Sensitive instances of the Constraint Satisfaction Problem

Libor Barto
joint work with Marcin Kozik, Johnson Tan, Matt Valeriote

Department of Algebra, Charles University, Prague
Department of Theoretical Computer Science, Jagiellonian University, Kraków
Department of Mathematics, University of Illinois, Urbana
Department of Mathematics and Statistics, McMaster University, Hamilton

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Constraint Satisfaction Problem (CSP)

Is it possible to assign domain elements to variables so that given local constraints are satisfied?

Strategy: $(k, k+1)$**-consistency algorithm**

Derive the strongest possible constraint on each set of k variables by considering $k+1$ variables at a time

How good is the algorithm?

"so so" no contradiction found $\Rightarrow$ solution exists

"great" every partial solution on $\geq k$ variables extends to a solution

"**good**" every partial solution on k variables extends to a solution
$=$ every sharpening of a constraint invalidates some solution
sensitivity

Instance of the CSP is a list of constraints $R(\mathbf{x})$

- $\mathbf{x}$ is a list of variables, called the scope
- R is a relation on a fixed domain A of appropriate arity

Example: $R(x_1, x_2), S(x_2, x_4, x_2), R(x_3, x_4)$, where
$R \subseteq \{0, 1, 2\}^2, S \subseteq \{0, 1, 2\}^3$

Solution: mapping variables $\to$ domain
that satisfies every constraint

Partial solution: partial mapping variables $\to$ domain
that satisfies every constraint with fully evaluated scope

Sensitive instance: every sharpening of a constraint
invalidates some solution

Fix: $k \geq 1$

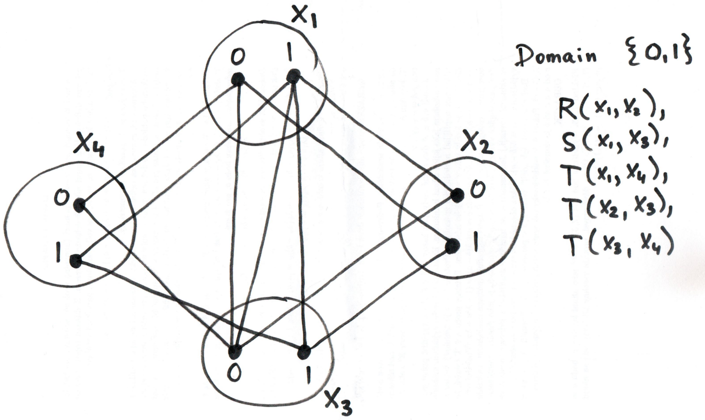
Assume: all constraint relations have arity $\leq k$

$(k, k+1)$-consistency algorithm produces a $(k, k+1)$-instance

- every k-element set of variables is constraint by a single constraint (and there are no other constraints)
- each partial solution on k variables can be extended to any additional variable

and

- the algorithm is polynomial
- the $(k, k+1)$-instance has the same solution set as the original one


x1
0 1
x4
0
1
x2
0
1
0 1
x3
Domain {0,1}
R(x1,x2),
S(x1,x3),
T(x1,x4),
T(x2,x3),
T(x3,x4)

Template is

- relational structure $\mathbb{A} = (A; R_1, R_2, \ldots)$, each $R_i \subseteq A^{k_i}$
- or algebra $\mathbf{A} = (A; f_1, f_2, \ldots)$, each $f_i : A^{k_i} \to A$

CSP over $\mathbb{A}$: constraint relations are from $\mathbb{A}$

Examples: 3-SAT, 3-LIN$_p$, HORN-3-SAT, 2-SAT

CSP over $\mathbf{A}$: constraint rel's are compatible with operations in $\mathbf{A}$

Examples:

CSP over $(\{0,1\}; (x,y,z) \mapsto x + y + z \pmod 2)$ is $\sim$ LIN$_2$
CSP over $(\{0,1\}; (x,y) \mapsto \min(x,y))$ is $\sim$ HORN-SAT
CSP over $(\{0,1\}; (x,y,z) \mapsto$ majority of $x,y,z)$ is $\sim$ 2-SAT

