Promise Constraint Satisfaction Problem

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3-coloring problem for graphs

Input: graph

Output: yes if it's 3-colorable; no otherwise

3SAT

Input: eg. $(x \vee y \vee \neg z) \wedge (\neg y \vee \neg w \vee u) \wedge \ldots$

Output: yes if it's satisfiable

Linear equations

Input: eg. $2x + 3y + z = 2,\ 3y + 2w + 37z = 1, \ldots$

Output: yes if it has a solution

What is the computational complexity?

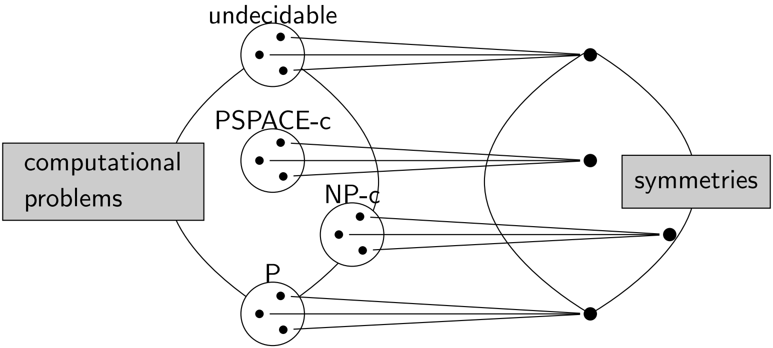
ie. How fast does the best algorithm run?

eg. **Can 3SAT or 3-coloring be solved in polynomial time?**

answer = \$1,000,000

CoolF: computational problems ⟶ objects capturing symmetry

CoolF(X) = CoolF(Y) iff X and Y have the same complexity



(P)CSPs over fixed finite templates

- tiny portion of problems on the left
- CoolF'(X) = CoolF'(Y) $\Rightarrow$ X, Y have the same complexity

Constraint Satisfaction Problems (CSPs) over finite templates

- class of computational problems
- goal: determine the computational complexity
- symmetry determines the complexity + improvements
- goal scored (two complexity classes: P, NP-complete)

Promise Constraint Satisfaction Problems (PCSPs)

- larger class of computational problems, goal not scored
- richer on both algorithmic and hardness side
  - algorithms need to be infinitary
  - hardness requires heavy tools
- further improvement to the basics

Barto, Bulín, Krokhin, Opršal: Algebraic approach to promise constraint satisfaction

CSP – definition and examples

Fix $\mathbb{A} = (A; R, S, \dots)$ relational structure, A finite

Definition ($\mathrm{CSP}(\mathbb{A})$)

Input: pp-sentence ϕ, eg. $(\exists x_1 \exists x_2 \dots) R(x_1, x_3) \wedge S(x_5, x_2) \wedge \dots$
Answer Yes: ϕ satisfied in $\mathbb{A}$
Answer No: ϕ not satisfied in $\mathbb{A}$

Search version: Find a satisfying assignment.
Search looks harder, but it's not [Bulatov, Jeavons, Krokhin'05]

Fact: Always in NP.

Example 1: 3-coloring

$\mathbb{K}_3 = (A; R)$ where

- $A = \{\mathit{lilac}, \mathit{mauve}, \mathit{cyclamen}\}$
- R = (binary) inequality relation on A

Input of $\mathrm{CSP}(\mathbb{K}_3)$ is, e.g.

$(\exists x_1 \exists x_2 \ldots \exists x_4) R(x_1, x_2) \wedge R(x_1, x_3) \wedge R(x_1, x_4) \wedge R(x_2, x_3) \wedge R(x_2, x_4)$

Viewpoint

- variables = vertices
- clauses (constraints) = edges

$\mathrm{CSP}(\mathbb{K}_3)$ is the 3-coloring problem for graphs

Fact: It is NP-hard (7-coloring NP-hard, 2-coloring in P)

