FINITELY TRACTABLE PCSPs

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OUTLINE

- PCSP
- Finite tractability
- Necessary condition for finite tractability
- PCSP(1-in-3, NAE-3) is not finitely tractable

$\boxed{\text{PCSP}}$

$\boxed{\text{Search version of PCSP}(\mathbb{A}, \mathbb{B})}$ (recall $\exists \mathbb{A} \to \mathbb{B}$)

INPUT: $\mathbb{X}$ such that $\exists\, \mathbb{X} \to \mathbb{A}$ (~~not given~~ the homo is not given to us)

OUTPUT: $\mathbb{X} \to \mathbb{B}$

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special cases
- $\mathbb{A}, \mathbb{B}$ graphs (many partial results)
- $\mathbb{A}, \mathbb{B}$ over 2-element domain
  - dichotomy for symmetric "folded" templates [Brakensiek, Guruswami '18]
  - dichotomy for symmetric templates [Ficak, Kozik, Olšák, Stankiewicz '19]

<u>Example</u>

$\mathbb{A} = (\{0,1\};\ \text{1-in-3})$ $\quad$ $\text{1-in-3} = \{001, 010, 100\}$

$\mathbb{B} = (\{0,1\};\ \text{NAE-3})$ $\quad$ $\text{NAE-3} = \{$ —"— $, 110, 101, 110\}$

INPUT: list $(x_1, x_3, x_7), (x_2, x_5, x_8), \dots$ which is 1-in-3 satisfiable, ie.

$(s(x_1), s(x_3), s(x_7)), \dots \in \text{1-in-3}$ for some s : variables $\to \{0,1\}$

OUTPUT: assignment s: variables $\to \{0,1\}$ such that

$(s(x_1), s(x_3), s(x_7)), \dots \in \text{NAE-3}$

$\boxed{\text{PCSP}(\text{1-in-3}, \text{NAE-3}) \text{ is in P}}$

given input $(x_1, x_3, x_7), (x_2, x_5, x_8), \ldots\ldots$ which is 1-in-3 satisfiable

solve system
$$\begin{aligned} x_1' + x_3' + x_7' &= 1 \\ x_2' + x_5' + x_8' &= 1 \\ &\ldots \end{aligned}$$
in $\mathbb{Q} \setminus \{1/3\}$ (solvable)

define $x_i = \begin{cases} 0 & \text{if } x_i' < \frac{1}{3} \\ 1 & \text{if } x_i' > \frac{1}{3} \end{cases}$

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the same in terms of homomorphisms:

- $\mathbb{A} \longrightarrow \mathbb{C} \longrightarrow \mathbb{B}$ where $\mathbb{C} = (\mathbb{Q} \setminus \{1/3\}; \{(x,y,z); x+y+z=1\})$
- $\text{CSP}(\mathbb{C})$ in P
- $\text{CSP}(\mathbb{A}, \mathbb{B})$ reducible to $\text{CSP}(\mathbb{C})$ by the trivial reduction

Remark: such a $\mathbb{C}$ exists for all known PCSPs in P

FINITE TRACTABILITY

assume $P \neq NP$ here

Def. $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is <u>finitely tractable</u> if
$\exists \mathbb{C}$ finite such that $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ and $\mathrm{CSP}(\mathbb{C})$ in P

<u>Fact</u> $\iff \exists \mathbb{C}$ finite such that $\exists \mathrm{Pol}(\mathbb{C}) \xrightarrow{\text{minion homo}} \mathrm{Pol}(\mathbb{A}, \mathbb{B})$ and $\mathrm{CSP}(\mathbb{C})$ in P

<u>Fact</u> $\mathrm{Pol}(\mathbb{A}', \mathbb{B}') \xrightarrow{\text{minion homo}} \mathrm{Pol}(\mathbb{A}, \mathbb{B})$ and $\mathrm{PCSP}(\mathbb{A}', \mathbb{B}')$ is finitely tractable
then so is $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

→ finite tractability depends only on height one identities

→ if $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is finitely tractable, then $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ satisfies height one identities at least as strong as $\underbrace{}_{\mathrm{Pol}(\mathbb{C}) \text{ for}}$ some tractable finite domain $\mathrm{CSP}(\mathbb{C})$

Theorem [B'19, B. Bulín, Krokhin, Opršal '21+] PCSP(1-in-3, NAE-3) is not finitely tractable

Theorem [AB'21+] "Many other Boolean PCSPs are not finitely tractable"

NECESSARY CONDITION FOR FINITE TRACTABILITY

again $P \neq NP$

<u>Theorem</u> [B, Kozik'12] If $\mathbb{C}$ is finite and $CSP(\mathbb{C})$ in P, then

$\forall p$ prime $p > |C|$ $\exists c \in Pol(\mathbb{C})$ <u>cyclic operation</u> of arity p

$$c(x_1, x_2, \dots, x_p) \approx c(x_2, \dots, x_p, x_1)$$

→ if $PCSP(\mathbb{A}, \mathbb{B})$ is finitely tractable, then $Pol(\mathbb{A}, \mathbb{B})$ contains a p-ary cyclic operation for every sufficiently large prime p

