An exponential lower bound on the size of primitive positive definition

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Primitive Positive Definition

$\mathcal{B}$ is a set of finitary relations on a set A.

Definition

A primitive positive formula (pp-formula) over $\mathcal{B}$:

$$R(x_1, \ldots, x_n) =$$
$$\exists y_1 \ldots \exists y_l \; R_1(z_{1,1}, \ldots, z_{1,n_1}) \wedge \ldots \wedge R_k(z_{k,1}, \ldots, z_{k,n_k}),$$

where $R_1, \ldots, R_k \in \mathcal{B}$, $z_{i,j} \in \{x_1, \ldots, x_n, y_1, \ldots, y_l\}$

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$A = \{0, 1, 2\}$
$R(x_1, x_2, x_3) = \exists y (x_1 \neq y \wedge x_2 \neq y \wedge x_3 \neq y)$

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Definition

For a relation R on a set A and a set of relations $\mathcal{B}$ (basis) put
$Q_{\mathcal{B}}(R) := \min\{Q(\Phi) \mid \Phi \text{ pp-defines R}\}$
$C_{\mathcal{B}}(R) := \min\{C(\Phi) \mid \Phi \text{ pp-defines R}\}$

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  Galois connection and so on

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Basis

$S_c = A^3 \setminus \{(c, c, c)\}$, $\mathcal{B} = \{S_c \mid c \in A\}$

Claim

Any relation on A can be pp-defined over $\mathcal{B}$.

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Theorem [Bashirov, 2015]

$|A|^{\frac{n-1}{3}} - n \leq Q_{\mathcal{B}}(n) \leq |A|^n(2|A|(n-3)+1)$
$\frac{|A|^n}{3\log_2(10n|A|^{n+3})} \leq C_{\mathcal{B}}(n) \leq |A|^n(2|A|(|A|-1)(n-3)+1)$

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Tell me if you know better bounds

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R. Willard, Testing expressibility is hard, in D. Cohen (Ed.): CP 2010, LNCS 6308, 9-23, 2010

For infinitely many n there exist a constraint language Γ_n and a relation R_n, both on a 22-element domain, such that $|R_n| = n$, R_n is expressible from Γ_n but every pp-definition of R_n instance expressing R_n has at least $2^{n/3}$ variables.

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Tell me if you know other results

My exponential lower bound

Basis

$$A = \{0, 1, 2\},\ R_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & \cdot \\ 0 & 1 & 0 & 1 & \cdot & 2 \\ 0 & 0 & 0 & 1 & \cdot & \cdot \end{pmatrix},\ R_2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & \cdot & \cdot \end{pmatrix},$$

$$\mathcal{B} = \{R_1, R_2\}$$

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Relation σ_n

$\sigma = \{0,1,2\}^2 \setminus \{(0,1),(1,0)\}$

$\sigma_n(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = \sigma(x_1, y_1) \vee \cdots \vee \sigma(x_n, y_n)$

- σ_n does not contain
  $(0,1,0,1,0,1,\ldots,0,1)$
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Theorem

$2^n \leq Q_{\mathcal{B}}(\sigma_n) \leq 2^n(n+2)$

$2^n \leq C_{\mathcal{B}}(\sigma_n) \leq 2^n(n+3)$

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R is pp-defined from $\mathcal{B}$ and f preserves $\mathcal{B} \Rightarrow f$ preserves R.

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Partial Operations are powerful!

Relation σ_n

$\sigma = \{0, 1, 2\}^2 \setminus \{(0, 1), (1, 0)\}$

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Theorem

Suppose $\mathcal{B}$ is preserved by $f_n(x_1, \ldots, x_n, y_1, \ldots, y_n) =$

$$\begin{cases} *, & \text{if } \{x_1, \ldots, x_n\} = \{0\} \text{ and } \{y_1, \ldots, y_n\} = \{1\} \\ 0, & \text{if } \{x_1, \ldots, x_n\} = \{0\} \text{ and } \{y_1, \ldots, y_n\} \neq \{1\} \\ 1, & \text{if } \{x_1, \ldots, x_n\} \in \{\{1\}, \{0,1\}\} \text{ and } \{y_1, \ldots, y_n\} = \{1\} \\ 2, & \text{otherwise} \end{cases}$$

Then $Q_{\mathcal{B}}(\sigma_n) \geq 2^n$.

Partial Operations are powerful!

Relation σ_n

$\sigma = \{0,1,2\}^2 \setminus \{(0,1),(1,0)\}$

$\sigma_n(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = \sigma(x_1, y_1) \vee \cdots \vee \sigma(x_n, y_n)$

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For $\mathcal{B} = \{R_1, R_2\}$ we have $Q_{\mathcal{B}}(\sigma_n) \geq 2^n$

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Suppose $|\mathcal{B}| < \infty$, $\mathcal{B}$ is preserved by all total operations from PartialClo($\{f_1, f_2, f_3, \dots\}$). Then $Q_{\mathcal{B}}(\sigma_n)$ is exponential on n.

Connection with Quantified Constraint Satisfaction Problem

QCSP(Γ):

Given a sentence $\exists y_1 \forall x_1 \dots \exists y_t \forall x_t (R_1(\dots) \wedge R_s(\dots))$, where $R_1, \dots, R_s \in \Gamma$.
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Counter-example

$$\Gamma = \left\{ \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & \cdot \\ 0 & 1 & 0 & 1 & \cdot & 2 \\ 0 & 0 & 1 & 1 & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & \cdot & \cdot \end{pmatrix} \right\}.$$

- Pol(Γ) has EGP property.
- QCSP(Γ) can be solved in polynomial time.

Thank you for your attention