Promise Constraint Satisfaction

Libor Barto

Department of Algebra, Charles University, Prague

ATCAGC 2020, Bedřichov, 29 January

CoCoSym: Symmetry in Computational Complexity

This project has received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No 771005)

Outline

Task from the organizers: talk about recent developments in the complexity of CSPs

It will be of interest to participants even if graph covers will not show up at all

Recent developments in fixed-template CSPs:

- computational complexity fully classified
- PCSP: promise CSP
  - new insight: LabelCover is everywhere (in (P)CSP and variants)
  - algorithmically more interesting
  - more tools are useful: algebraic topology, analysis

CSP

Fix $\mathbb{A} = (A; R, S, \dots)$ finite relational structure, eg. digraph $(A; R)$

Definition ($\mathrm{CSP}(\mathbb{A})$)

Input: $\mathbb{X}$ of the same signature as $\mathbb{A}$
Answer Yes: $\mathbb{X} \to \mathbb{A}$ (homomorphism)
Answer No: $\mathbb{X} \not\to \mathbb{A}$

Definition (search version of $\mathrm{CSP}(\mathbb{A})$)

Input: $\mathbb{X}$ such that $\mathbb{X} \to \mathbb{A}$
Task: Find $\mathbb{X} \to \mathbb{A}$

Examples

- $\mathbb{A} = \mathbb{K}_3$: 3-coloring problem
- $\mathbb{A} = (Z_p;$ affine subspaces): solving linear equations in $\mathbb{Z}_p$
- $\mathbb{A} = (\{0, 1\}; \dots)$: 3-SAT, HORN-3-SAT, NAE-3-SAT, 1-in-3-SAT

Variants of fixed-template CSPs

Different questions:

- counting (solved [Bulatov'08] [Dyer,Richerby'10])
- optimization (solved [Thapper, Živný'13])
- approximation (part solved modulo UGC [Raghavendra'08])

Generalizations:

- valued CSP (solved [Kolmogorov,Krokhin,Rolínek'15])
- infinite domains
- PCSP

Restrictions:

- restricted inputs: planar, bounded-degree
- restricted homomorphisms: covers

CSP and symmetry

polymorphism of $\mathbb{A}$: homomorphism $f : \mathbb{A}^n \to \mathbb{A}$

Pol($\mathbb{A}$): the set of all polymorphisms (it is a "clone")
$= $ set of multivariable symmetries of $\mathbb{A}$

Example: $f(x_1, \ldots, x_4) = 2x_1 + 3x_2 + 3x_3 + 3x_4$ is a polymorphism of $(Z_5; \text{affine subspaces})$ because affine subspaces are closed under affine combinations (note $2 + 3 + 3 + 3+ = 1$)

Example: a projection $f(x_1, \ldots, x_n) = x_i$
is always a polymorphism

Example: $f(x_1, \ldots, x_n) = \alpha(x_i)$ for a bijection α
are the only polymorphisms of $\mathbb{K}_3$

Jeavons'98: On the algebraic structure of combinatorial problems

motiv.: Feder,Vardi'98:The Computational Structure of Monotone Monadic SNP...

Theorem

Complexity of $\mathrm{CSP}(\mathbb{A})$ *is determined by* $\mathrm{Pol}(\mathbb{A})$*:*

If $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$ *then* $\mathrm{CSP}(\mathbb{B})$ *reduces to* $\mathrm{CSP}(\mathbb{A})$.

Proof.

If $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$, then relations in $\mathbb{B}$ can be defined from relations in $\mathbb{A}$ by a pp-formula.

[Geiger'69, Bondarčuk, Kalužnin, Kotov, Romov'69]

This gives a computational reduction of $\mathrm{CSP}(\mathbb{B})$ to $\mathrm{CSP}(\mathbb{A})$. □

So: 3-coloring is NP-complete because $\mathbb{K}_3$ has few symmetries

System of functional equations is, e.g.

$$f(g(x,y),z) = g(x,h(y,z))$$
$$m(y,x,x) = m(y,y,y)$$
$$m(x,x,y) = m(y,y,y)$$

Satisfied in $\mathcal{M}$, where $\mathcal{M}$ is a set of functions:
symbols can be interpreted in $\mathcal{M}$ so that
each equality is (universally) satisfied

Example: The above system is satisfied in
Pol(Z_5; affine subspaces)

- take $f(x,y) = g(x,y) = h(x,y) = x$
- take $m(x,y,z) = x - y + z$

Bulatov, Jeavons, Krokhin'05: Classifying the complexity of constraints using finite algebras + Bodirsky'08: PhD thesis

Theorem

Complexity of $\mathrm{CSP}(\mathbb{A})$ *is determined by*
systems of functional equations satisfied in $\mathrm{Pol}(\mathbb{A})$*:*
If each system satisfied in $\mathrm{Pol}(\mathbb{A})$ *is satisfied in* $\mathrm{Pol}(\mathbb{B})$*,*
then $\mathrm{CSP}(\mathbb{B})$ *reduces to* $\mathrm{CSP}(\mathbb{A})$*.*

Proof.

