An exponential lower bound on the size of primitive positive definition

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 \mathcal{B} is a set of finitary relations on a set A.

Definition

A primitive positive formula (pp-formula) over \mathcal{B} :

$$R(x_1,\ldots,x_n) = \\ \exists y_1\ldots\exists y_l\ R_1(z_{1,1},\ldots,z_{1,n_1})\wedge\ldots\wedge R_k(z_{k,1},\ldots,z_{k,n_k}),$$

where $R_1, ..., R_k \in \mathcal{B}, z_{i,j} \in \{x_1, ..., x_n, y_1, ..., y_l\}$

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$$A = \{0, 1, 2\}$$

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where $R_1, ..., R_k \in \mathcal{B}, z_{i,j} \in \{x_1, ..., x_n, y_1, ..., y_l\}$

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$$R(x_1, x_2) = (x_2 < x_1).$$

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How to measure size of a pp-formula?

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Example

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Definition

For a relation R on a set A and a set of relations \mathcal{B} (basis) put

$$Q_{\mathcal{B}}(R) := \min\{Q(\Phi) \mid \Phi \text{ pp-defines R}\}\$$

 $C_{\mathcal{B}}(R) := \min\{C(\Phi) \mid \Phi \text{ pp-defines R}\}\$



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- we can use Universal Algebra Clone Theory Galois connection and so on

 $Q_{\mathcal{B}}(n) := \max\{Q(R) \mid R \text{ is an } n\text{-ary relation on } A\}$ $C_{\mathcal{B}}(n) := \max\{C(R) \mid R \text{ is an } n\text{-ary relation on } A\}$

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Basis

$$S_c = A^3 \setminus \{(c,c,c)\}, \ \mathcal{B} = \{S_c \mid c \in A\}$$

Claim

Any relation on A can be pp-defined over \mathcal{B} .

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Theorem [Bashirov, 2015]

$$|A|^{\frac{n-1}{3}} - n \le Q_{\mathcal{B}}(n) \le |A|^n (2|A|(n-3)+1)$$

$$\frac{|A|^n}{3\log_2(10n|A|^{n+3})} \le C_{\mathcal{B}}(n) \le |A|^n (2|A|(|A|-1)(n-3)+1)$$

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Tell me if you know better bounds

R. Willard, Testing expressibility is hard, in D. Cohen (Ed.): CP 2010, LNCS 6308, 9-23, 2010

For infinitely many n there exist a constraint language Γ_n and a relation R_n , both on a 22-element domain, such that $|R_n|=n$, R_n is expressible from Γ_n but every pp-definition of R_n instance expressing R_n has at least $2^{n/3}$ variables.

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Tell me if you know other results

My exponential lower bound

Basis

$$A = \{0, 1, 2\}, R_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & \cdot \\ 0 & 1 & 0 & 1 & \cdot & 2 \\ 0 & 0 & 0 & 1 & \cdot & \cdot \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & \cdot & \cdot \end{pmatrix},$$

$$\mathcal{B} = \{R_1, R_2\}$$

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$$\mathcal{B}=\{R_1,R_2\}$$

Relation σ_n

$$\sigma = \{0, 1, 2\}^2 \setminus \{(0, 1), (1, 0)\}$$

$$\sigma_n(x_1, y_1, x_2, y_2, \dots, x_n, y_n) = \sigma(x_1, y_1) \vee \dots \vee \sigma(x_n, y_n)$$

•
$$\sigma_n$$
 does not contain $(0, 1, 0, 1, 0, 1, \dots, 0, 1)$

$$(0,1,0,1,0,1,\dots,0,1)$$

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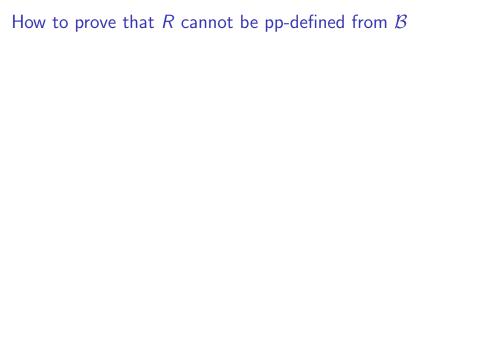
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Theorem

$$2^{n} \leq Q_{\mathcal{B}}(\sigma_{n}) \leq 2^{n}(n+2)$$

$$2^{n} \leq C_{\mathcal{B}}(\sigma_{n}) \leq 2^{n}(n+3)$$



▶ Find an operation f preserving \mathcal{B} but not preserving R.

