# Combinatorial Value & Gap Amplification

L. Barto, M. Kozik : Combinatorial Gap Theorem and Reductions between Promise CSPs, SODA'22

F. Bialas : in progress

CoCoSym: Symmetry in Computational Complexity

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# GAP PROBLEMS

## NO GAP

CSPs

IN: 3-SAT instance

YES
satisfiable

NO
¬ satisfiable

IN: Graph

YES
3-colorable

NO
¬ 3-colorable

## COMBINATORIAL GAP

PCSPs

IN: Graph

YES
3-colorable

gap

NO
¬ 5-colorable

IN: 3-uniform hypergraph

YES
2-colorable

NO
¬ 10000-colorable

## ANALYTICAL GAP

PVCSPs

IN: CSP instance

YES
satisfiable
$\Leftrightarrow$ value 1

NO
¬ value $\geq 0.9$

optimum fraction of satisfied constraints

# REDUCTIONS



IN: 3-SAT instance

YES: satisfiable

NO: ¬ satisfiable

$\xrightarrow{\text{completeness}}$

$\rightsquigarrow$

$\xrightarrow[\text{soundness}]{}$

IN: Graph

YES: 3-colorable

NO: ¬ 3-colorable

IN: 3-uniform hypegraph

YES: 2-colorable

NO: ¬ 10000-colorable

$\rightsquigarrow$

IN: Graph

YES: 3-colorable

NO: ¬ 5-colorable

**Fact** $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}') \rightsquigarrow \mathrm{PCSP}(\mathbb{B}, \mathbb{B}')$ whenever $\mathrm{Pol}(\mathbb{B}, \mathbb{B}') \xrightarrow{\text{minion homo}} \mathrm{Pol}(\mathbb{A}, \mathbb{A}')$

IN: 3-SAT instance

YES: satisfiable

NO: ¬ satisfiable

gap amplification $\rightsquigarrow$

PCP Theorem

IN: CSP instance (small domain sizes)

YES: satisfiable ⇔ value 1

small gap

NO: ¬ value $\geq 0.999999$

gap amplification $\rightsquigarrow$

Parallel Repetition Theorem

IN: CSP instance (bigger domain sizes)

