PSpace-hard vs Π_2^P Dichotomy of the QCSP

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Question

What is the complexity of QCSP(Γ) for different Γ?

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Theorem [Schaefer 1978 + Creignou et al. 2001/ Dalmau 1997.]

Suppose Γ is a constraint language on $\{0, 1\}$. Then

- QCSP(Γ) is in P if Γ is preserved by an idempotent WNU operation,
- QCSP(Γ) is PSPACE-complete otherwise.

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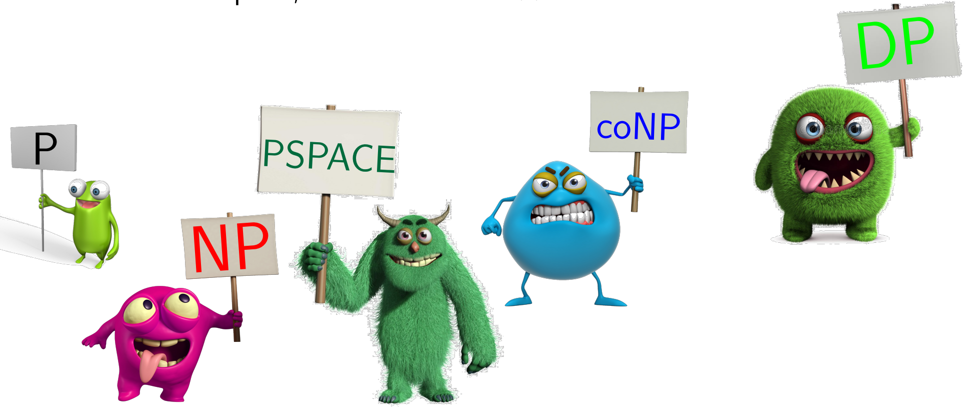
- there exists Γ on a 4-element domain such that QCSP(Γ) is DP-complete, where DP $=$ NP $\wedge$ coNP.



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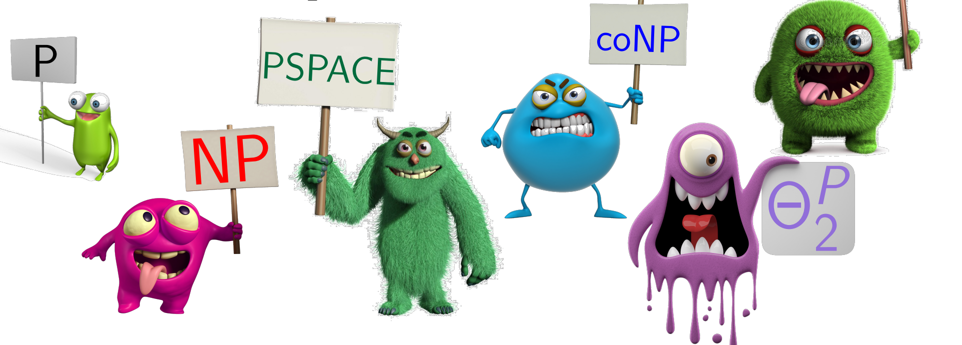
- there exists Γ on a 10-element domain such that QCSP(Γ) is Θ_2^P-complete.



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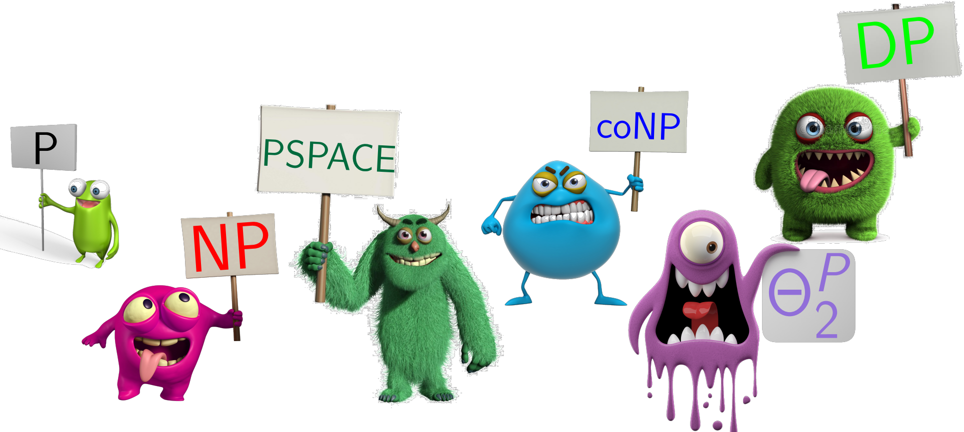

QCSP Complexity Classes

Theorem [Zhuk, Martin, 2019]

Suppose Γ is a constraint language on $\{0,1,2\}$ containing $\{x = a \mid a \in \{0,1,2\}\}$. Then QCSP($\Gamma$) is

- in P, or
- NP-complete, or
- coNP-complete, or
- PSPACE-complete.