- $3\mathbb{NAE}_2 = (\{0,1\}; 3NAE_2)$ where
  $3NAE_2 =$ all but $\{(0,0,0),(1,1,1)\}$
  $\mathrm{CSP}(3\mathbb{NAE}_2) =$ positive not-all-equal 3-SAT
  $=$ 2-coloring problem for 3-uniform hypergraphs
- $3\mathbb{NAE}_4 = (\{0,1,2,3\}; 3NAE_4)$, where $3NAE_4$ still ternary
  $\mathrm{CSP}(3\mathbb{NAE}_4) =$ 4-coloring problem for 3-uniform hypergraphs
- $1\mathbb{IN}3 = (\{0,1\}; 1in3)$ where
  $1in3 = \{(0,0,1),(0,1,0),(1,0,0)\}$
  $\mathrm{CSP}(1\mathbb{IN}3) =$ positive 1-in-3 SAT

Fact: All NP-hard

$3\mathbb{LIN}_5 = (\mathbb{Z}_5; L_{0000}, L_{0001}, \ldots, L_{4444})$ where e.g.

$L_{1234} = \{(x, y, z) \in \mathbb{Z}_5^3 : 1x + 2y + 3z = 4\}$

(note: relations are affine subspaces of $\mathbb{Z}_5^3$)

$\mathrm{CSP}(3\mathbb{LIN}_5)$ = solving systems of linear equations in $\mathbb{Z}_5$

Fact: In P

CSP and symmetry

polymorphism of $\mathbb{A}$: mapping $f : A^n \to A$
compatible with every relation

compatible with R: f applied component-wise to tuples in R
is a tuple in R

Example: $f(x_1, \ldots, x_4) = 2x_1 + 3x_2 + 3x_3 + 3x_4$ $\quad f : \mathbb{Z}_5^4 \to \mathbb{Z}_5$
is compatible with each L_{abcd}
because $f(\mathbf{v}_1, \ldots, \mathbf{v}_4)$ is an affine combination of these vectors (as $2 + 3 + 3 + 3 = 1$)
and L_{abcd} is an affine subspace

Pol($\mathbb{A}$): the set of all polymorphisms (it is a "clone")
$=$ set of **multivariable** symmetries of $\mathbb{A}$

Jeavons'98: On the algebraic structure of combinatorial problems

Theorem

Complexity of $\mathrm{CSP}(\mathbb{A})$ *is determined by* Pol($\mathbb{A}$)*:*

If Pol($\mathbb{A}$) $\subseteq$ Pol($\mathbb{B}$) *then* $\mathrm{CSP}(\mathbb{B})$ *reduces to* $\mathrm{CSP}(\mathbb{A})$*.*

So: $\mathrm{CSP}(3\mathbb{LIN}_5)$ is in P because $3\mathbb{LIN}_5$ has a lot of polymorphs
$\mathrm{CSP}(1\mathbb{IN}3)$ is NP-complete because $1\mathbb{IN}3$ has few

System of functional equations is, e.g.

$$f(g(x,y),z) = g(x,h(y,z))$$
$$m(y,x,x) = m(y,y,y)$$
$$m(x,x,y) = m(y,y,y)$$

Satisfied in $\mathcal{M}$, where $\mathcal{M}$ is a set of functions:
symbols can be interpreted in $\mathcal{M}$ so that
each equality is (universally) satisfied

Example: The above system is satisfied in $\mathrm{Pol}(3\mathbb{LIN}_5)$:

- take $f(x,y) = g(x,y) = h(x,y) = x$
  (note: projections are always polymoprhisms)
- take $m(x,y,z) = x - y + z$

Bulatov, Jeavons, Krokhin'05: Classifying the complexity of constraints using finite algebras + Bodirsky'08: PhD thesis

Theorem

Complexity of $\mathrm{CSP}(\mathbb{A})$ *is determined by systems of functional equations satisfied in* $\mathrm{Pol}(\mathbb{A})$*:*

So: $\mathrm{CSP}(3\mathbb{LIN}_5)$ is in P because $\mathrm{Pol}(3\mathbb{LIN}_5)$ satisfies strong systems of functional equations.