Previous theorem, pp-definitions $\rightarrow$ pp-interpretations,
the HSP theorem [Birkhoff'35] □

So: solving linear equations over $\mathbb{Z}_5$ is in P because
their template satisfies strong systems of functional equations

Barto, Opršal, Pinsker'18: The wonderland of reflections

minor condition = system of functional equations, each of the form

symbol(*variables*) = *symbol*(*variables*),

e.g. $m(y,x,x) = m(y,y,y)$, $m(x,x,y) = m(y,y,y)$

Theorem

Complexity of $\mathrm{CSP}(\mathbb{A})$ *determined by*

minor conditions satisfied in $\mathrm{Pol}(\mathbb{A})$*:*

If each minor condition satisfied in $\mathrm{Pol}(\mathbb{A})$ *is satisfied in* $\mathrm{Pol}(\mathbb{B})$*,*

then $\mathrm{CSP}(\mathbb{B})$ *reduces to* $\mathrm{CSP}(\mathbb{A})$*.*

Proof.

pp-interpretation $\to$ pp-construction,
version of the HSP theorem. □

The classification result

Minor condition is trivial:

satisfied in every $\mathrm{Pol}(\mathbb{A})$

$=$ satisfied in $\mathcal{P}$, the set of projections on $\{0, 1\}$

Corollary

If $\mathrm{Pol}(\mathbb{A})$ *satisfies only trivial minor conditions, then* $\mathrm{CSP}(\mathbb{A})$ *is NP-hard.*

Theorem ([Bulatov'17], [Zhuk'17])

If $\mathrm{Pol}(\mathbb{A})$ *satisfies some non-trivial minor condition, then* $\mathrm{CSP}(\mathbb{A})$ *is in P.*

Proof.

Both complex

News: the 2 approaches are closer [Barto, Bulatov, Kozik, Zhuk] □

(Barto,) Bulín, Krokhin, Opršal: Algebraic approach to promise constraint satisfaction

Definition ($\mathrm{MinorCond}(N, \mathcal{M})$)

Input: minor condition **X** with symbols of arity N
Answer Yes: **X** is trivial (=satisfied in $\mathcal{P}$)
Answer No: **X** not satisfied in $\mathcal{M}$

Theorem

Let $\mathcal{M} = \mathrm{Pol}(\mathbb{A})$. *The following computational problems are equivalent for a large enough* N.

(i) $\mathrm{CSP}(\mathbb{A})$

(ii) $\mathrm{MinorCond}(N, \mathcal{M})$

Consequence: 3rd step
Proof: direct, simple, known
Note: No $\neq \neg$Yes

What can we do for NP-complete $\mathrm{CSP}(\mathbb{A})$?

1. Try to satisfy only some fraction of the constraints, eg.

   for a satisfiable 3SAT instance,
   find an assignment satisfying at least 90% of the clauses

2. Try to satisfy a relaxed version of all constraints, eg.

   for a 3-colorable graph,
   find a 37-coloring

Approximation and LabelCover

satisfying a fraction of constraints

Theorem (Håstad'01)

The following problem is NP-complete for every $\epsilon > 0$
Input: *3SAT instance, eg.* $(x_1 \vee \neg x_4 \vee x_3) \wedge (\neg x_2 \vee x_5 \vee \neg x_3) \wedge \ldots$
Answer Yes: *it is satisfiable*
Answer No: *no* $(7/8 + \epsilon)$*–fraction of clauses is satisfiable*

Corollary: It is NP-hard to satisfy 90% of clauses of a satisfiable 3SAT instance.

Proof.

Reduction from a version of the Label Cover problem
(reduction uses Fourier analysis of Boolean functions. □

$\mathrm{LaberCover}(N)$ is $\mathrm{CSP}(A; \langle \mathrm{Gr}_\phi \rangle_{\phi : A \to A})$ where $|A| = N$ and

$$\mathrm{Gr}_\phi = \{(a, \phi(a)) : a \in [N]\}$$

Definition ($\mathrm{GapLabelCover}(N, \epsilon)$)

Input: like $\mathrm{LaberCover}(N)$
Answer Yes: ϕ is satisfiable
Answer No: no ϵ–fraction of constraints is satisfiable

Theorem

For every $\epsilon > 0$ there exists N such that $\mathrm{GapLabelCover}(N, \epsilon)$ is NP–complete

Proof: The PCP theorem [Arora, Lund, Motwani, Sudan, Szegedy'98]
Parallel Repetition Thoerem [Raz'98]

The following two problems are the same!