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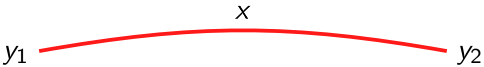

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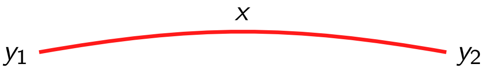

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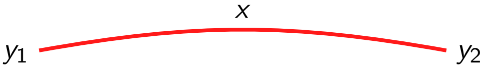

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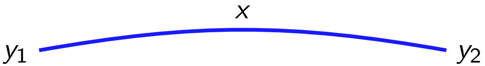

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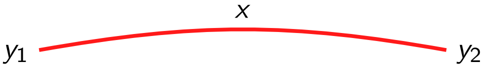

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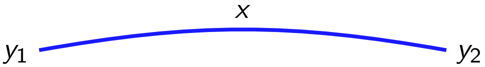
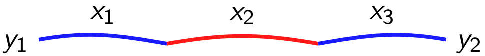


$$\exists u_1 \exists u_2 R_1(y_1, u_1, x_1) \wedge R_0(u_1, u_2, x_2) \wedge R_1(u_2, y_2, x_3)$$



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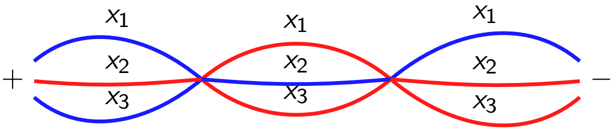

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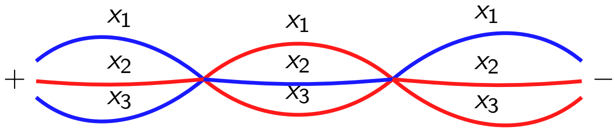

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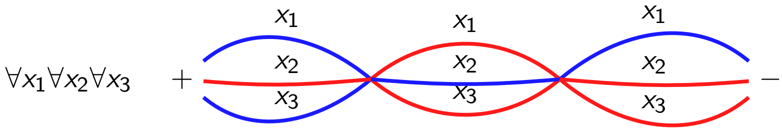

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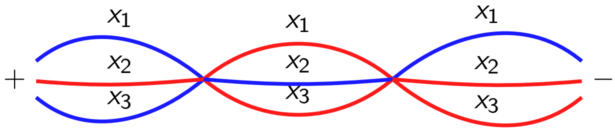

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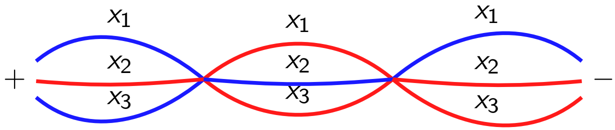
$\forall x_1 \forall x_2 \forall x_3$ 

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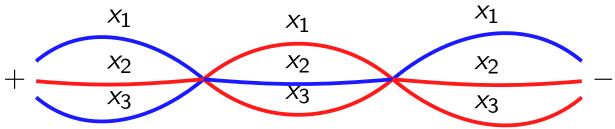

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Claim

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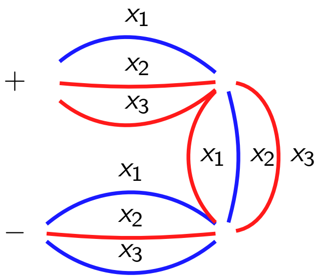

$$\neg\left((x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_2 \vee \overline{x}_3)\right)$$

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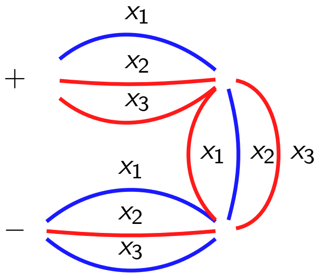

$$\forall x_1 \exists x_2 \forall x_3 \ \neg\big((x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_2 \vee \overline{x}_3)\big)$$

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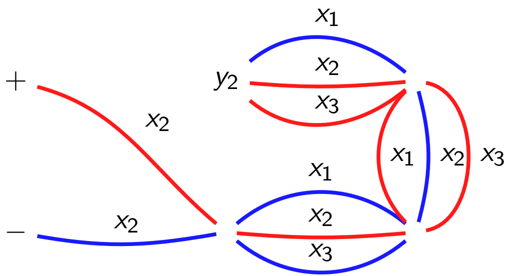

$$\forall x_1 \exists x_2 \forall x_3 \;\; \neg((x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_2 \vee \overline{x}_3))$$

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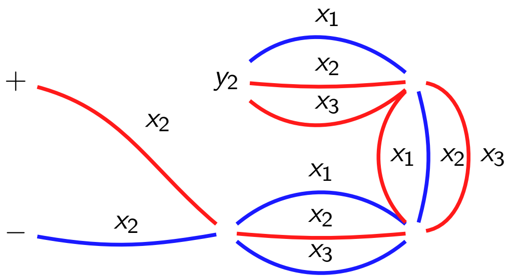

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$$\Updownarrow$$

$$\neg\big(\exists x_1 \forall x_2 \exists x_3 \;\; ((x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_2 \vee \overline{x}_3))\big)$$

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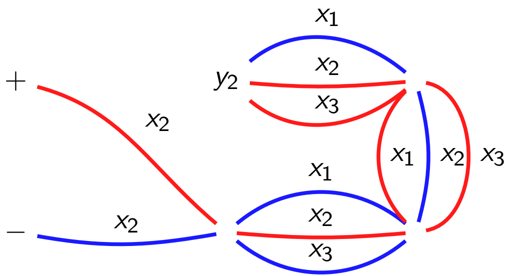

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Claim

QCSP(Γ) is PSpace-hard.

PSpace-hardness

PSpace-hardness

Let $A = \{+, -, \boxed{0}, \boxed{1}\}$, $\Gamma = \{R_0, R_1, \{+\}, \{-\}\}$.