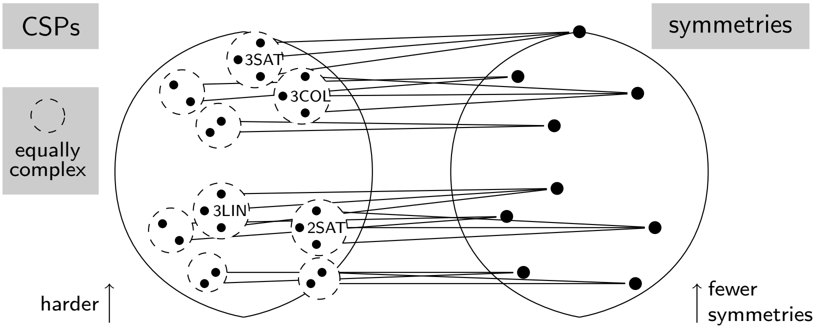
Barto, Opršal, Pinsker'18: The wonderland of reflections

minor condition = system of functional equations, each of the form
symbol(*variables*) = *symbol*(*variables*),
e.g. $m(y, x, x) = m(y, y, y)$, $m(x, x, y) = m(y, y, y)$

Theorem

Complexity of $\mathrm{CSP}(\mathbb{A})$ *determined by*
minor conditions satisfied in $\mathrm{Pol}(\mathbb{A})$*:*

So: $\mathrm{CSP}(\mathbb{EQ}_5)$ is in P because it satisfies strong minor conditions.

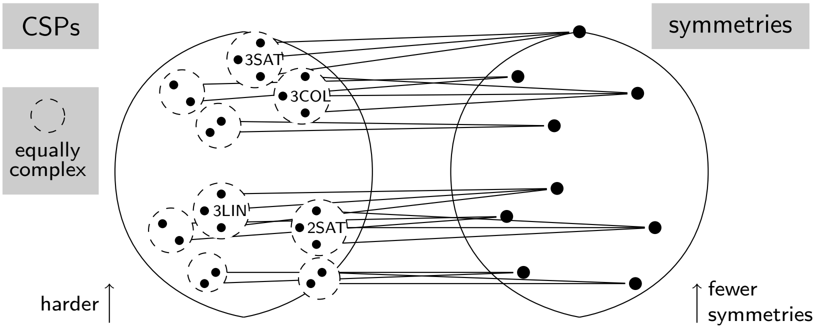


(1) polymorphisms

(2) systems of functional equations satisfied by polymorphisms

(3) minor conditions satisfied by polymorphisms

Where are the borderlines between complexity classes?



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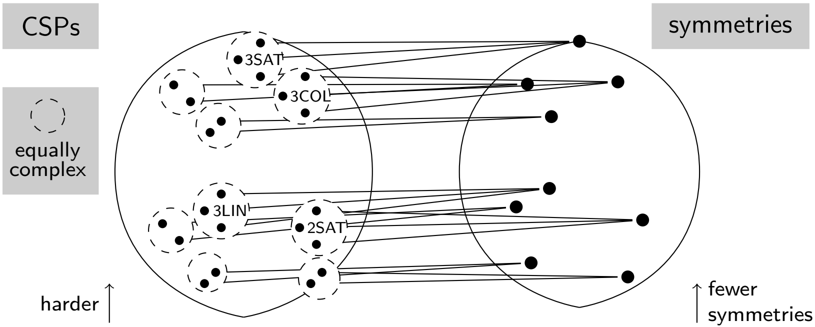

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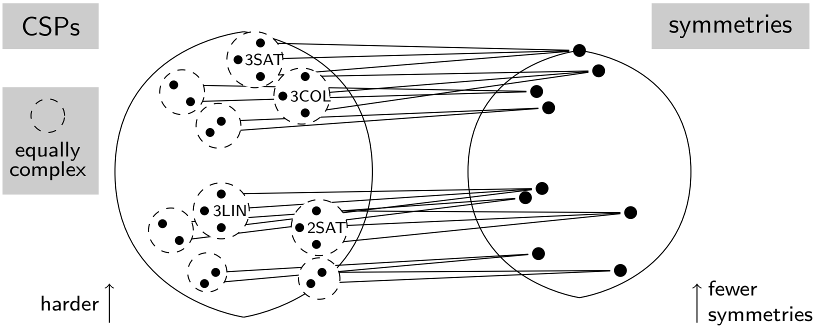

