mathbb{B}')$ *then* $\mathrm{PCSP}(\mathbb{A}', \mathbb{B}')$ *is not harder than* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

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Symmetric PCSP

What are we studying? The complexity of $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{R})$ where $\mathbb{R} = (\{0,1,2\}; R)$ and R is a ternary relation
Fact: WLOG R is symmetric

Example: If $R = NAE = \{\overleftrightarrow{001}, \overleftrightarrow{110}\}$ (where $\{\overleftrightarrow{001}\} = \{(0,0,1),(0,1,0),(1,0,0)\}$)
then we know that $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{R})$ is in P

Fact: If $\mathbb{R}$ has an homomorphism to $\mathbb{S}$, then $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{S})$ is easier than $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{R})$. Then

- we can draw the poset of all the possible $\mathbb{R}$;
- the higher the structure is, the simpler the PCSP is.

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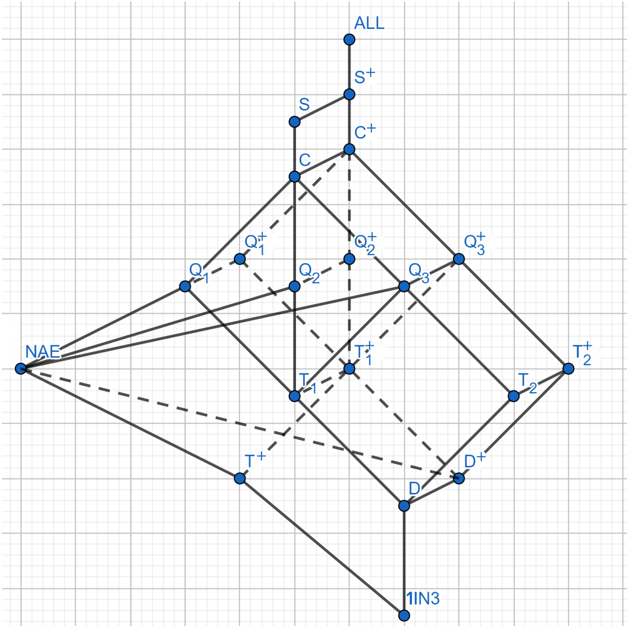
Poset



What is done, what to do

Done:

- $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{NAE})$ is in P [Brakensiek, Guruswami '18]
- $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{D})$ is NP-hard [Kazda '19 - Unpublished] $\quad D = \{\overleftrightarrow{001}, \overleftrightarrow{112}\}$
- $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_2)$ is in P [Barto, B. '19 - Unpublished] $\quad T_2 = \{\overleftrightarrow{001}, \overleftrightarrow{112}, \overleftrightarrow{220}\}$
- $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_1)$ is NP-hard [Barto, B. '19 - Unpublished] $\quad T_1 = \{\overleftrightarrow{001}, \overleftrightarrow{002}, \overleftrightarrow{112}\}$
- $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}^+)$ is NP-hard [Barto, B., Few days ago] $\quad T^+ = \{\overleftrightarrow{001}, \overleftrightarrow{002}, \overleftrightarrow{012}\}$

Work in progress:

- $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{D}^+)$ $\quad D^+ = \{\overleftrightarrow{001}, \overleftrightarrow{112}\} \cup \{\overleftrightarrow{012}\}$
- $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_1^+)$ $\quad T_1^+ = \{\overleftrightarrow{001}, \overleftrightarrow{002}, \overleftrightarrow{112}\} \cup \{\overleftrightarrow{012}\}$

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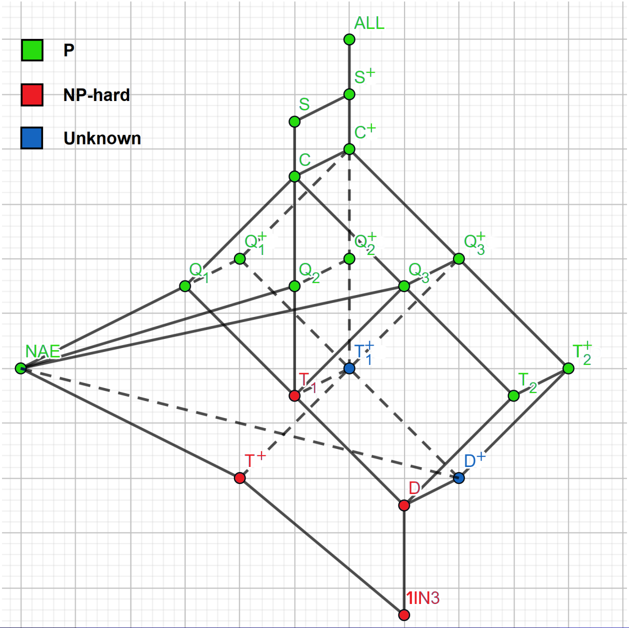
P
NP-hard
Unknown
ALL
S^+
S
C^+
C
Q_1^+
Q_2^+
Q_3^+
Q_1
Q_2
Q_3
NAE
T_1^+
T_2^+
T_1
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T^+
D^+
D
1IN3