$R_0(y_1, y_2, x) = (y_1, y_2 \in \{+, -\}) \wedge (y_1 = y_2 \vee x \neq 0)$

$R_1(y_1, y_2, x) = (y_1, y_2 \in \{+, -\}) \wedge (y_1 = y_2 \vee x \neq 1)$

PSpace-hardness

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Theorem

Suppose

1. Γ contains $\{x = a \mid a \in A\}$
2. QCSP(Γ) is PSpace-hard.

Then there exist

- $D \subseteq A$
- a nontrivial equivalence relation σ on D
- $B, C \subsetneq A$ with $B \cup C = A$

s.t. $\sigma(y_1, y_2) \vee B(x)$ and $\sigma(y_1, y_2) \vee C(x)$ are pp-definable over Γ.

QCSP Dichotomy

Theorem [Folklore]

CSP(Γ)

- is either NP-complete,
- or in P.

Theorem

QCSP(Γ)

- is either PSpace-complete,
- or in Π_2^P.

- Prove hardness
- Find fast algorithm

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Reduction to CSP

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QCSP Instance

$\Psi = \exists y_1 \forall x_1 \exists y_2 \forall x_2 \ldots \exists y_n \forall x_n \Phi.$

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Solution

Ψ holds $\Rightarrow \exists f_1, \ldots, f_n$ such that $y_i = f_i(x_1, \ldots, x_{i-1})$ satisfies Φ for every $x_1, \ldots, x_n$.

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- Denote $f_i(a_1, \ldots, a_{i-1})$ by $y_i^{a_1, \ldots, a_{i-1}}$

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- Denote $f_i(a_1, \ldots, a_{i-1})$ by $y_i^{a_1,\ldots,a_{i-1}}$
- Write all the constraints

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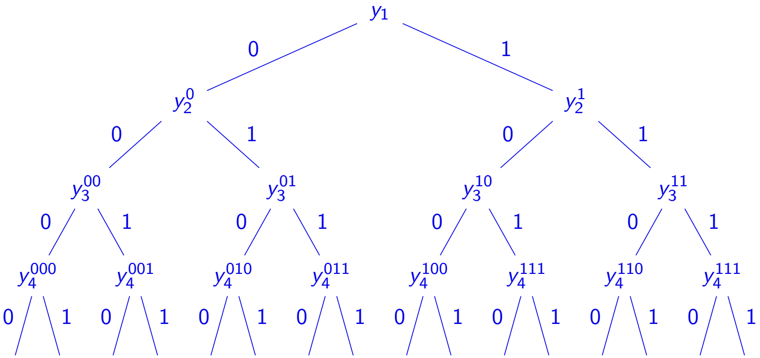

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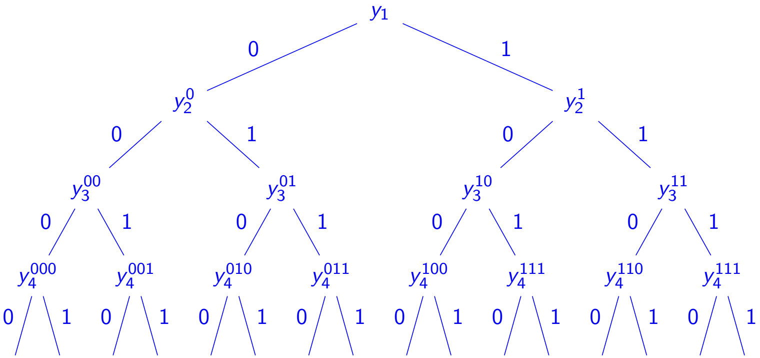

$R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0, 0, 0, 0) \wedge$

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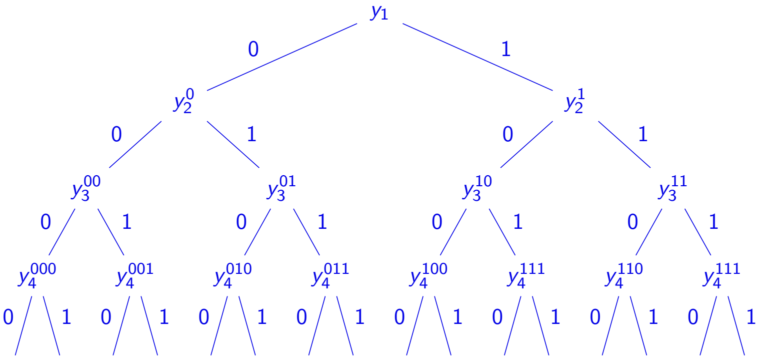

$R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0, 0, 0, 0) \wedge R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0, 0, 0, 1) \wedge$

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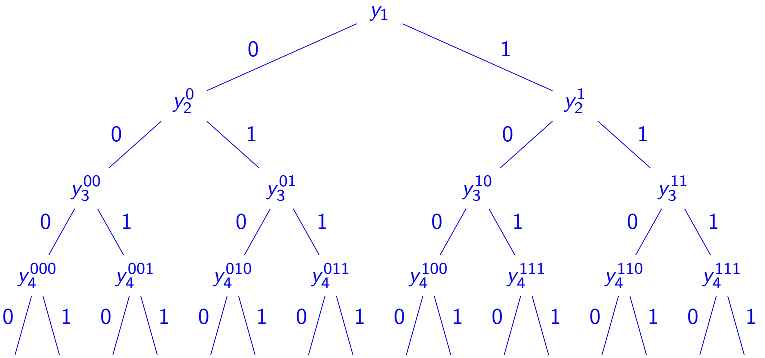