- not good enough for $PCSP$(1-in-3, NAE-3):

  for each n not divisible by 3, the following operation is in Pol(1-in-3, NAE-3)

$$c(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } \sum x_i / n < \frac{1}{3} \\ 1 & \text{if } \sum x_i / n > \frac{1}{3} \end{cases}$$

BETTER NECESSARY CONDITION

Fact If $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is finitely tractable, then $\exists b$ $\forall$sufficiently large prime p $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ contains p^2-ary t which is

- doubly cyclic and
- b-bounded

$$t\underbrace{\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & & \\ \vdots & & \ddots & \\ x_{p1} & \cdots & \cdots & x_{pp} \end{pmatrix}}_{X} \approx t\underbrace{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}_{Y}$$

where Y is obtained from X by

- cyclically shifting variables in columns

and

- cyclically shifting columns

$\exists$ equivalence $\sim$ on the set of p-ary x/y-tuples such that with $\leq b$ blocks

$$t(\bar{x}_1, \dots, \bar{x}_p) \approx t(\bar{y}_1, \dots, \bar{y}_p)$$

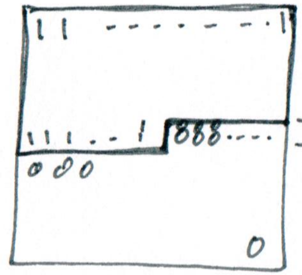
whenever $\bar{x}_1 \sim \bar{y}_1, \dots, \bar{x}_p \sim \bar{y}_p$

Proof: • enough to find such an operation in $\mathrm{Pol}(\mathbb{C})$ for $\mathbb{C}$ finite, $\mathrm{CSP}(\mathbb{C})$ in P

• define $t\begin{pmatrix} & \end{pmatrix} := c\left(\overset{c}{\vdots}, \overset{c}{\vdots}, \dots\right)$

PCSP(1-in-3, NAE-3)

- Assume it is finitely tractable
- Take b-bounded doubly cyclic polymorphism $t : \{0,1\}^{p \times p} \to \{0,1\}$ of sufficiently large arity p^2
- WLOG $\quad t\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & & \\ & & & 0 \end{bmatrix} = 0 \qquad t\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & \\ \vdots & & \ddots & \\ & & & 1 \end{bmatrix} = 1$
- $A \in \{0,1\}^{p \times p}$ is <u>almost rectangle</u> if $A =$



step size is small ($\le 5b$)

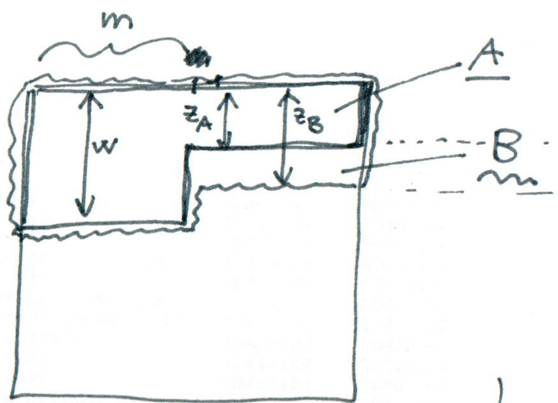
- will show that almost rectangles are <u>tame</u>:

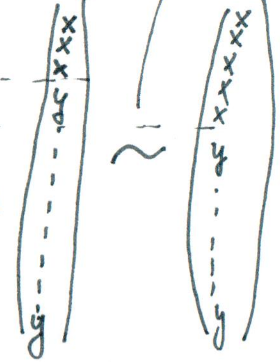
main part of the proof

$$t(A) = \begin{cases} 0 & \text{if } \sum a_{ij}/p^2 < \frac{1}{3} \\ 1 & \text{if } \text{—''—} > \frac{1}{3} \end{cases}$$

area of A

END OF PROOF

- Find A, B such that $\bullet_1$ 

, $\bullet_2$ 

,

  from def. of b-boundedness

  - $\bullet_3$ A, B almost rectangles
  - $\bullet_4$ $\text{area}(A) < 1/3$
  - $\bullet_5$ $\text{area}(B) > 1/3$

- tame $\Rightarrow t(A) = 0,\ t(B) = 1$
- b-bounded $\Rightarrow t(A) = t(B)$

$\Big\}\ \lightning$

How to find such:

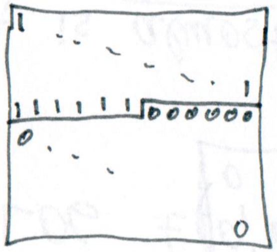
- $m = P/2 -$
- $z_A < z_B < P/3$ so that $\bullet_2$ (pigeonhole principle) & close to $P/3$
- max w so that $\bullet_4$ $(\text{area}(A) < 1/3)$, then $\bullet_5$ $(\text{area}(B) > 1/3)$
- P is large, z_A, z_B close to $P/3$ $\Rightarrow$ $\bullet_3$ (A, B almost rectangles)

  m close to $P/2$

TAMENESS – STEP SIZE 1

- $s(x_1, x_2, \ldots\ldots, x_{p^2}) := t \begin{bmatrix} x_1 & x_2 & \cdots & x_p \\ x_{p+1} & x_{p+2} & \cdots & x_{2p} \\ & & \ddots & \\ & & & x_{p^2} \end{bmatrix}$

- s is cyclic (from double cyclicity of t)

- t  step size 1 $= s(\underbrace{1\,1 \ldots\ldots 1}_{i}\,0\,0 \ldots\ldots 0)$

  $=: s\langle i \rangle$

- enough to show

$$s\langle i \rangle = \begin{cases} 0 & \text{if } i/p^2 < 1/3 \\ 1 & \text{if } i/p^2 > 1/3 \end{cases}$$

STEP SIZE 1 CNTD.