Outline

- $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_2)$ in P since $T_2 = \{(x, y, z) : x + y + z = 1 \mod 3\}$, so we can use Gaussian elimination in $\mathbb{Z}_3$
- To show that $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_1)$ is NP-hard we:
  1. describe completely $\mathrm{Pol}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_1)$
  2. use an NP-hardness criterion (described in Barto's talk)
- **Next:** is $\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_1^+)$ NP-hard?
  It is this problem: given a 3-uniform hypergraph which admits a 2-coloring in which exactly one vertex per hyperedge is colored with the color 1, find a 3-coloring such that if two colors in a hyperedge agree, the third one must be higher

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Our aim: to find what exactly is $\mathrm{Pol}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_1)$

- Identifying 1 and 2, we obtain a homomorphism $g : \mathbb{T}_1 \longrightarrow \mathbb{T}_1^*$ where $T_1^* = \{(x,y,z) : x+y+z = 1 \mod 2\}$
- $f \in \mathrm{Pol}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_1)$ induces $f^* = gf \in \mathrm{Pol}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_1^*)$
- $\mathrm{Pol}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_1^*)$ contains only operations that are affine. Namely, if $f \in \mathrm{Pol}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_1^*)^{(n)}$, there is $I_f \subseteq [n]$ such that

$$f(x_1, \dots, x_n) = \begin{cases} \sum_{i \in I_f} x_i & \text{mod } 2, \text{ if } |I_f| \text{ odd} \\ \sum_{i \in I_f} x_i + 1 & \text{mod } 2, \text{ if } |I_f| \text{ even} \end{cases}$$

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$\mathrm{PCSP}(1\text{-}\mathbb{IN}\text{-}3, \mathbb{T}_1)$ $\quad T_1 = \{\overleftrightarrow{001}, \overleftrightarrow{002}, \overleftrightarrow{112}\}$

Notation: for $A \subseteq [n]$ we write $f(A)$ meaning $f(x_1, \ldots, x_n)$ where $x_i = 1$ iff $i \in A$.

- From what we know about f^*, if $|I_f| \geq 6$ and odd we can derive that for every $A \subseteq [n]$,

$$f(A) = \begin{cases} 0, & \text{if } |A \cap I_f| \text{ is even} \\ 1 \text{ or } 2, & \text{if } |A \cap I_f| \text{ is odd} \end{cases}$$

- We can show then that there exists $k \in [n]$ (that we will call *important coordinate*) such that if $|A \cap I_f|$ is odd,

$$f(A) = \begin{cases} 2, & \text{if } k \in A \\ 1, & \text{if } k \notin A \end{cases}$$

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Example

If $|I_f|$ odd and $k \in I_f$ s.t. $f(\{k\}) = 2$, then k is the *important coordinate* for f.
Fix $A \subseteq I_f$ such that $|A|$ is odd (there is $j \in I_f \setminus A$) and B is arbitrary, then

$$\begin{array}{ccccccccc} \multicolumn{4}{c}{\overbrace{\hphantom{1 \quad 0 \quad \dots \quad 0}}^{I_f}} & & & & & \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 & \longrightarrow & 2 \\ 0 & & A & & & B & & \longrightarrow & 1 \\ 0 & & \neg A & & & \neg B & & \longrightarrow & 1 \\ \uparrow & & & & & & & & \\ k & & & & & & & & \end{array}$$

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0 & & A & & & B & & \longrightarrow 1 \\
0 & & \neg A & & & \neg B & & \longrightarrow 1 \\
\uparrow & & & & & & & \\
k & & & & & & &
\end{array}
$$

Example

$$
\begin{array}{ccccccccc}
\multicolumn{5}{c}{\overbrace{\hphantom{0\quad 1\quad 0\quad \ldots\quad 0}}^{I_f}} & & & & \\
0 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & \longrightarrow 1 \\
0 & 0 & & A & & & \neg B & & \longrightarrow 1 \\
1 & 0 & & \neg A & & & B & & \longrightarrow 2 \\
\uparrow & \uparrow & & & & & & & \\
k & j & & & & & & &
\end{array}
$$

We proved that k is an *important coordinate*.

Similarly: there is one and only one *important coordinate* if $|I_f| \geq 6$. Using this and the criterion explained in Barto's talk, we have that the problem is NP-hard.

Example

$$
\begin{array}{ccccccccc}
\multicolumn{5}{c}{\overbrace{\begin{array}{ccccc}0 & 1 & 0 & \dots & 0\end{array}}^{I_f}} & 0 & \dots & 0 & \longrightarrow 1 \\
0 & 0 & & A & & & \neg B & & \longrightarrow 1 \\
1 & 0 & & \neg A & & & B & & \longrightarrow 2 \\
\uparrow & \uparrow & & & & & & & \\
k & j & & & & & & &
\end{array}
$$

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Thank you for your attention!