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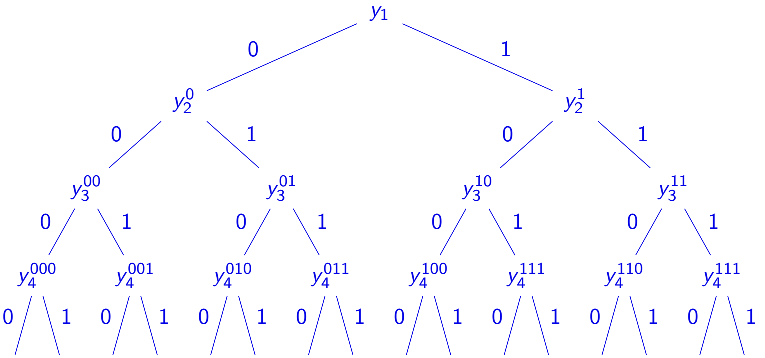

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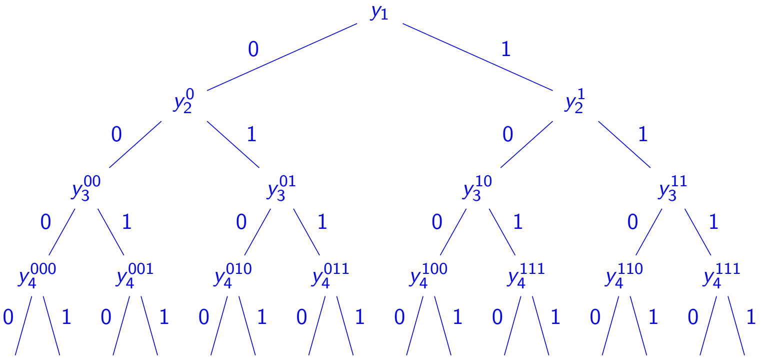

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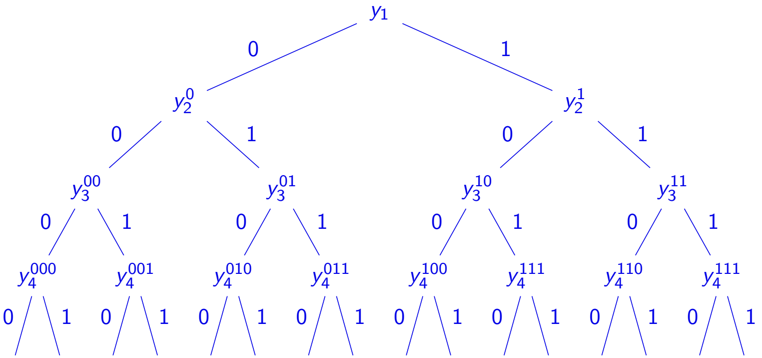

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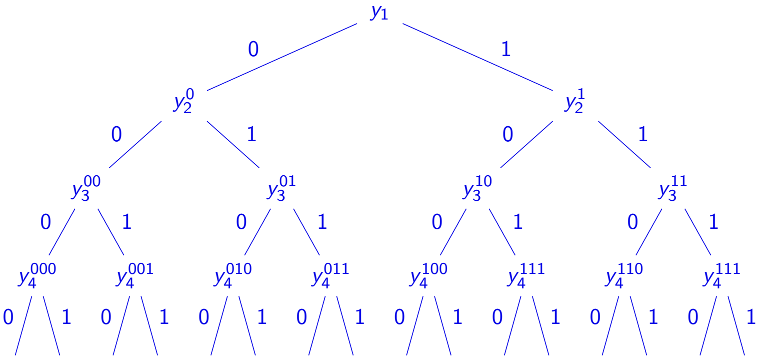

$$\begin{aligned}\mathrm{ExpCSP}_R^n = {} & R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0,0,0,0) \wedge R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0,0,0,1) \wedge \\ & R(y_1, y_2^0, y_3^{00}, y_4^{001}, 0,0,1,0) \wedge R(y_1, y_2^0, y_3^{00}, y_4^{001}, 0,0,1,1) \wedge \ldots \\ & \wedge R(y_1, y_2^1, y_3^{11}, y_4^{111}, 1,1,1,0) \wedge R(y_1, y_2^1, y_3^{11}, y_4^{111}, 1,1,1,1).\end{aligned}$$

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$$\mathcal{U}(Z) = \forall X^{|X|<p(|Z|)} \exists Y^{|Y|<q(|Z|)} \mathcal{V}(X, Y, Z),$$

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Theorem

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1. QCSP(Γ) is not PSpace-hard.
2. ExpCSP_R^n has no solutions

$\Rightarrow$ $\exists$ polynomial-size subinstance of ExpCSP_R^n without a solution.

$$\Psi \Leftrightarrow \forall \Omega \subseteq \mathrm{ExpCSP}_R^n \;{}^{|\Omega|<p(|\Phi|)}\; (\exists (y_1, y_2^0, y_2^1, y_3^{00}, \dots)\ \Omega)$$

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- $R(y_1, \ldots, y_n, x_1, \ldots, x_n) = \Phi$.
- $\widetilde{R}(y_1, \ldots, y_n, x_1, \ldots, x_n) =$

$\bigwedge_{a \in A, i=1,\ldots,n} (\exists y'_{i+1} \ldots \exists y'_n \; R(y_1, \ldots, y_i, y'_{i+1}, \ldots, y'_n, x_1, \ldots, x_i, a, \ldots, a))$.

