Algebras with polynomially-many homomorphisms

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STORY

- we encountered a very natural algebraic question
- we found an answer
- we wonder whether it was studied / known

tell ~~me~~ us
after the talk
!

OUTLINE

- the origin
- the question

+ partial answers, etc.

- the answer

UNIVERSES ARE FINITE

in this talk

THE **ORIGIN**

$\boxed{\text{CSP}}$ Constraint Satisfaction Problem

- CSP over $\mathbb{A}$ ... relational structure

  INPUT: $\mathbb{I}$ of the same signature

  QUESTION: $\exists$ homomorphism $\mathbb{I} \to \mathbb{A}$?

[Schaefer '78]
[Feder, Vardi '98]
[Bulatov, Jeavons, Krokhin '05]
[Bulatov '07] [Zhuk '17]

why not? • CSP over $\mathcal{A}$ ... structure

  INPUT: $\mathcal{I}$ — " —

  QUESTION: — " —

[Feder, Madelaine, Stewart '04]

in particular • CSP over $\underline{\mathbf{A}}$ ... algebra

  INPUT: $\underline{\mathbf{I}}$ of the same signature

  QUESTION: $\exists$ homomorphism $\underline{\mathbf{I}} \to \underline{\mathbf{A}}$?

$\boxed{\text{Fact}}$ $\forall \mathbb{A}$

$\exists \underline{\mathbf{B}}$ with 2 unary op.

$\mathrm{CSP}(\mathbb{A}) \sim \mathrm{CSP}(\underline{\mathbf{B}})$

Turing reduction

FIRST STEPS

- **Fact** [FMS'04] $\mathcal{A}$ contains only unary / constant operations

  $\Rightarrow$ CSP over $\mathcal{A}$ $\sim$ CSP over graph of $\mathcal{A}$

  graph of $f : A^2 \to A$ is $\{(a, b, f(a,b)) ; a, b \in A\}$

- Let's consider $\underline{A}$ with 2-element universes
  - $\uparrow$ unary algebras ✓
  - algebras with an idempotent element ✓

    $e \in A$ such that $\forall f \in \mathrm{Clo}(\underline{A})\ f(e, \dots, e) = e$

  - $\underline{A} = (\{0,1\}; *)$

$*$	0	1
0	1	0
1	0	0

    ✓

    - tried consistency methods

      ...

    - can enforce $\underline{I}$ satisfies identities true in $\underline{A}$
    - adding term operations to $\underline{A}$ does not change complexity
    - $\underline{A}$ is term-equivalent to the 2-element Boolean algebra

TWO

Suppose $\underline{\mathbf{A}} \models s \approx t$. Then there is a polynomial algorithm transforming $\underline{\mathbf{I}}$ to $\underline{\mathbf{I}}'$ so that $(\exists\, \underline{\mathbf{I}} \to \underline{\mathbf{A}}$ iff $\exists\, \underline{\mathbf{I}}' \to \underline{\mathbf{A}})$ & $\underline{\mathbf{I}}' \models s \approx t$

If $\mathrm{Clo}(\underline{\mathbf{A}}) = \mathrm{Clo}(\underline{\mathbf{B}})$ then CSP over $\underline{\mathbf{A}}$ $\sim$ CSP over $\underline{\mathbf{B}}$

$\boxed{\text{A question}}$ How hard is to find the smallest $\alpha \in \mathrm{Con}(\underline{\mathbf{I}})$ such that $\underline{\mathbf{I}}/\alpha \in \mathrm{var}(\underline{\mathbf{A}})$? ($\ldots \in SP(\underline{\mathbf{A}})$?)

not "the"

- $\uparrow$ in P for fixed finitely based $\underline{\mathbf{A}}$ (of finite signature)

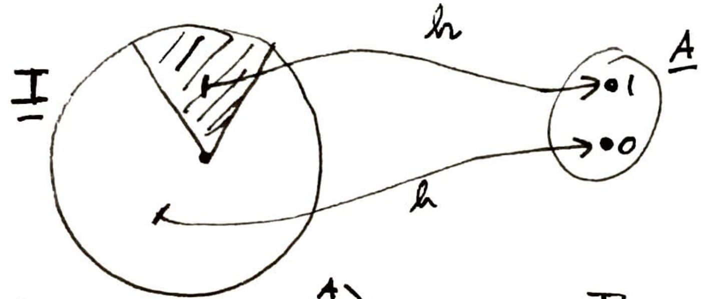
- back to 2-element algebras
- $\boxed{\text{Fact}}$ if not essentially unary, then contains a semilattice, majority, or minority term-operation $\rightsquigarrow$ goal: understand homomorphisms to these