- $\mathrm{MinorCond}(N, \mathcal{P})$ ie. deciding whether a given minor condition is trivial
- $\mathrm{LaberCover}(N)$ ie. deciding whether a given label cover input is satisfiable

Because:

- interpretation of f and g by projections making the following equation true
  $f(x_3, x_1, x_1, x_2, x_1) = g(x_1, x_2, x_3, x_4, x_5)$
- corresponds to a satisfying assignment of $\mathrm{Gr}_\phi(f, g)$ where
  $\phi : 1 \mapsto 3, \quad 2, 3, 5 \mapsto 1, \quad 4 \mapsto 2$
- under the correspondence
  $i \;\leftrightarrow\;$ projection onto the ith coordinate

Remark: often implicitely used ("long code")

Input: bipartite minor condition (symbols of arity N)

Answer Yes: it is trivial

Answer No:

$(\mathrm{GapLabelCover}(N, \epsilon))$ no ϵ–fraction of equations is trivial

$(\mathrm{MinorCond}(N, \mathcal{M}))$ not satisfied in $\mathcal{M}$

- 1st is crucial problem for hardness of approximation
- 2nd is equivalent to $CSP(\mathbb{A})$ if $\mathcal{M} = \mathrm{Pol}(\mathbb{A})$
- **Single source of hardness** (no ad-hoc reductions)
  1st with $\epsilon = 1$ ie. $\mathrm{LaberCover}(N)$
  trivially reduces to every NP-complete CSP

PCSP

satisfying a relaxed version of all constraints

Definition

Fix 2 finite relational structures $\mathbb{A} \to \mathbb{B}$

Definition ($\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$)

Input: $\mathbb{X}$
Answer Yes: $\mathbb{X} \to \mathbb{A}$
Answer No: $\mathbb{X} \not\to \mathbb{B}$

Definition (search version of $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$)

Input: $\mathbb{X}$ such that $\mathbb{X} \to \mathbb{A}$
Task: Find $\mathbb{X} \to \mathbb{B}$

(it may be a harder problem, we don't know)

Example: $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_4)$ is 4-coloring of a 3-colorable graph polymorphism of $(\mathbb{A}, \mathbb{B})$: homomorphism $\mathbb{A}^n \to \mathbb{B}$

$\mathrm{Pol}(\mathbb{A}, \mathbb{B})$: the set of all polymorphisms (it is a "minion")
= set of multivariable symmetries of $(\mathbb{A}, \mathbb{B})$

Theorem

Let $\mathcal{M} = \mathrm{Pol}(\mathbb{A}, \mathbb{B})$. The following computational problems are equivalent for a large enough N.

(i) $\mathrm{CSP}(\mathbb{A}, \mathbb{B})$

(ii) $\mathrm{MinorCond}(N, \mathcal{M})$

Shows that PCSP is in some sense more natural than CSP.

$\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_4)$

Input: 3-colorable graph

Task: find a 4-coloring

Conjectures

- $\mathrm{PCSP}(\mathbb{K}_k, \mathbb{K}_l)$ NP-hard ($l \geq k \geq 3$), **(3,6) open**
- Stronger: $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ NP-hard for any non-bipartite
- Enough to show: $\mathrm{PCSP}(\mathbb{C}_{\mathrm{odd}}, \mathbb{K}_k)$ NP-hard

Recent hardness results:

- $\mathrm{PCSP}(\mathbb{K}_n, \mathbb{K}_{2n-2})$ [Brakensiek, Guruswami'16]
- $\mathrm{PCSP}(\mathbb{K}_n, \mathbb{K}_{2n-1})$ [Bulín, Krokhin, Opršal'19]
- $\mathrm{PCSP}(\mathbb{K}_n, \mathbb{K}_{\binom{n}{\lfloor n/2 \rfloor}-1})$, $n \geq 4$ [Wrochna, Živný'20]
- $\mathrm{PCSP}(\mathbb{C}_{\mathrm{odd}}, \mathbb{K}_3)$ [Opršal, Krokhin'19]

$3\mathbb{NAE}_k$ ternary not-all-equal relation on a k-element set

$\mathrm{PCSP}(3\mathbb{NAE}_2, 3\mathbb{NAE}_{137})$

Input: a 3-uniform hypergraph
Answer Yes: it is 2-colorable
Answer No: it is not 137-colorable