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Solving $\text{ExpCSP}^n_{\widetilde{R}}$

- Given an sentence $\Psi = \exists y_1 \forall x_1 \exists y_2 \forall x_2 \dots \exists y_n \forall x_n \Phi$.
- $R(y_1, \dots, y_n, x_1, \dots, x_n) = \Phi$.
- $\widetilde{R}(y_1, \dots, y_n, x_1, \dots, x_n) =$

$$\bigwedge_{a \in A, i=1,\dots,n} (\exists y'_{i+1} \dots \exists y'_n \; R(y_1, \dots, y_i, y'_{i+1}, \dots, y'_n, x_1, \dots, x_i, a, \dots, a)).$$

- Ψ is equivalent to ExpCSP_R^n and to $\text{ExpCSP}_{\widetilde{R}}^n$.

Solving $\text{ExpCSP}_{\widetilde{R}}^n$

Check 1-consistency.

- Given an sentence $\Psi = \exists y_1 \forall x_1 \exists y_2 \forall x_2 \ldots \exists y_n \forall x_n \Phi$.
- $R(y_1, \ldots, y_n, x_1, \ldots, x_n) = \Phi$.
- $\widetilde{R}(y_1, \ldots, y_n, x_1, \ldots, x_n) =$

$$\bigwedge_{a \in A, i=1,\ldots,n} (\exists y'_{i+1} \ldots \exists y'_n \; R(y_1, \ldots, y_i, y'_{i+1}, \ldots, y'_n, x_1, \ldots, x_i, a, \ldots, a)).$$

- Ψ is equivalent to ExpCSP^n_R and to $\mathrm{ExpCSP}^n_{\widetilde{R}}$.

Solving $\mathrm{ExpCSP}^n_{\widetilde{R}}$

Check 1-consistency. If not, we seek for 1-consistency.

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Solving $\text{ExpCSP}^n_{\widetilde{R}}$

Check 1-consistency. If not, we seek for 1-consistency.

- no 1-consistent reduction $\Rightarrow$

No 1-consistent reduction

No 1-consistent reduction

- Consider a tree-instance of $\mathrm{CSP}(\widetilde{R})$ giving a contradiction.

No 1-consistent reduction

- Consider a tree-instance of CSP($\widetilde{R}$) giving a contradiction.
- Strengthen/Relax/Remove constraints while no solutions

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  $R(y_1, \ldots, y_4, 0, 0, 1, 0)$

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$$\begin{aligned} &R(y_1, \ldots, y_4, 0, 0, 1, 0) \\ \forall x\ &R(y_1, \ldots, y_4, 0, 0, 1, x) \end{aligned}$$

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$$
\begin{aligned}
&R(y_1,\ldots,y_4,0,0,1,0)\\
\forall x\ &R(y_1,\ldots,y_4,0,0,1,x)\\
\exists y_4 \forall x\ &R(y_1,\ldots,y_4,0,0,1,x)\\
\forall x \exists y_4\ &R(y_1,\ldots,y_4,0,0,1,x)\\
\forall x' \forall x \exists y_4\ &R(y_1,\ldots,y_4,0,0,x',x)
\end{aligned}
$$

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$$\exists y_2 \forall x \exists y_3 \exists y_4\ R(y_1, \ldots, y_4, 0, x, x, x)$$

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$$\forall x \exists y_2 \exists y_3 \exists y_4\ R(y_1,\dots,y_4,0,x,x,x)$$

- If there exists a path of length $> 2^{2|A|}$, then we can pp-define a relations $\sigma(y_1,y_2) \vee B(x)$ and $\sigma(y_1,y_2) \vee C(x)$.

No 1-consistent reduction

- Consider a tree-instance of $\text{CSP}(\widetilde{R})$ giving a contradiction.
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- If any path is of length $< 2^{2|A|}$, then the tree-instance is of polynomial size.

No 1-consistent reduction

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- Ψ is equivalent to ExpCSP^n_R and to $\text{ExpCSP}^n_{\widetilde{R}}$.

Solving $\text{ExpCSP}^n_{\widetilde{R}}$

Check 1-consistency. If not, we seek for 1-consistency.

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Solving $\mathrm{ExpCSP}^n_{\widetilde{R}}$

Check 1-consistency. If not, we seek for 1-consistency.

- no 1-consistent reduction $\Rightarrow$exists a polynomial witness (L1).

Lemma 1

$\mathrm{ExpCSP}^n_{\widetilde{R}}$ has no 1-consistent reduction $\Rightarrow$ polynomial-size subinstance of ExpCSP^n_R witnesses this.

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Solving $\mathrm{ExpCSP}^n_{\widetilde{R}}$

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Solving $\text{ExpCSP}^n_{\widetilde{R}}$

Check 1-consistency. If not, we seek for 1-consistency.

- no 1-consistent reduction $\Rightarrow$exists a polynomial witness (L1).
- exists a 1-consistent reduction $\Rightarrow$ exists a solution (L2).

Lemma 1

$\text{ExpCSP}^n_{\widetilde{R}}$ has no 1-consistent reduction $\Rightarrow$ polynomial-size subinstance of ExpCSP^n_R witnesses this.

Lemma 2

$\text{ExpCSP}^n_{\widetilde{R}}$ has a 1-consistent reduction $\Rightarrow$ $\text{ExpCSP}^n_{\widetilde{R}}$ has a solution.

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B is a nice subuniverse of D

Lemma 2

$\mathrm{ExpCSP}^n_{\widetilde{R}}$ has a 1-consistent reduction $\Rightarrow$ $\mathrm{ExpCSP}^n_{\widetilde{R}}$ has a solution.

B is a nice subuniverse of D if there exists $\mathcal{U} \leq D \times A^n$ s.t.