YES: satisfiable ⇔ value 1

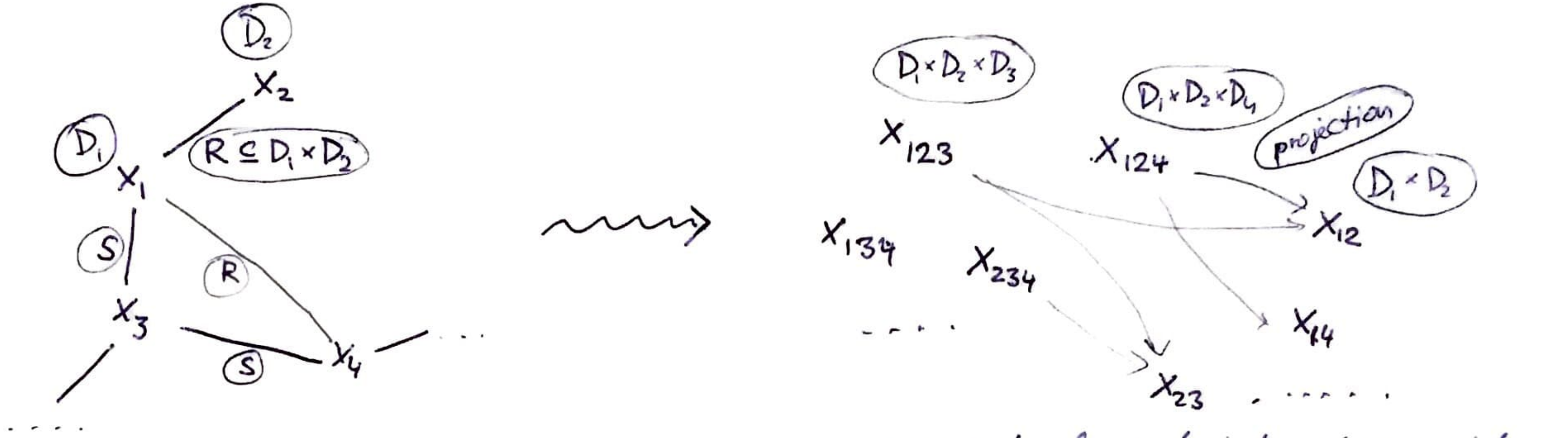
big gap

NO: ¬ value $\geq 0.001$

# TWO TYPES OF TRANSFORMATIONS



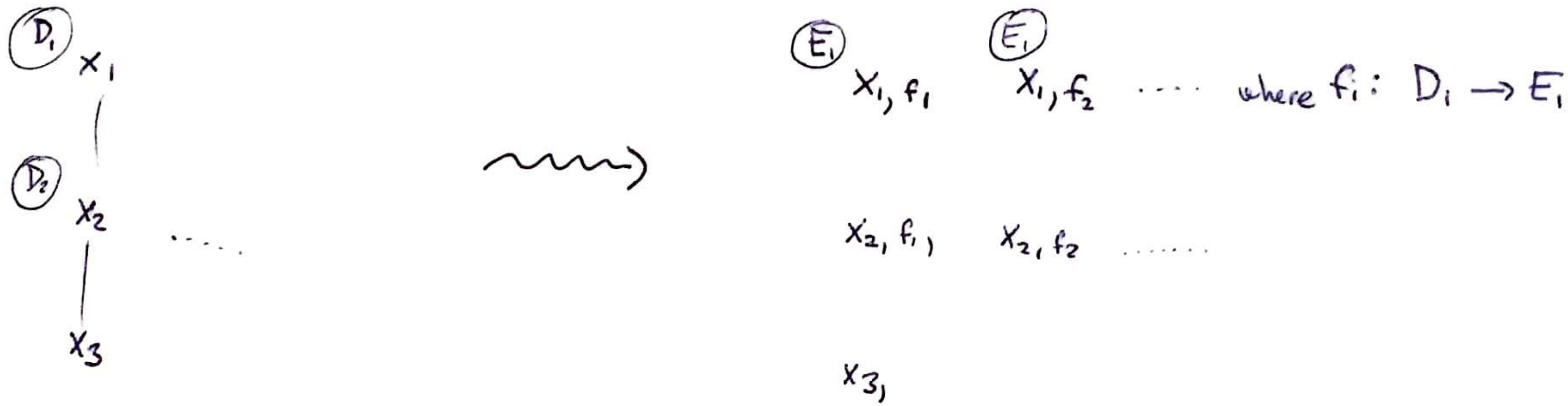
## Rep




some natural constraints about consistency

## Hash

aka Long Code, ....




some natural constraints about consistency

# REDUCTIONS IN MORE DETAIL

$\boxed{\text{Nice}}$ $\mathrm{Pol}(\mathbb{B},\mathbb{B}') \xrightarrow{\text{minion homo}} \mathrm{Pol}(\mathbb{A},\mathbb{A}') \Rightarrow \mathrm{PCSP}(\mathbb{A},\mathbb{A}') \overset{\text{Hash}}{\rightsquigarrow} \mathrm{PCSP}(\mathbb{B},\mathbb{B}')$

$\boxed{\text{Not so nice}}$ To get 3-SAT $\rightsquigarrow$ 3 vs. ¬5-coloring

3-SAT

| | |
|---|---|
| YES | satisfiable |
| NO | ¬ satisfiable |

$\overset{\text{Rep}}{\rightsquigarrow}$ $\overset{\text{Hash}}{\rightsquigarrow}$ $\overset{\text{Rep}}{\rightsquigarrow}$ $\overset{\text{Hash}}{\rightsquigarrow}$ . . . . . . . . .