Main result

Operation $t : A^m \to A$ is

- idempotent if $(\forall a \in A)\ t(a, a, \dots, a) = a$
- near unanimity of arity m, or NU(m) if $(\forall a, b \in A)$
  $t(b, a, \dots, a) = t(a, b, a, \dots, a) = \dots = t(a, \dots, a, b)$

Theorem ([BKTV])

Let $k \geq 2$ and $\mathbf{A}$ a finite idempotent algebra. TFAE

(i) *$\mathbf{A}$ has an NU($k+2$) term operation.*

(ii) *Every $(k, k+1)$-instance of CSP over $\mathbf{A}^2$ is sensitive.*

- idempotency and square in $\mathbf{A}^2$ necessary for (ii) $\Rightarrow$ (i)
- not necessary for (i) $\Rightarrow$ (ii)
- more general version for infinite idempotent algebras

How good is the $(k, k+1)$-consistency algorithm?

Consider $k \geq 2$, $\mathbb{A}$ a finite structure with relations of arity $\leq k$

If $\mathbb{A}$ has a compatible NU(m) (for some m), the alg. is "so so"

for any instance of CSP over $\mathbb{A}$
if the associated $(k, k+1)$-instance is non-trivial,
then there exists a solution [B., Kozik'09, B.'16]

If $\mathbb{A}$ has a compatible NU($k+1$), then the algorithm is "great"

for any instance of CSP over $\mathbb{A}$
in the associated $(k, k+1)$-instance
every partial solution on $\geq k$ variables extends to a solution

[Bergman'77, Feder,Vardi'99]

If $\mathbb{A}$ has a compatible NU($k+2$), then the algorithm is "good"

for any instance of CSP over $\mathbb{A}$
the associated $(k, k+1)$-instance is sensitive [BKTV]

Note: NU(3) $\Rightarrow$ NU(4) $\Rightarrow$ NU(5) $\Rightarrow \ldots$

- 3-LIN_p tractable, but not "so so" for any k
- HORN-3-SAT is "so so" but not "good" (for any k)
- 2-SAT is "great" ($k \geq 2$)
- the following structure is "good" but not "great" for $k = 2$

  $\mathbb{A} = (\{0,1\}^2; R_1, R_2, R_3)$, where $((a,b),(c,d)) \in R_i$ iff

  (i=1) $a + b + c + d \geq 2$
  (i=2) $a = c$
  (i=3) $a = d$

Theorem ([BKTV])

Let $k \geq 2$ and $\mathbf{A}$ a finite idempotent algebra. TFAE

(i) *$\mathbf{A}$ has an NU($k+2$) term operation.*

(ii) *Every $(k, k+1)$-instance of $\mathrm{CSP}(\mathbf{A}^2)$ is sensitive.*

(ii) $\Rightarrow$ (i):

- careful choices of $(k, k+1)$-instances give "very local" $\mathrm{NU}(k+2)$'s
- $\mathrm{NU}(k+2)$ can be assembled from these [Horowitz'13]

(i) $\Rightarrow$ (ii): we apply a new loop lemma, improvement of [Olšák'17]

Theorem ([BKTV]): If $S \subseteq A^2$ contains a directed closed walk and absorbs all the loops, then S has a loop.

For $\mathbb{A}$ with ≤ 2-ary relations compatible with NU($k+2$), $k \geq 2$

"so so" after enforcing $(2,3)$-consistency, no contradiction found $\Rightarrow$ solution

"**good**" after enforcing $(k, k+1)$-consistency, every partial solution on k variables extends to a solution

"great" after enforcing $(k+1, k+2)$-consistency, every partial solution on $\geq k$ variables extends to a solution

Questions:

- gap between "so so" and "good" – $\exists$ natural conditions in between?
- "so so" and "great" (holding for every instance) can be characterized by compatible operations, what about "good"?
- "so so" and "great" have natural versions for higher arity relations, is there such for "good"?
- characterization of "great" has a generalization to a class of infinite domain structures (by means of oligopotent quasi-NUs), is it possible to generalize our result to oligopotent quasi-NUs?