1. $(\forall x_1 \ldots \forall x_s\, \mathcal{U}(y, x_1, \ldots, x_s)) = (y \in B)$
2. $(\forall x\, \mathcal{U}(y, x, \ldots, x)) = (y \in D)$

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Lemma 3

Suppose

- $\mathrm{ExpCSP}^n_{\widetilde{R}}$ has no solutions
- $D_1, D_2^0, D_2^1, \ldots, D_n^{11\ldots,1}$ is a 1-consistent reduction of $\mathrm{ExpCSP}^n_{\widetilde{R}}$.

Then there exists a nice subuniverse on some D_i^{α}.

Lemma 2

$\text{ExpCSP}^n_{\widetilde{R}}$ has a 1-consistent reduction $\Rightarrow$ $\text{ExpCSP}^n_{\widetilde{R}}$ has a solution.

B is a nice subuniverse of D if there exists $\mathcal{U} \leq D \times A^n$ s.t.

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Then there exists a nice subuniverse on some D_i^{α}.

Lemma 4

Suppose

- $D_1, D_2^0, D_2^1, \dots, D_n^{11\dots,1}$ is a 1-consistent reduction of $\text{ExpCSP}^n_{\widetilde{R}}$.
- there exists a proper nice subuniverse on some D_i^{α}.

Then there exists a 1-consistent reduction $B_1, B_2^0, B_2^1, \dots, B_n^{11\dots,1}$ of $\text{ExpCSP}^n_{\widetilde{R}}$ s.t. B_i^{α} is a nice subuniverse of D_i^{α} for all i, α.

- Given an sentence $\Psi = \exists y_1 \forall x_1 \exists y_2 \forall x_2 \dots \exists y_n \forall x_n \Phi$.
- $R(y_1, \dots, y_n, x_1, \dots, x_n) = \Phi$.
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Solving $\text{ExpCSP}^n_{\widetilde{R}}$ (an instance of $\text{CSP}(\widetilde{R})$)

Check 1-consistency. If not, we seek for 1-consistency.

- no 1-consistent reduction $\Rightarrow$ exists a polynomial witness (L1).
- exists a 1-consistent reduction $\Rightarrow$ there exists a solution (L2).

Lemma 1

$\text{ExpCSP}^n_{\widetilde{R}}$ has no 1-consistent reduction $\Rightarrow$ polynomial size subinstance of ExpCSP^n_R witnesses this.

Lemma 2

$\text{ExpCSP}^n_{\widetilde{R}}$ has a 1-consistent reduction $\Rightarrow$ $\text{ExpCSP}^n_{\widetilde{R}}$ has a solution.

Theorem

Suppose

1. QCSP(Γ) is not PSpace-hard.
2. ExpCSP_R^n has no solutions

$\Rightarrow \exists$ polynomial-size subinstance of ExpCSP_R^n without a solution.

Solving $\text{ExpCSP}_{\widetilde{R}}^n$ (an instance of $\text{CSP}(\widetilde{R})$)

Check 1-consistency. If not, we seek for 1-consistency.

- no 1-consistent reduction $\Rightarrow$ exists a polynomial witness (L1).
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$\text{ExpCSP}_{\widetilde{R}}^n$ has a 1-consistent reduction $\Rightarrow$ $\text{ExpCSP}_{\widetilde{R}}^n$ has a solution.

Theorem (Π_2^P vs PSpace)

QCSP(Γ)

- *is either PSpace-hard*
- *or in Π_2^P.*

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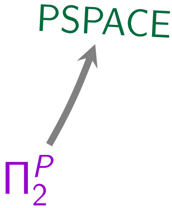

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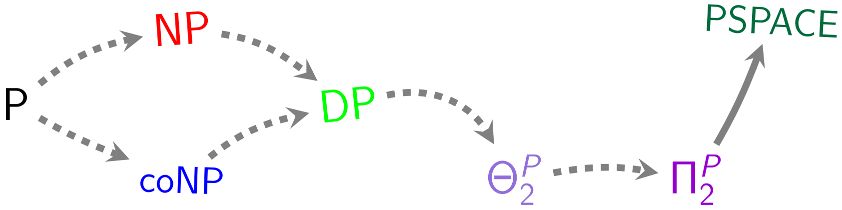

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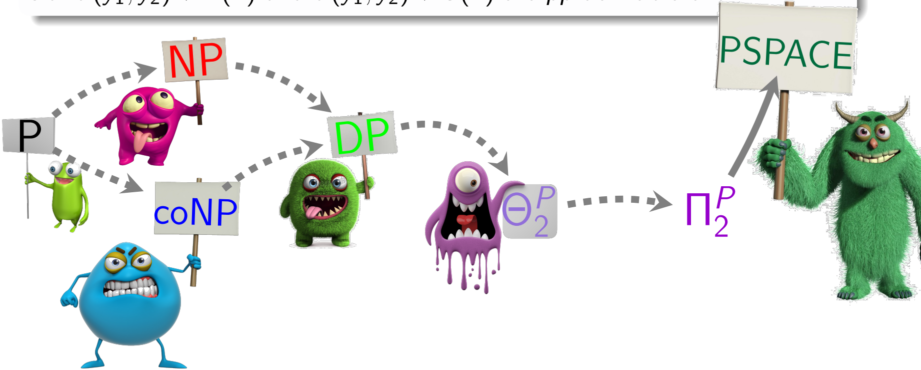