2-ELEMENT ALGEBRAS

every $\mathbf{A}$ that contains such a term operation is ✓

- $\underline{A} = (\{0,1\}; x+y+z)$ term equivalent to 1-dim affine space over $\mathbb{Z}/2\mathbb{Z}$ ✓+
- $\underline{A} = (\{0,1\}; \wedge^{\underline{A}})$, $\underline{I} = (I; \wedge^{\underline{I}})$ WLOG semilattice

  $a \wedge b = \min\{a,b\}$



$\forall h: \underline{I} \to \underline{A}$
$\exists i \in I \quad h(j) = 1$ iff $j \geq i$ ✓+

- $\underline{A} = (\{0,1\}; maj^{\underline{A}})$ $\quad \underline{I} = (I; maj^{\underline{I}})$

  ternary majority

  semilattice!

  if $h(i) = 0$ then h preserves $\begin{cases} s^{\underline{A}}(x,y) := maj^{\underline{A}}(0,x,y) \\ s^{\underline{I}}(x,y) := maj^{\underline{I}}(i,x,y) \end{cases}$ $\rightsquigarrow$ ✓+

$\Rightarrow$ classification of computational complexity for CSPs over 2-element <u>structures</u>

THE **QUESTION**

THE QUESTION: WHAT IS $\mathcal{K}$?

$$\mathcal{K} = \{ \underline{A} : \exists \text{polynomial } p \ \ \forall \underline{I} \text{ of the same signature as } \underline{A} \ \ |\{h : \underline{I} \to \underline{A} : h \text{ homomorphism}\}| < p(|I|) \}$$

(effective variant)

☺ $\mathcal{K}$ is closed under

- S $\underline{B} \in \mathcal{K},\ \underline{A} \leq \underline{B} \Rightarrow \underline{A} \in \mathcal{K}$
- P $\underline{A}, \underline{B} \in \mathcal{K}$ (same sign.) $\Rightarrow \underline{A} \times \underline{B} \in \mathcal{K}$
- E $\underline{B} \in \mathcal{K} \quad \mathrm{Clo}(\underline{B}) \subseteq \mathrm{Clo}(\underline{A}) \Rightarrow \underline{A} \in \mathcal{K}$

not closed under quotients

☹ $\underline{A} \in \mathcal{K} \Leftrightarrow \forall \underline{B} \leq \underline{A} \quad \underline{B} + \text{constants} \in \mathcal{K}$

→ bad news & good news

Examples

- 2-element semilattice $\in \mathcal{K}$
  $\forall$ semilattice $\in \mathcal{K}$
- 2-element majority algebra $\in \mathcal{K}$
- $(\mathbb{Z}_p; x-y+z) \in \mathcal{K}$

($\Rightarrow$ 2-element "non-unary" algebra $\in \mathcal{K}$)

ROCK - PAPER - SCISSORS

$\underline{A} = (\{0,1,2\};\ \cdot^{\underline{A}},\ 0^{\underline{A}}, 1^{\underline{A}}, 2^{\underline{A}})$

$x \cdot^{A} y$ = looser

$0\cdot 1 = 1 \cdot 0 = 0 = 0 \cdot 0$
$1 \cdot 2 = 2 \cdot 1 = 1 = 1 \cdot 1$
$0 \cdot 2 = 2 \cdot 0 = 2 = 2 \cdot 2$

