Constraint Satisfaction Problem: what makes the problem easy

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International Congress of Matematicians

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CoCoSym: Symmetry in Computational Complexity

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Example

Check whether there exists a solution $x_1, x_2, x_3, \ldots \in \{0, 1\}$.

$$\begin{cases} x_1 + x_2 + x_3 = 0 \mod 2 \\ x_1 + x_3 + x_5 = 0 \mod 2 \\ x_2 + x_4 + x_5 = 0 \mod 2 \\ x_2 + x_3 + x_5 = 1 \mod 2 \end{cases}$$

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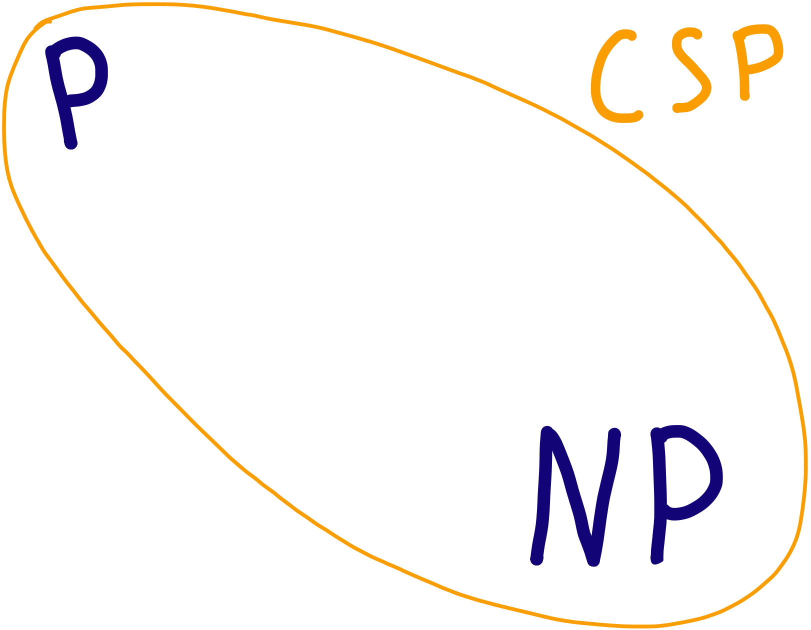
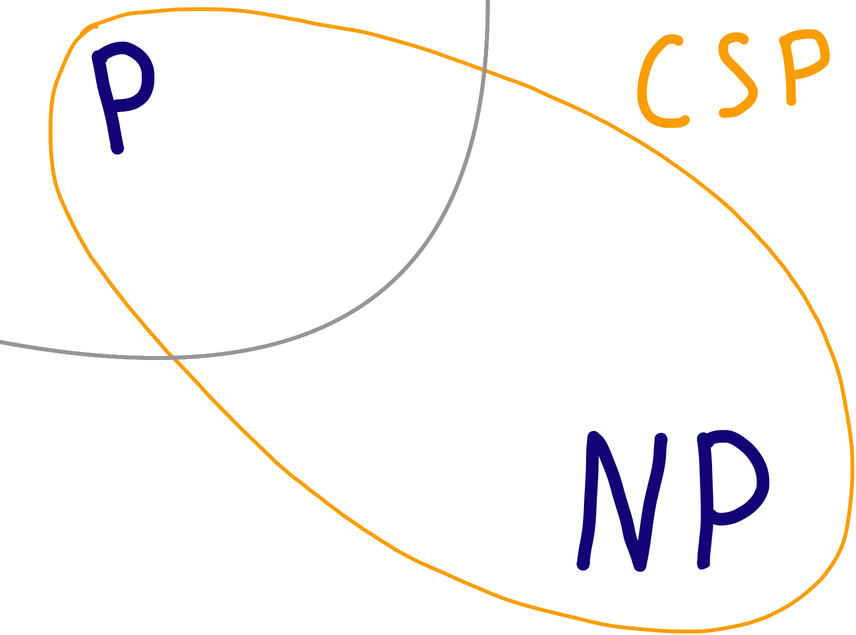
What is the complexity of this problem? Nobody knows!

P

P

NP


CSP
P
NP


P
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What is CSP?

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Constraint Satisfaction Problem
is a triple $\langle \mathbf{X}, \mathbf{D}, \mathbf{C} \rangle$, where

- $\mathbf{X} = \{x_1, \ldots, x_n\}$ is a set of variables,
- $\mathbf{D} = \{D_1, \ldots, D_n\}$ is a set of the respective domains of values, and
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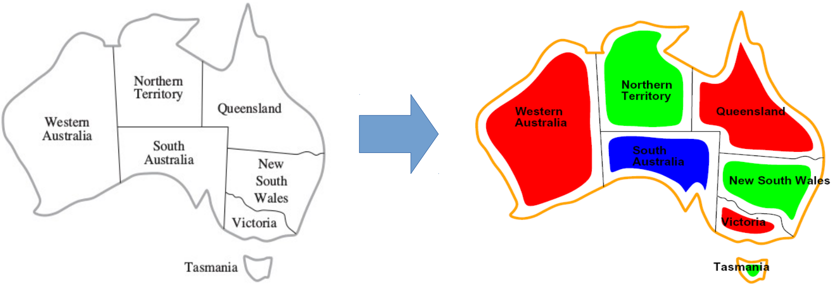
CSP example: map coloring