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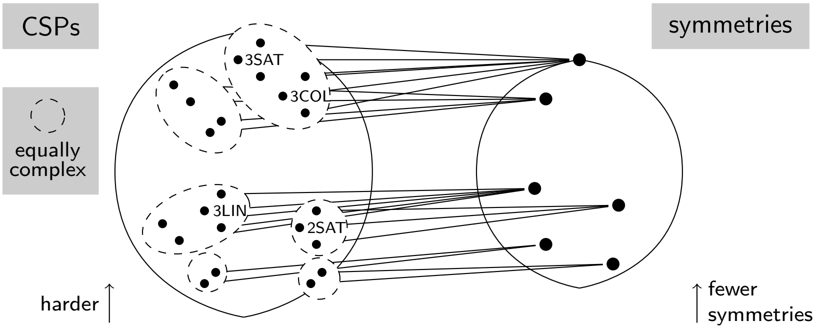
The Three Steps (movie)



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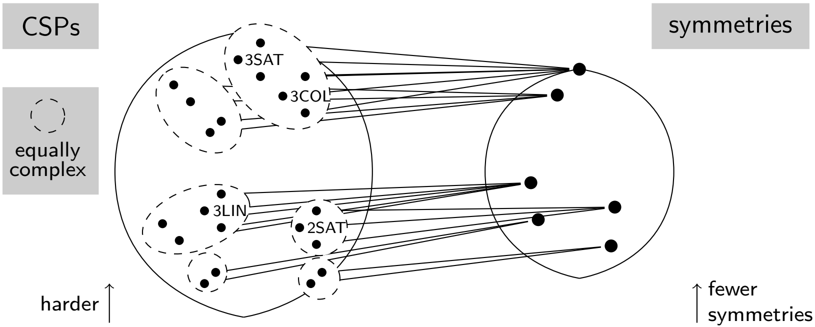

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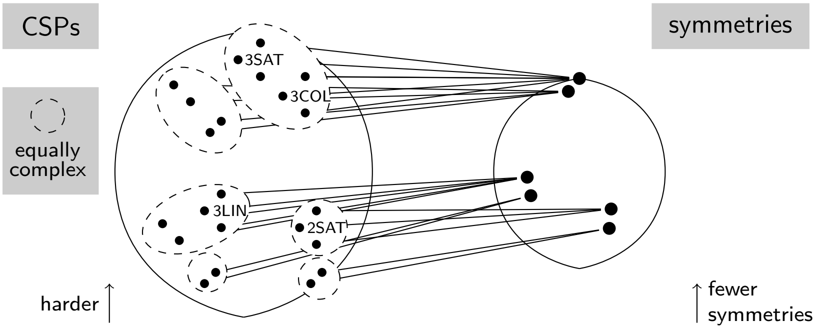

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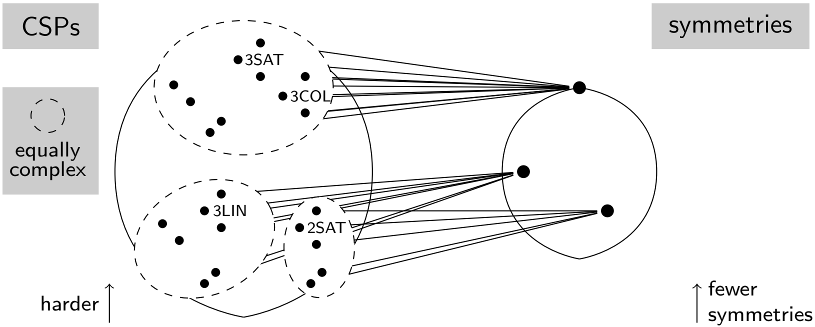


(1) polymorphisms

(2) systems of functional equations satisfied by polymorphisms

(3) minor conditions satisfied by polymorphisms

Where are the borderlines between complexity classes?

