Infinite Nature of Finite PCSPs

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CSP - Constraint Satisfaction Problem

- homomorphism problem
  - $$\left.\begin{array}{r}\mathcal{E} = (E; S_1, S_2, \ldots, S_n) \\ \mathcal{D} = (D; R_1, R_2, \ldots, R_n)\end{array}\right\}$$ similar relational structures
  - $h : E \to D$ is a homomorphism from $\mathcal{E}$ to $\mathcal{D}$ if $(a_1, a_2, \ldots, a_k) \in S_i \Rightarrow (h(a_1), h(a_2), \ldots, h(a_k)) \in R_i$
  - $\mathrm{CSP}(\mathcal{D})$
    - Decision: Given $\mathcal{E}$ is there a homomorphism $\mathcal{E} \to \mathcal{D}$?
    - Search: Find a homomorphism.

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Examples:

- 3-SAT (NP-complete)
- 1-in-3-SAT (NP-complete)
  $\text{1-in-3} = (\{0,1\};\{(1,0,0),(0,1,0),(0,0,1)\})$
- NAE-3-SAT (NP-complete)
  $\text{NAE-3} = (\{0,1\};\{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\})$
- 3-coloring of a graph (NP-complete)
  $(\{0,1,2\};\neq)$
- 2-coloring of a graph (P)
  $(\{0,1\};\neq)$

Theorem ([Bulatov '17];[Zhuk '17])

$\mathrm{CSP}(\mathcal{A})$, *$\mathcal{A}$ - finite, is in P or NP-complete*

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- $\mathrm{PCSP}(\mathcal{A}, \mathcal{B})$
- $\mathcal{A}, \mathcal{B}$ - relational structures, $\mathcal{A} \to \mathcal{B}$
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- $\mathrm{PCSP}(\mathcal{A}, \mathcal{A}) = \mathrm{CSP}(\mathcal{A})$
- $(\mathcal{A}, \mathcal{B}), (\mathcal{A}', \mathcal{B}')$ - PCSP templates
  $\mathcal{A}' \to \mathcal{A}, \mathcal{B} \to \mathcal{B}'$ - $(\mathcal{A}', \mathcal{B}')$ is a homomorphic relaxation of $(\mathcal{A}, \mathcal{B})$
  $\mathrm{PCSP}(\mathcal{A}', \mathcal{B}') \leq \mathrm{PCSP}(\mathcal{A}, \mathcal{B})$
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PCSP dichotomy?

- Dichotomy for PCSP?
- Yes for symmetric Boolean PCSPs (allowing negations)
- $\mathrm{PCSP}(\Gamma)$
- Γ allows negations: $(\neq, \neq) \in \Gamma$ where $\neq = \{(0,1),(1,0)\}$

Theorem (Brakensiek, Guruswami '17)

Let Γ be a symmetric collection of Boolean relation pairs that allows negations. Then $\mathrm{PCSP}(\Gamma)$ is either in P or NP-hard.

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Classification

$$\text{odd-in-k} = \{(x_1, x_2, \ldots, x_k) : \Sigma x_i = 1 \pmod 2\}$$
$$\text{even-in-k} = \{(x_1, x_2, \ldots, x_k) : \Sigma x_i = 0 \pmod 2\}$$
$$\leq \text{j-in-k} = \{(x_1, x_2, \ldots, x_k) : \Sigma x_i \leq j\}$$
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Theorem (Brakensiek, Guruswami '17)

$\Gamma = \{(\mathrm{P}, \mathrm{Q}), (\neq, \neq)\}$. *If* $(\mathrm{P}, \mathrm{Q}) =$
a) (odd-in-k, odd-in-k), (even-in-k, even-in-k)
b) ($\leq$ j-in-k, $\leq$ (2j-1)-in-k), $j < \frac{k}{2}$
c) (j-in-k, NAE-k)
then $\mathrm{PCSP}(\Gamma)$ *is tractable.*

All tractable cases: relaxations and modifications of the upper cases

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All tractable cases: relaxations and modifications of the upper cases

$\mathrm{PCSP}(1\text{-in-}3, \mathrm{NAE}\text{-}3)$

- $\mathrm{PCSP}(1\text{-in-}3, \mathrm{NAE}\text{-}3)$ is in P
- $(\mathbb{Z}; \{(x, y, z) : x + y + z = 1\}) =: \mathcal{Z}$
- $1\text{-in-}3 \to \mathcal{Z}$
  $\mathcal{Z} \to \mathrm{NAE}\text{-}3$
- $\mathrm{PCSP}(1\text{-in-}3, \mathrm{NAE}\text{-}3) \leq \mathrm{CSP}(\mathcal{Z})$
- This finite-to-infinite transition is unavoidable.