CSP small domains

| | |
|---|---|
| YES | value 1 |
| NO | ¬ value ≥ 0.99999 |

$\overset{\text{Rep}}{\rightsquigarrow}$

CSP big domain

| | |
|---|---|
| YES | value 1 |
| NO | ¬ value ≥ 0.00001 |

$\overset{\text{Hash}}{\rightsquigarrow}$

3-unif. hypergrap

| | |
|---|---|
| YES | 2-colorable |
| NO | ¬ 10000-colorable |

~~Rep~~ $\overset{\text{Hash}}{\rightsquigarrow}$

Graph

| | |
|---|---|
| YES | 3-colorable |
| NO | ¬5-colorable |

$\boxed{\text{This work}}$

- Rep increases <u>combinatorial value</u> => Baby PCP theorem,

not this talk

$\Rightarrow$ $\mathrm{Pol}(\mathbb{B},\mathbb{B}') \xrightarrow{\text{weaker minion homo}} \mathrm{Pol}(\mathbb{A},\mathbb{A}') \Rightarrow \mathrm{PCSP}(\mathbb{A},\mathbb{A}') \overset{\text{Rep}}{\rightsquigarrow} \overset{\text{Hash}}{\rightsquigarrow} \mathrm{PCSP}(\mathbb{B},\mathbb{B}')$

not needed

# COMBINATORIAL VALUE

→ related concept studied: "MinRep"

**instance**

$$\begin{array}{l} (D_1)\ x_1 \\ (R)\ | \\ (D_2)\ x_2 \\ \quad | \\ \quad \vdots \end{array}$$

**solution**

$x_1 \longmapsto s(x_1) \in D_1$

$x_2 \longmapsto s(x_2) \in D_2$

$\vdots$

$(s(x_1), s(x_2)) \in R$

$\vdots$

**3-weak solution**

$x_1 \longmapsto s(x_1) \in \binom{D_1}{3}$

$x_2 \longmapsto s(x_2) \in \binom{D_2}{3}$

$\vdots$

$s(x_1) \times s(x_2) \cap R \neq \emptyset$

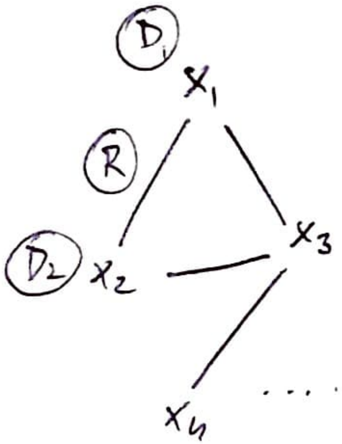
$\vdots$

**combinatorial value** of an instance = minimum $d$ such that $\exists\, d$-weak solution

**Fact** value $\geq 1/(\text{combinatorial value})^2$ ← for $\leq 2$-ary constraints

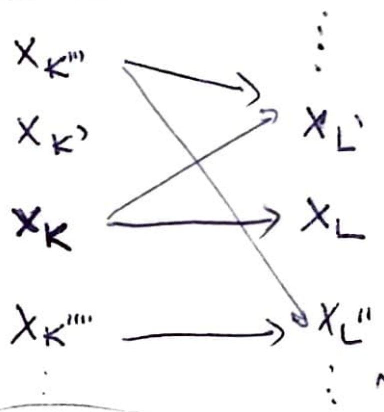
# GAP AMPLIFICATION

## instance




$k > l$

## Label Cover instance




one variable $x_K$ for each $K \subseteq$ original vars $|K| = k$

— " — $|L| = l$

domain of $x_K$ : the set of all partial solutions on $K$

$x_K \xrightarrow{\pi_{K,L}} x_L$ exists iff $L \subseteq K$

$\pi_{K,L}$ = the projection

there is a version with more layers

## Theorem

$\forall$ bound on domain sizes & arity of constraints $\forall d \ \exists k > l$