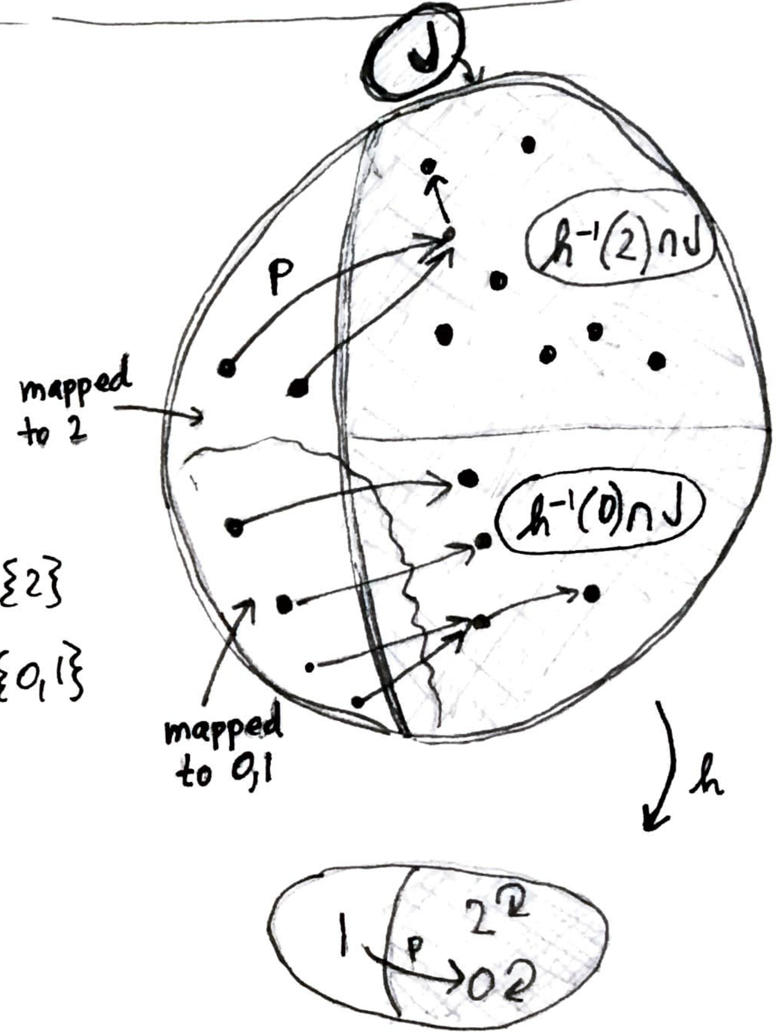
rock 0 $\leq$ 1 paper
$0 \geq 2$ scissors
$2 \geq 1$

- take $\underline{I} = (I;\ \cdot^{\underline{I}}, 0^{\underline{I}}, 1^{\underline{I}}, 2^{\underline{I}})$, $h: \underline{I} \to \underline{A}$
- consider $p(x) = x \cdot 0$

x	0	1	2
$p^{\underline{A}}(x)$	0	0	2

- $J := p^{\underline{I}}(I)$ $\quad h(J) \subseteq \{0,2\}$
- Fix $h_{\restriction J}$, for $j \in J$ $\cdot\ h(j) = 2 \Rightarrow h(p^{-1}(j)) = \{2\}$
  $\cdot\ h(j) = 0 \Rightarrow h(p^{-1}(j)) \subseteq \{0,1\}$

$\rightsquigarrow$ polynomially many $h_{\restriction J}$, for each $h_{\restriction J}$ only polynomially many extensions



images of polynomials, separation, ... $\rightsquigarrow$ Tame Congruence Theory [Hobby, McKenzie '88]

TAYLOR ALGEBRAS IN $\mathcal{K}$

with all constants

Fact $\underline{\mathbf{A}} \in \mathcal{K}$ provided that $\exists \alpha \in \mathrm{Con}(\underline{\mathbf{A}})$ such that

- $\underline{\mathbf{A}}/\alpha \in \mathcal{K}$
- $\forall a \neq b \ (a,b) \in \alpha \ \exists f \in \mathrm{Clo}_1 \underline{\mathbf{A}}$ such that $f(a) \neq f(b)$ and $\underline{\mathbf{A}} \restriction f(a/\alpha) \in \mathcal{K}$

Fact $\underline{\mathbf{A}}$ with constants is in $\mathcal{K}$ <u>if</u> $\exists$ chain of congruences $0 \prec \alpha_1 \prec \alpha_2 \prec \dots \prec 1$ $(*)$
such that $\mathrm{typ}(\alpha_{i+1}, \alpha_i) \neq 1$

$\boxed{\Rightarrow}$ if $\underline{\mathbf{A}}$ satisfies a non-trivial idempotent Mal'cev condition,
then $\underline{\mathbf{A}} \in \mathcal{K}$

Fact $\underline{\mathbf{A}}$ with constants, and

- $\underline{\mathbf{A}}$ is simple (tame) or
- $\underline{\mathbf{A}}$ is 3-element