Theorem (Barto)

Let $\mathcal{C}$ be a finite relational structure such that $(1\text{-in-}3, \mathrm{NAE}\text{-}3)$ is a homomorphic relaxation of $(\mathcal{C}, \mathcal{C})$. Then $\mathrm{CSP}(\mathcal{C})$ is NP-complete.

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Infinity is relevant

a) $(\text{odd-in-k}, \text{odd-in-k})$, $(\text{even-in-k}, \text{even-in-k})$ and relaxations: reducible to finite CSP

b) $(\leq \text{j-in-k}, \leq \text{(2j-1)-in-k})$, $j < \frac{k}{2}$

- $(\leq \text{2-in-k}, \leq \text{3-in-k})$, $k \geq 5$: infinitary
- $(\text{2-in-k}, \leq \text{3-in-k})$, $k \geq 5$: infinitary
- remaning cases: open

c) $(\text{j-in-k}, \text{NAE-k})$

- $(\text{1-in-3}, \text{NAE-3})$: infinitary
  $(\text{1-in-k}, \text{NAE-k})$, k-odd: probably infinitary
- $(\text{1-in-k}, \text{NAE-k})$, k-even: reducible to finite CSP
- remaining cases: open

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- $(\leq \text{2-in-k}, \leq \text{3-in-k})$, $k \geq 5$: infinitary
- $(\text{2-in-k}, \leq \text{3-in-k})$, $k \geq 5$: infinitary
- remaning cases: open

c) $(\text{j-in-k}, \text{NAE-k})$

- $(\text{1-in-3}, \text{NAE-3})$: infinitary
  $(\text{1-in-k}, \text{NAE-k})$, k-odd: probably infinitary
- $(\text{1-in-k}, \text{NAE-k})$, k-even: reducible to finite CSP
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Infinity is relevant

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- remaining cases: open

Idea of the proof for $(2\text{-in-}5, \leq 3\text{-in-}5)$

Definition

Let $\mathcal{C}$ be a CSP template. $s : C^n \to C$ is a *polymorphism* of $\mathcal{C}$ if for each relation R in $\mathcal{C}$

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \in R, \dots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \in R \Rightarrow \begin{bmatrix} s(a_{11}, \dots, a_{1n}) \\ \vdots \\ s(a_{m1}, \dots, a_{mn}) \end{bmatrix} \in R.$$

Definition

$s : C^n \to C$ is cyclic if

$$s(a_1, a_2, \dots, a_n) = s(a_2, \dots, a_n, a_1)$$

Theorem (Barto, Kozik '12)

Let $\mathcal{C}$ be a finite CSP template. If $\mathrm{CSP}(\mathcal{C})$ is not NP-complete, then $\mathcal{C}$ has a cyclic polymorphism of arity p for every prime number $p > |C|$.

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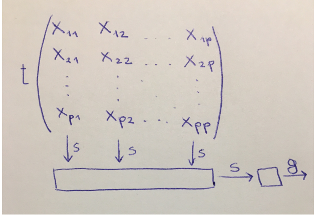
Sketch of the proof

Theorem

Let $\mathcal{C} = (C; R, \neq)$ *be a finite relational structure with 5-ary* $R \subseteq C^5$ *and binary* $\neq = \{(0,1),(1,0)\}$ *such that* $(\{0,1\}; 2\text{-in-}5, \neq) \to \mathcal{C}$ *and* $\mathcal{C} \to (\{0,1\}, \leq 3\text{-in-}5, \neq)$. *Then* $\mathrm{CSP}(\mathcal{C})$ *is NP-complete.*

- assume $\mathrm{CSP}(\mathcal{C})$ is not NP-complete
- s - cyclic polymorphism of prime arity p big enough
- $g : \mathcal{C} \to (\{0,1\}, \leq 3\text{-in-}5, \neq)$



- $\lambda(X) = \left(\sum_{i,j} x_{ij}\right)/p^2$
- $\lambda(X_1) < \frac{1}{2}$ and $\lambda(X_2) > \frac{1}{2} \Rightarrow t(X_1) \neq t(X_2)$
- find X_1 and X_2 such that $\lambda(X_1) < \frac{1}{2}$ and $\lambda(X_2) > \frac{1}{2}$ and $t(X_1) = t(X_2)$
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Thanks for your attention!