Minor condition is trivial:

satisfied in every $\mathrm{Pol}(\mathbb{A})$

$=$ satisfied by some projections

Corollary

If $\mathrm{Pol}(\mathbb{A})$ *satisfies only trivial minor conditions,*
then $\mathrm{CSP}(\mathbb{A})$ *is NP-hard.*

Conjecture ([Bulatov, Jeavons, Krokhin'05])

If $\mathrm{Pol}(\mathbb{A})$ *satisfies some non-trivial minor condition,*
then $\mathrm{CSP}(\mathbb{A})$ *is in P.*

Theorem

Let $\mathcal{M} = \mathrm{Pol}(\mathbb{A})$. *The following are equivalent.*

- $\mathcal{M}$ *satisfies some nontrivial minor condition*
- *There is no mapping* $\xi : \mathcal{M} \to \mathbb{N}$
  - *if* f *is of arity* n*, then* $\xi(f) \in \{1, 2, \ldots, n\}$
    *(**think**: an important coordinate of* f*)*
  - ξ *behaves nicely with minors*
- $\mathcal{M}$ *satisfies, for some* $n \geq 2$*, the minor condition*

$$c(x_1, x_2, \ldots, x_n) = c(x_2, \ldots, x_n, x_1)$$

[Barto, Kozik'12]

- $\ldots$
- $\ldots$ *zillion other characterizations* $\ldots$
- $\ldots$

Results

Characterizations of the conjectured borderline

- classic Universal Algebra [Taylor'77], [Hobby, McKenzie'88]
- numerous new [Maróti, McKenzie'08],[Siggers'10], [BK'12], ...

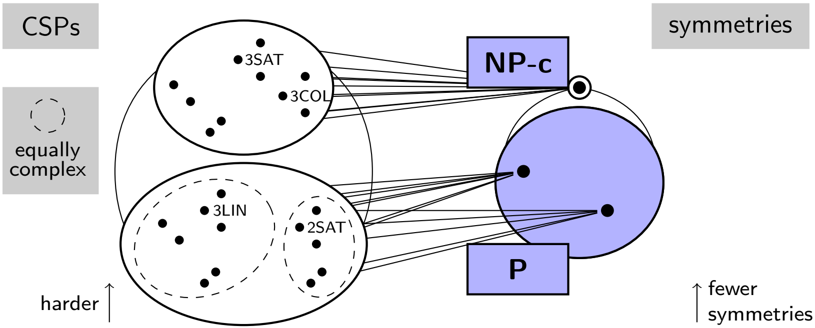
Applicability of algorithms

- describing all solutions
  [Idziak, Marković, McKenzie, Valeriote, Willard'07]
- constraint propagation algorithms [Barto, Kozik'09], [Bulatov]

Conjecture for special classes

- 2-element domain [Schaefer'78]
- graphs [Hell, Nešetřil'90]
- 3-element domain [Bulatov'06]
- conservative structures [Bulatov'03 '16], [Barto'11]
- digraphs without sources or sinks [Barto,Kozik,Niven'09]

Conjecture confirmed [Bulatov'17], [Zhuk'17]



- **only trivial minor conditions $\Rightarrow$ NP-complete**
  single and simple reason for hardness
- **some nontrivial minor condition $\Rightarrow$ P**
  $+$ concrete minor conditions

Classifications in variants of CSP

- optimization [Kolmogorov, Krokhin, Rolínek'15]
- counting [Bulatov'08], [Dyer, Richerby'10]
- robust satisfiability [Barto,Kozik'12]

What next?

- infinite domains
- approximation
- PCSP

PCSP

$\mathrm{CSP}(\mathbb{A})$ is often NP-complete

What can we do?