Theorem: It is NP-hard [Dinur,Regev,Smyth'05]
(more generally $\mathrm{PCSP}(3\mathbb{NAE}_l, 3\mathbb{NAE}_k)$ NP-hard
for every $k \geq l \geq 2$)

PCSP(1-in-3-SAT,NAE-SAT) (combinatorial formulation):

Input: a 3-uniform hypergraph which has
a 2-coloring such that
exactly one vertex in each hyperedge receives 1

Task: find a 2-coloring

Fact: It is in P. Algorithm for finding a 2-coloring of:

- for each hyperedge $\{x, y, z\}$ write $x + y + z = 1$
- solve the system over $\mathbb{Q} \setminus \{\frac{1}{3}\}$ (it is solvable in $\{0, 1\}$)
- assign $x \mapsto 1$ iff $x > 1/3$

Note: algorithm uses infinite domain CSP
Theorem: infinity is necessary [Barto'19]

Shows that PCSPs are algorithmically more interesting

Hardness proofs

Label Cover and topology

- Denote $\mathcal{M} = \mathsf{Pol}(\mathbb{A},\mathbb{B})$
- **Strategy:** Find ϵ so that $\mathrm{GapLabelCover}(N,\epsilon) \leq \mathrm{MinorCond}(N,\mathcal{M})$ trivially

Input: minor condition **M** (symbols of arity N)
Answer Yes: it is trivial
Answer No:
$(\mathrm{GapLabelCover}(N,\epsilon))$ no ϵ–fraction of equations is trivial
$(\mathrm{MinorCond}(N,\mathcal{M}))$ not satisfied in $\mathcal{M}$

- **enough:** **M** satisfied in $\mathcal{M}$
  $\Rightarrow$ some ϵ-fraction of equations is trivial
- **enough:** for each $f \in \mathcal{M}$ find a small (constant-size) set of "important coordinates"

  if the choice behaves somewhat nicely with minors, then probabilistic argument gives us $\Rightarrow$

- every $f : \mathbb{K}_3^n \to \mathbb{K}_4$ is close to an essentially unary function:

  $(\exists i)\ (\exists c \in K_4)\ (\exists \alpha)\ (\forall x \in K_3^n)$
  $f(x_1, \ldots, x_n) \neq c \Rightarrow f(x_1, \ldots, x_n) = \alpha(x_i)$

- such an i is unique

- $\{i\}$ is the small set of important coordinates

- $f : \mathbb{C}_{137}^n \to \mathbb{K}_3$ is topologically $S^n \to S$ (S a circle)
- define $w_i^f \in \mathbb{Z}$ for $i \in \{1, \ldots, n\}$
  - fix all coordinates but i arbitrarily, call it $f_i : S \to S$
  - w_i^f is the winding number of f_i
- behave very nicely with minors, eg. if $g(x_1, x_2, x_3) = f(x_1, x_1, x_2, x_3)$ then $w_1^g = w_1^f + w_2^f$
- winding number of unary f is bounded above by a constant C
- therefore $\sum w_i \leq C$, actually $\sum |w_i| \leq C$
- important coordinates of f := those i with $w_i^f \neq 0$

- Hardness of hypergraph coloring
  - proof (now) follows the same strategy
  - needs a better version of GapLabelCover
  - combinatorial core to get important coordinates: high chromatic number of Kresner's graphs [Lovász'78]
- Hardness of $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$
  - almost for free since $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$ satisfies less minor conditions than $\mathrm{Pol}(\mathbb{NAE}_2, \mathbb{NAE}_{10000})$
- $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_6)$?
  - people mostly tried analytic approach to analyze polymorphisms

Summary

Summary

- Label Cover madness
  - CSP (and PCSP) is equivalent to a gap version of Label Cover
  - a different gap version of Label Cover crucial in the hardness proofs (both approximation and PCSP)
  - in progress: intermediate problems
- PCSP algorithmically more interesting
  - linear programming, linear equations over $\mathbb{Z}$
  - requires infinite-domain CSP
- topology is implicitely or explicitely in most PCSP NP-hardness proofs (CSP hardness is easy)
- **question:** what about other kind of homomorphisms, like covers or harmonic morphisms?

Summary

- Label Cover madness
  - CSP (and PCSP) is equivalent to a gap version of Label Cover
  - a different gap version of Label Cover crucial in the hardness proofs (both approximation and PCSP)
  - in progress: intermediate problems
- PCSP algorithmically more interesting
  - linear programming, linear equations over $\mathbb{Z}$
  - requires infinite-domain CSP
- topology is implicitely or explicitely in most PCSP NP-hardness proofs (CSP hardness is easy)
- **question:** what about other kind of homomorphisms, like covers or harmonic morphisms?

Thank you!