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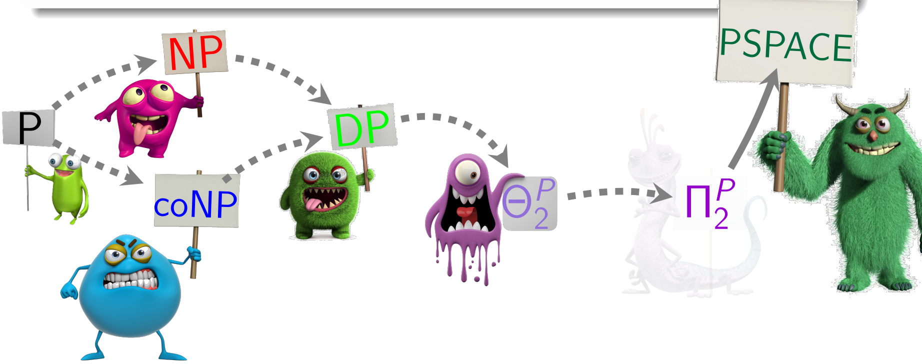

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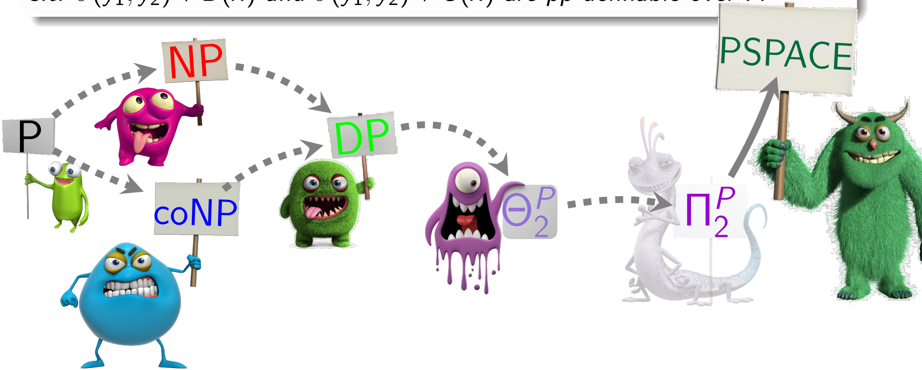

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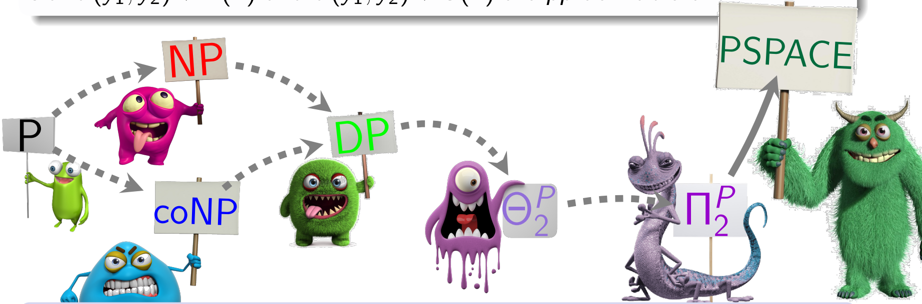

Lemma

There exists Γ on a 6-element set such that QCSP(Γ) is Π_2^P-complete.

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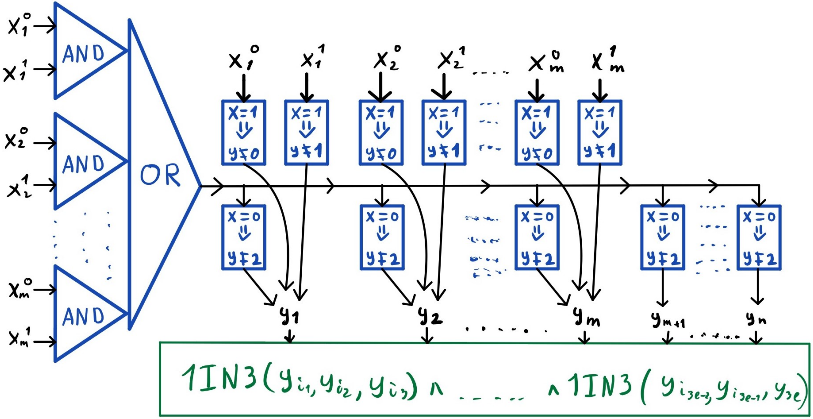
$\forall x_1^0 \forall x_1^1 \forall x_2^0 \forall x_2^1 \ldots \forall x_m^0 \forall x_m^1 \exists y_1 \exists y_2 \ldots \exists y_n$



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$$\mathrm{1IN3} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

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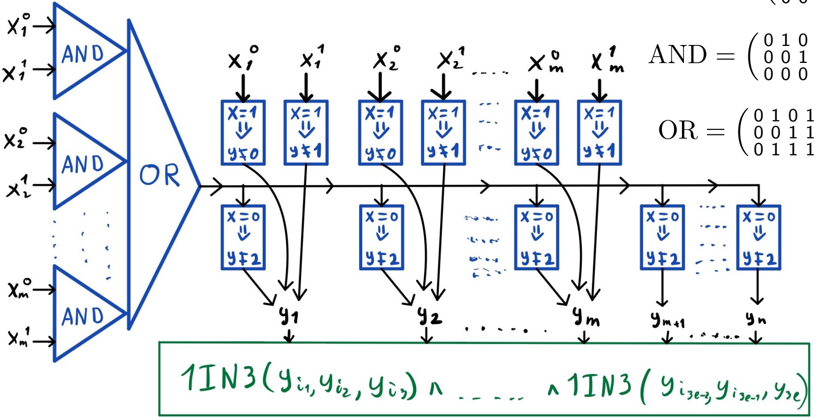