Then $\underline{\mathbf{A}} \in \mathcal{K}$ <u>iff</u> ~~$(*)$~~ for all minimal α, $\mathrm{typ}(0,\alpha) \neq 1$

non-membership in $\mathcal{K}$

→ studying subreducts of matrix powers

→ ad hoc construction

THE **ANSWER**

FREE ALGEBRAS

- $\underline{\mathbf{I}}$ is "$\underline{(X,A)\text{-free}}$" for $\underline{\mathbf{A}}$ if $\forall$map $\varphi: X \to A$ can be extended to a homomorphism $\hat{\varphi}: \underline{\mathbf{I}} \to \underline{\mathbf{A}}$

  here $X \subseteq I$

  - $\boxed{\text{Example}}$: $\underline{\mathbf{I}} =$ the free algebra for $HSP(\underline{\mathbf{A}})$ over X

    $$\cong Clo_{|X|}(\underline{\mathbf{A}}) \leq \underline{\mathbf{A}}^{A^{|X|}}$$

  $\Rightarrow$ if $|Clo_n(\underline{\mathbf{A}})| \in \mathcal{O}(n^k)$ then $\underline{\mathbf{A}} \notin \mathcal{K}$ (#homos is $\Omega(2^{\sqrt[k]{|\mathbf{I}|}})$)

- $\underline{\mathbf{I}}$ is "$\underline{(X, ab)\text{-free}}$" for $\underline{\mathbf{A}}$ if $\forall$map $\varphi: X \to \{a,b\}$ can be extended to a homomorphism $\hat{\varphi}: \underline{\mathbf{I}} \to \underline{\mathbf{A}}$

  $\{a,b\} \subseteq A$, $a \neq b$

  - $\boxed{\text{Example}}$ $\underline{\mathbf{I}} = \{ f \restriction_{\{a,b\}^{|X|}} : f \in Clo_{|X|}(\underline{\mathbf{A}}) \} \leq \underline{\mathbf{A}}^{\{a,b\}^{|X|}}$

  $\Rightarrow$ if $|$"$Clo_n(\underline{\mathbf{A}}) \restriction_{\{a,b\}}$"$| \in \mathcal{O}(n^k)$ then $\underline{\mathbf{A}} \notin \mathcal{K}$

Is it a standard concept?

NON-MEMBERSHIP IN $\mathcal{K}$

Def $\alpha \in \mathrm{Con}(\underline{\mathbf{A}})$ is <u>strongly abelian</u> if

$$f(x_0, x_1, \dots, x_{n-1}) = f(y_0, y_1, \dots, y_{n-1}) \Rightarrow f(x_0, z_1, \dots, z_{n-1}) = f(y_0, z_1, \dots, z_{n-1})$$

whenever $f \in \mathrm{Clo}_n(\underline{\mathbf{A}} + \text{constants})$, $(x_0, y_0) \in \alpha$, $(x_i, y_i) \in \alpha$, $(y_i, z_i) \in \alpha$, $i = 1, \dots, n-1$

α strongly abelian, $(a, b) \in \alpha$, $a \neq b \Rightarrow f\restriction_{\{a,b\}^n}$ depends on at most $\log_2 |A|$ coordinates

$\Rightarrow \left| \text{"} \mathrm{Clo}_n(\underline{\mathbf{A}} + \text{consts})\restriction_{\{a,b\}} \text{"} \right| \in \mathcal{O}(n^{\log_2 |A|}) \Rightarrow \underline{\mathbf{A}} \notin \mathcal{K}$

Remark: enough to have (a,b) generating a strongly rectangular tolerance (we get $\mathcal{O}(n^{|A|})$ instead of $\mathcal{O}(n^{\log_2 |A|})$) [Kearnes, Kiss '13]

THE ANSWER

THEOREM For a finite algebra $\underline{A}$ the following are equivalent.

(1) $\underline{A} \in \mathcal{K}$ $\quad$ (# homos $\underline{I} \to \underline{A}$ $\leq p(|\underline{I}|)$)

(2) no subalgebra of $\underline{A}$ has a nonzero strongly abelian congruence

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- if $\underline{A} \notin \mathcal{K}$ $\quad$ # homos $\underline{I} \to \underline{A}$ is $\cancel{2^{\Omega(|I|)}}$ $2^{|I|^{\Omega(1)}}$
- $\underline{A}$ finite signature: $\quad$ (2) can be tested in P ; $\quad$ effective enumeration of homos
  - $\underline{A} \in \mathcal{K} \Rightarrow$ CSP over any expansion of $\underline{A}$ is ~~NP-complete~~ in P
  - $\underline{A} \notin \mathcal{K} \Rightarrow$ CSP over some expansion of $\underline{A}$ is NP-complete (1 ternary relation suffices)

QUESTIONS

- was it known/studied?
- complexity of CSP over algebras remains open
- what is $\mathcal{L} = \{\underline{A} ;\ \forall \underline{I} \in HSP(\underline{A})$ has small generating set$\}$?

THANK YOU