CSP instance

| | |
|---|---|
| YES | satisfiable |
| NO | $\neg$ satisfiable |

$k > l$

Label Cover inst.

| | |
|---|---|
| YES | satisfiable |
| NO | $\neg \le d$-weakly sat. |

= $\neg$ combinatorial value $\le d$

## Proof

- YES $\longrightarrow$ YES ✓
- NO $\longrightarrow$ NO contrapositive

solution $\longleftarrow$ $d$-weak solution

- strategy $d$-weak $\Rightarrow$ $(d-1)$-weak
  $k > l$ with smaller $k' > l'$

- we do not look at the original instance

$\forall U$ size $a$ $\quad S_U \subseteq D^U$ size $\not\geq q$

$\forall V$ size $b$ $\quad S_V \subseteq D^V$ size $\not\geq r$

weakly con...

big. 6



Strategy

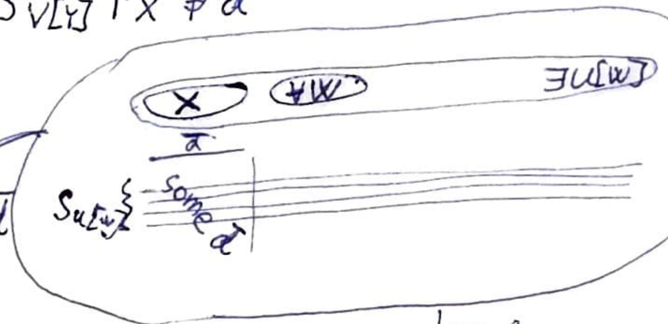
$(q,r) \leftarrow (q-1,r)$ I, II

$(1,r) \leftarrow (1,r-1)$

# 7 From a differen talk


- Assume $(\exists X \text{ size } a')(\forall \bar d \in D^X)(\forall Y \supseteq X \text{ size} \le c)(\exists V[Y] \text{ size } b)\ S_{V[Y]} \restriction X \not\ni \bar d$

- Fix such $X$

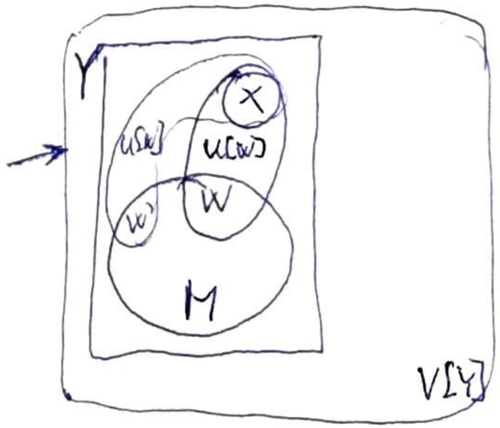



- Find $\bar d \in D^X$: $(\forall W \text{ size } a')(\exists U[W] \supseteq X \cup W \text{ size } a)\ S_{U[W]} \restriction X \ni \bar d$

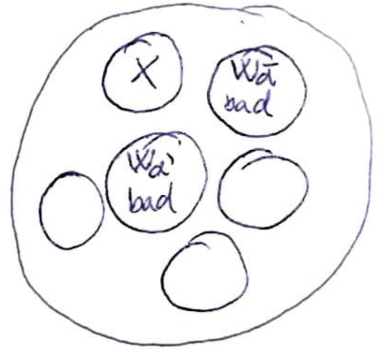
- For $W$ size $a'$ define $S'_W := (S_{U[W]} - \text{tuples with } \restriction X = \bar d) \restriction W$

- For $M$ size $b'$ take $Y := \bigcup_{\substack{W \subseteq Y \\ \text{size } a'}} U[W]$

  define $S'_M := S_{V[Y]} \restriction M$




↓proof




Remark: seems different from Dinur's proof of PCP theorem

# BABY PCP THEOREM



there is also an intermediate theorem!

PCP theorem + Parallel repetition thm.