1. **Approximation:** Try to satisfy only some fraction of the constraints, eg.

   for a satisfiable 3SAT instance,
   find an assignment satisfying at least 90% of the clauses
   Theorem: NP-hard [Håstad'01]

2. **PCSP:** Try to satisfy a relaxed version of all constraints, eg.

   for a 3-colorable graph,
   find a 37-coloring

Fix 2 relational structures in the same language

- $\mathbb{A} = (A; R^{\mathbb{A}}, S^{\mathbb{A}}, \dots)$
- $\mathbb{B} = (B; R^{\mathbb{B}}, S^{\mathbb{B}}, \dots)$
- there is a homomorphism $\mathbb{A} \to \mathbb{B}$ (eg. $A \subseteq B, R^{\mathbb{A}} \subseteq R^{\mathbb{B}}, \dots$)

Definition ($\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$)

Input: pp-sentence ϕ, eg. $(\exists x_1 \exists x_2 \dots) R(x_1, x_3) \wedge S(x_5, x_2) \wedge \dots$
Answer Yes: ϕ satisfied in $\mathbb{A}$
Answer No: ϕ not satisfied in $\mathbb{B}$

Search version: Find a $\mathbb{B}$-satisfying assignment
given an $\mathbb{A}$-satisfiable input.
(it may be a harder problem, we don't know)

Recall: $\mathbb{K}_n = (\{1, 2, \ldots, n\}; \text{inequality})$

$\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_4)$ search version

Input: a graph
Promise: it is 3-colorable
Task: find a 4-coloring

Fun facts:

- **Theorem:** it is NP-hard [Brakensiek, Guruswami'16]
  (more generally $\mathrm{PCSP}(\mathbb{K}_n, \mathbb{K}_{2n-2})$ is NP-hard)
- $\mathrm{PCSP}(\mathbb{K}_n, \mathbb{K}_{2n-1})$ [Bulín, Krokhin, Opršal'19]
- $\mathrm{PCSP}(\mathbb{K}_n, \mathbb{K}_{\binom{n}{\lfloor n/2 \rfloor}-1})$, $n \geq 4$ [Wrochna, Živný]
- 6-coloring 3-colorable graph: complexity not known
- **Conjecture**: k-coloring l-colorable graph NP-hard ($k \geq l \geq 3$)

Recall: $3\mathbb{NAE}_k$ ternary not-all-equal relation on a k-element set

$\mathrm{PCSP}(3\mathbb{NAE}_2, 3\mathbb{NAE}_{137})$ search version

Input: a 3-uniform hypergraph
Promise: it is 2-colorable
Task: find a 137-coloring

Theorem: It is NP-hard [Dinur,Regev,Smyth'05]
(more generally $\mathrm{PCSP}(3\mathbb{NAE}_l, 3\mathbb{NAE}_k)$ NP-hard for every $k \geq l \geq 2$)

Proof uses

- the PCP theorem [Arora, Lund, Motwani, Sudan, Szegedy'98]
- $+$ the Parallel Repetition Theorem [Raz'98]
- Lovász's theorem on Kneser's graphs [Lovász'78]

Recall: $1\mathbb{IN}3 = (\{0,1\}; \{(1,0,0),(0,1,0),(0,0,1)\})$

$\mathrm{PCSP}(1\mathbb{IN}3, 3\mathbb{NAE}_2)$ search version

Input: a 3-uniform hypergraph
Promise: there is a 2-coloring such that
exactly one vertex in each hyperedge receives 1
Task: find a 2-coloring

Fact: It is in P. Algorithm:

- for each hyperedge $\{x, y, z\}$ write $x + y + z = 1$
- solve the system over $\mathbb{Q} \setminus \{\frac{1}{3}\}$ (it is solvable in $\{0, 1\}$)
- assign $x \mapsto 1$ iff $x > 1/3$

Note: algorithm uses infinite domain CSP
Theorem: infinity is necessary

Barto'19: Promises make finite problems infinitary

PCSP and symmetry

polymorphism of $(\mathbb{A}, \mathbb{B})$: mapping $f : A^n \to B$
compatible with every relation-pair

compatible with $(R^{\mathbb{A}}, R^{\mathbb{B}})$: f applied to tuples in $R^{\mathbb{A}}$
is a tuple in $R^{\mathbb{B}}$