Π_2^P-example

Π_2^P-complete problem on $\{0,1\}$

$\forall x_1 \dots \forall x_m \exists x_{m+1} \dots \exists x_n \; 1\mathrm{IN}3(x_{i_1}, x_{i_2}, x_{i_3}) \wedge \dots \wedge 1\mathrm{IN}3(x_{i_{3l-2}}, x_{3l-1}, x_{3l})$

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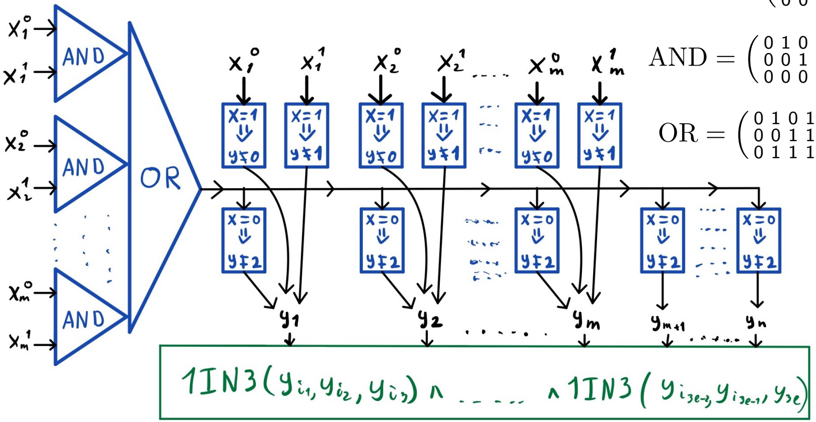
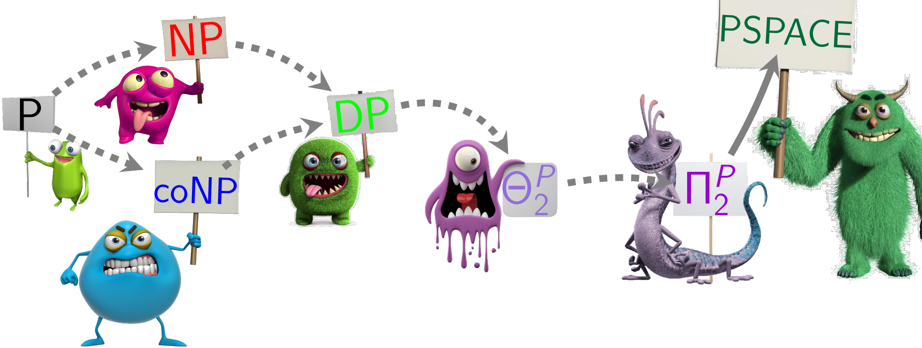
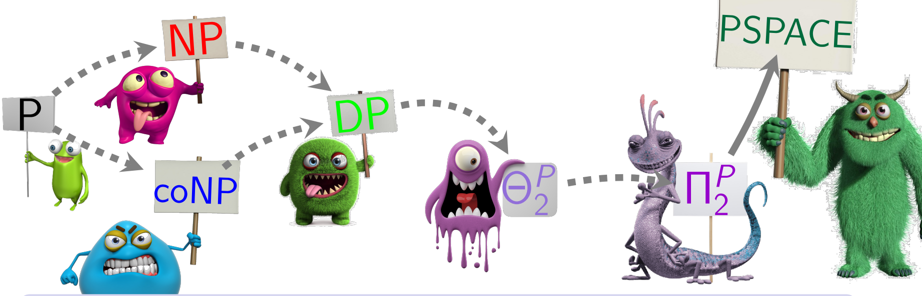



P
NP
coNP
DP
Θ_2^P
Π_2^P
PSPACE

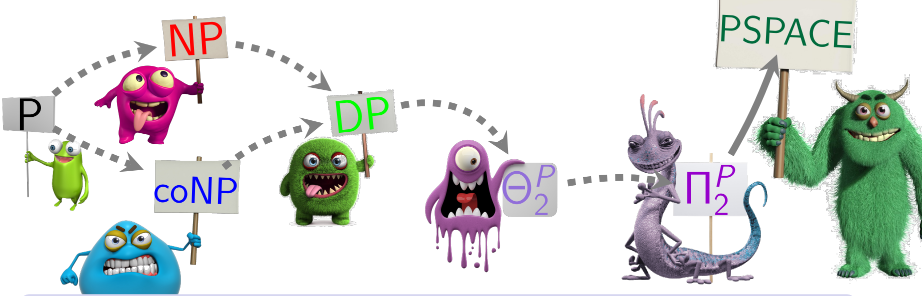


QCSP Hepta-chotomy

P: All moves are trivial.
NP: Only EP plays, the play of UP is trivial.
coNP: Only UP plays, the play of EP is trivial.
DP = **NP** $\wedge$ **coNP**: Each plays its own game. Yes-instance: EP wins and UP loses.
$\mathbf{\Theta_2^P}$ = (**NP** $\vee$ **coNP**) $\wedge \cdots \wedge$ (**NP** $\vee$ **coNP**): Each plays many games (no interaction). Yes-instance: any boolean combination.

$\mathbf{\Pi_2^P}$: First, UP plays, then EP plays.

PSpace: EP and UP play against each other. No restrictions.



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Thank you for your attention