(10)

proof in case $p = 5$ ... $p^2 = 25$

polymorphism of (1-in-3, NAE-3)

- $$\left(\begin{matrix} \overbrace{1\,1\,1\,1\,1\,1\,1\,1}^{8} & \overbrace{0\,0\,0\,0\,0\,0\,0\,0}^{8} & \overbrace{0\,0\,0\,0\,0\,0\,0\,0\,0}^{9} \\ 0\,0\,0\,0\,0\,0\,0\,0 & 1\,1\,1\,1\,1\,1\,1\,1 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0 & 1\,1\,1\,1\,1\,1\,1\,1\,1 \end{matrix}\right) \begin{matrix} \xmapsto{s} s\langle 8 \rangle \\ \xmapsto{s} s\langle 8 \rangle \\ \xmapsto{s} s\langle 9 \rangle \end{matrix} \Bigg\} \text{ from cyclicity}$$

  $(s(8), s(8), s(9)) \in NAE\text{-}3 \implies s(8) \neq s(9)$

- similarly $(7+9+9=25)$ $\quad s(7) \neq s(9) \implies s(7) = s(8) \neq s(9)$
- similarly $(7+8+10=25)$ $\quad s(7) = s(8) \neq s(9) = s(10)$

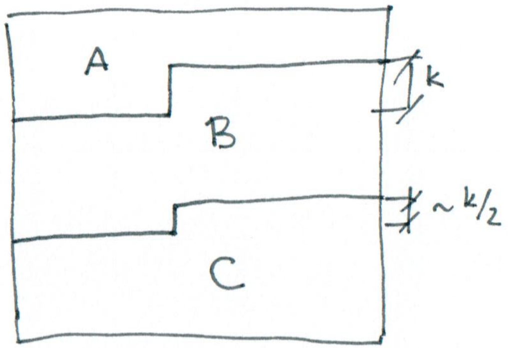
$\vdots$

$$\begin{matrix} s(0) = s(1) = \ldots = s(8) \neq s(9) = s(10) = \ldots = s(25) \\ \shortparallel \\ 0 \end{matrix}$$

STEP SIZE > 1

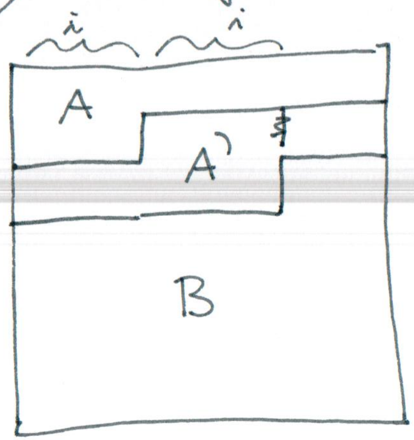
- by induction on step size
- take almost rectangle A
- Case 1: area of A is not too close to $1/3$ (say $\ll 1/3$)

areas of 1's are drawn for A, B, C



- double cyclicity $\Rightarrow t(B)=t(B'),\ t(C)=t(C')$ for almost rectangles B', C'
- step size of $B', C' < k$
- not too close to $1/3 \Rightarrow$ area of $B', C' > 1/3$
- t polymorphism $\Rightarrow (t(A), t(B), t(C)) \in NAE\text{-}3$ by induction

- Case 2 area of A is close to $1/3$



- chosen so that $t(A)=t(A')$ by double cyclicity
- t polymorphism $\Rightarrow t(B) \neq t(A)$
- $|\text{area}(B) - 1/3| > |\text{area}(A) - 1/3|$
- repeat if necessary and use case 1

□

DIRECTIONS

- other templates, e.g. which of the following are finitely tractable

  $\mathbb{A} = (\{0,1\}; \neq, R)$ $\quad R \subseteq \leq r\text{-in-}s$

  $\mathbb{B} = (\{0,1\}; \neq, S)$ $\quad S \supseteq \leq(2r-1)\text{-in-}s$

- say $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$, $CSP(\mathbb{C})$ in P

  How "finite" can $\mathbb{C}$ be

  - for $|A| = |B| = 2$ sometimes finite $\mathbb{C}$ exists

    but necessarily $|C| \geq 3$ [Deng, El Sai, Manders, Mayr, Nakkirt, Sparks '20+]

    $|C| \geq$ any fixed number [Kazda + ? Mayr, Zhuk]

  - when infinite $\mathbb{C}$ necessary, can it be e.g. ω-categorical ?

Thank you!