Example: $f(x_1, \dots, x_{97}) = 1$ iff $\frac{\sum x_i}{97} > \frac{1}{3}$ $\quad f : \{0,1\}^{97} \to \{0,1\}$
is compatible with $(1in3, 3NAE_2)$

$\mathrm{Pol}(\mathbb{A}, \mathbb{B})$: the set of all polymorphisms (it is a "minion")
$=$ set of multivariable symmetries of $(\mathbb{A}, \mathbb{B})$

Algebraic theory, 3 steps

1st step (polymorphisms):
can be generalized [Brakensiek, Guruswami'18]
using [Pippenger'02]

2nd step (systems of functional equations):
makes no sense
since polymorphisms can no longer be composed

3rd step (minor conditions): the same as in CSP!

Definition ($\mathrm{MinorCond}(N, \mathcal{M})$)

Input: minor condition **X** with symbols of arity N
Answer Yes: **X** is trivial (=satisfied in $\mathcal{P}$)
Answer No: **X** not satisfied in $\mathcal{M}$

Theorem ([Bulín, Krokhin, Opršal'19])

Let $\mathcal{M} = \mathrm{Pol}(\mathbb{A}, \mathbb{B})$. *The following computational problems are equivalent for a large enough* N.

(i) $\mathrm{CSP}(\mathbb{A})$
(ii) $\mathrm{MinorCond}(N, \mathcal{M})$

Consequence: 3rd step
Proof: direct, simple, known

No conjectured borderlines

Algorithms

- applicability of some algorithms understood [BBKO]
- new algorithms [Brakensiek,Guruswami]

Classification for special classes

- 2-element domain – far from finished
  known for symmetric relations
  partially [Brakensiek,Guruswami'18]
  fully [Ficak,Kozik,Olšák, Stankiewicz'19]
- graphs – major open problem
  partial results

Theorem (BBKO)

Let $\mathcal{M} = \mathrm{Pol}(\mathbb{A}, \mathbb{B})$. *If there exists* $C \in \mathbb{N}$ *and a mapping* $\xi : \mathcal{M} \to P(\mathbb{N})$ *such that*

- *if* f *is of arity* n, *then* $\xi(f) \subseteq \{1, 2, \ldots, n\}$, $|\xi(f)| \leq C$ *(**think**: a small set of important coordinates of f)*
- ξ *behaves nicely with minors*

Then $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ *is NP-complete.*

This criterion (more precisely, a slightly stronger one) is good enough for all known cases...

Summary

Summary

CSP

- = problem about minor conditions
- Complexity captured by a piece of information about polymorphisms
- Single, simple reason for hardness

PCSP is cool and fun

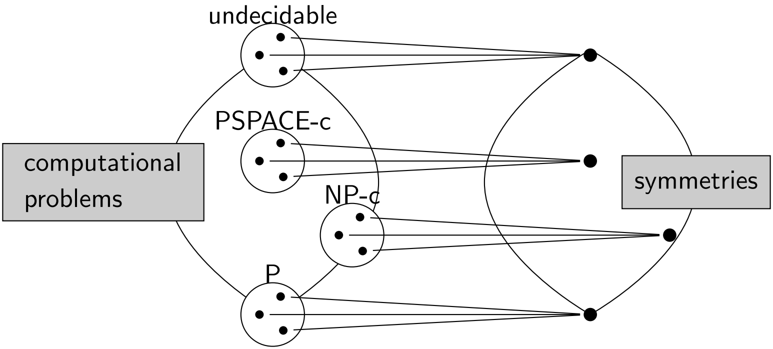
- Basics work but a lot is open: eg. borderlines, special cases
- More algorithms needed
- More interesting hardness proofs (PCP, topology)
- Bridge between CSP and approximation

Message to TCS

- Concrete problems $\to$ classes of problems
- Unary symmetries $\to$ multivariate symmetries
- Analysis $\to$ geometry

$\mathrm{CoolFunc}$: computational problems $\longrightarrow$ objects capturing symmetry

kernel of $\mathrm{CoolFunc}$ $=$ polynomial time reducibility



Thank you for your patience!