Label Cover instance

| YES |
|---|
| satisfiable |

NP-hard

| NO |
|---|
| ∃ value ≥ 0.0001 |

Baby PCP theorem

☹ weaker

☺
- very simple reduction from any CSP
- simpler proof

Label Cover instance

| YES |
|---|
| satisfiable |

NP-hard

| NO |
|---|
| ∃ combinatorial value ≤ 10000 |

2-to-1 conjecture

Label Cover instance
constraint relations are
2-to-1 functions

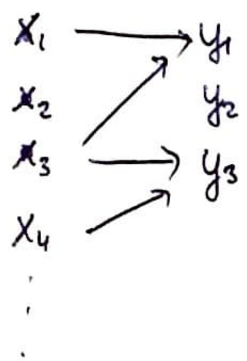
YES/NO dtto NP-hard

Baby 2-to-1 conjecture

- simpler intermediate step
- would be enough to prove Theorem with $l=k-1$
- our proof strategy does not work

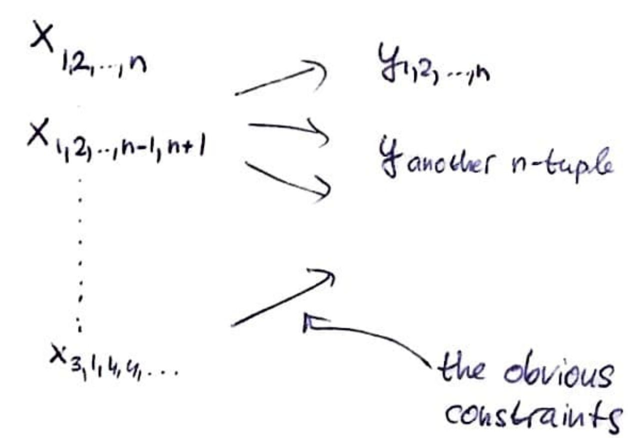
# PARALLEL REPETITION

Label Cover instance




n-fold parallel repetition $\rightsquigarrow$

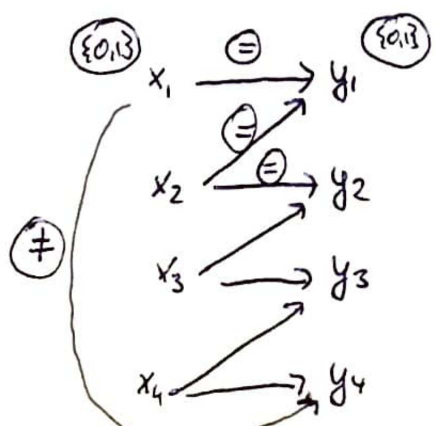
Label Cover instance




Parallel repetition theorem: value $\leq (1-\varepsilon) \rightsquigarrow$ value $\leq (1-\varepsilon)^{\Omega(n)}$

Combinatorial version: combinatorial value $\geq d \rightsquigarrow$ comb. value $\geq d^{\Omega(n)}$

FALSE




comb. value 2 $\rightsquigarrow$ comb. value $\leq n+1$

# SUMMARY

$\boxed{\text{Thm}}$ simple transformation produces gap in combinatorial value

$\Longrightarrow$ Baby PCP theorem

did not talk about

$\Longrightarrow$ $\mathrm{Pol}(\mathbb{B}, \mathbb{B}') \xrightarrow{\text{some homo}} \mathrm{Pol}(\mathbb{A}, \mathbb{A}') \Rightarrow \mathrm{PCSP}(\mathbb{A}, \mathbb{A}') \overset{\text{simple}}{\rightsquigarrow} \mathrm{PCSP}(\mathbb{B}, \mathbb{B}')$

e.g. $3\text{-SAT} \overset{\text{simple}}{\rightsquigarrow} \mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$

– not satisfactory

$\boxed{\text{Todo}}$

- Baby ~~th~~ 2-to-1 conjecture
- generalize to valued setting