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$$R(x_{1}, x_{2}, ..., x_{n}) = \exists y_{1} \exists y_{2} ... \exists y_{s} (R_{1}(...) \land \cdots \land R_{s}(...))$$

$$(a_{1}^{1}, a_{2}^{1}, ..., a_{n}^{1}) \in R$$

$$(a_{1}^{2}, a_{2}^{2}, ..., a_{n}^{2}) \in R$$

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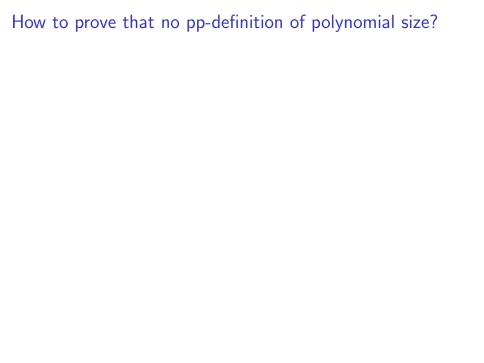
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▶ Find a partial operation f preserving \mathcal{B} but not preserving R.

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$$(c_1, c_2, \dots, c_n) \notin R \qquad d_1 \ * \dots d_s$$

Relation σ_n $\sigma = \{0, 1, 2\}^2 \setminus \{(0, 1), (1, 0)\}$ $\sigma_n(x_1, y_1, x_2, y_2, \dots, x_n, y_n) = \sigma(x_1, y_1) \vee \dots \vee \sigma(x_n, y_n)$

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Theorem

Suppose
$$\mathcal{B}$$
 is preserved by $f_n(x_1, \dots, x_n, y_1, \dots, y_n) = \begin{cases} *, & \text{if } \{x_1, \dots, x_n\} = \{0\} \text{ and } \{y_1, \dots, y_n\} = \{1\} \\ 0, & \text{if } \{x_1, \dots, x_n\} = \{0\} \text{ and } \{y_1, \dots, y_n\} \neq \{1\} \\ 1, & \text{if } \{x_1, \dots, x_n\} \in \{\{1\}, \{0, 1\}\} \text{ and } \{y_1, \dots, y_n\} = \{1\} \\ 2, & \text{otherwise} \end{cases}$

Then $Q_{\mathcal{B}}(\sigma_n) \geq 2^n$.

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Then $Q_{\mathcal{B}}(\sigma_n) \geq 2^n$.

Corollary

For $\mathcal{B} = \{R_1, R_2\}$ we have $Q_{\mathcal{B}}(\sigma_n) \geq 2^n$

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Corollary

Suppose $|\mathcal{B}| < \infty$, \mathcal{B} is preserved by all total operations from PartialClo($\{f_1, f_2, f_3, \dots\}$). Then $Q_{\mathcal{B}}(\sigma_n)$ is exponential on n.

Connection with Quantified Constraint Satisfaction Problem

$QCSP(\Gamma)$:

Given a sentence $\exists y_1 \forall x_1 \dots \exists y_t \forall x_t (R_1(\dots) \land R_s(\dots))$, where $R_1, \dots, R_s \in \Gamma$.

Decide whether it holds.

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Chen's Conjecture

If $Pol(\Gamma)$ has EGP property, then $QCSP(\Gamma)$ is PSPACE-complete.

*EGP - we need exponentially many tuples to generate A^n

Connection with Quantified Constraint Satisfaction Problem

$QCSP(\Gamma)$:

Given a sentence $\exists y_1 \forall x_1 \dots \exists y_t \forall x_t (R_1(\dots) \land R_s(\dots))$, where $R_1, \dots, R_s \in \Gamma$.

Decide whether it holds.

Chen's Conjecture

If $Pol(\Gamma)$ has EGP property, then $QCSP(\Gamma)$ is PSPACE-complete.

*EGP - we need exponentially many tuples to generate A^n

Counter-example

$$\Gamma = \left\{ \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & \cdot \\ 0 & 1 & 0 & 1 & \cdot & 2 \\ 0 & 0 & 1 & 1 & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & \cdot & \cdot \end{pmatrix} \right\}.$$

- Pol(Γ) has EGP property.
- QCSP(Γ) can be solved in polynomial time.

Thank you for your attention