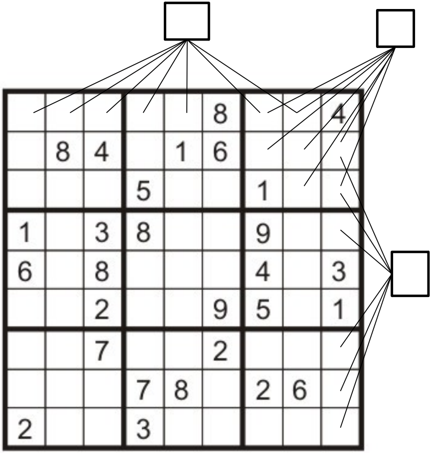
Problem: assign each territory a color such that no two adjacent territories have the same color

Variables: $X = \{WA, NT, Q, NSW, V, SA, T\}$

Domain of variables: $D = \{r, g, b\}$

Constraints: $C = \{SA \neq WA, SA \neq NT, SA \neq Q, \ldots\}$

Another example: sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

9-way alldiff for each column

9-way alldiff for each row

9-way alldiff for each region

(or can have a bunch of pairwise inequality constraints)

Slide: Berkeley CS188 course notes (downloaded Summer 2015)

Constraint Satisfaction Problem parameterized by a constraint language

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Γ is a set of relations on a finite set A.

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CSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

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$A = \{0, 1, 2\}, \Gamma = \{x < y, x \leq y\}$.

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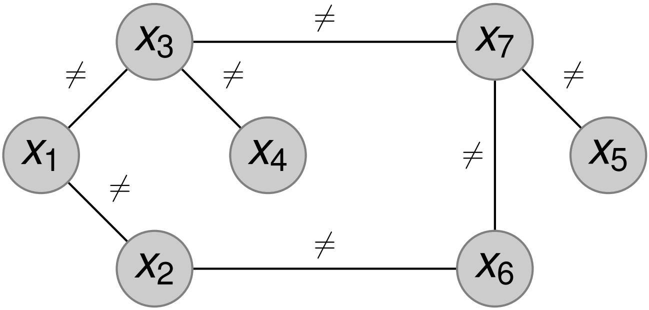
Question

What is the complexity of CSP(Γ) for different Γ?

Graph coloring (two colors)



Domain $D = \{$■, ■$\}$
Constraint language $\Gamma = \{\neq\}$.

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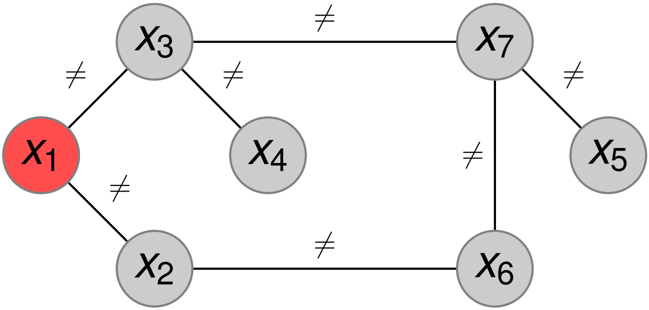

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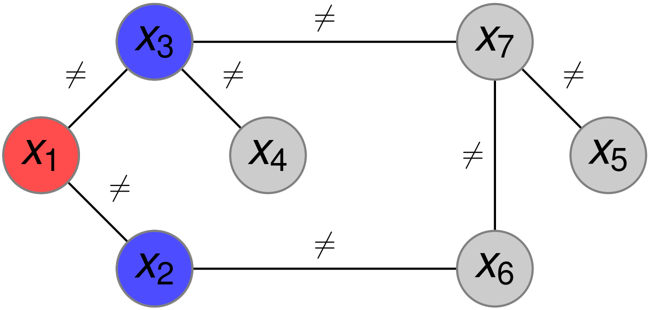

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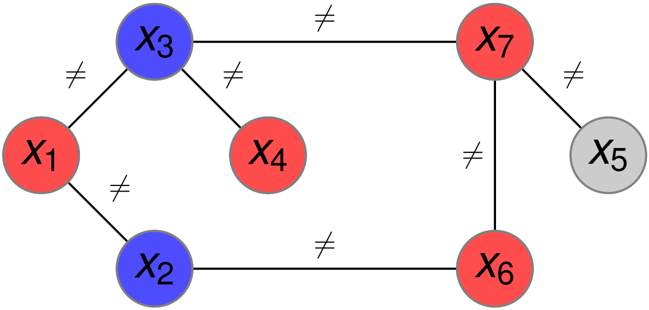

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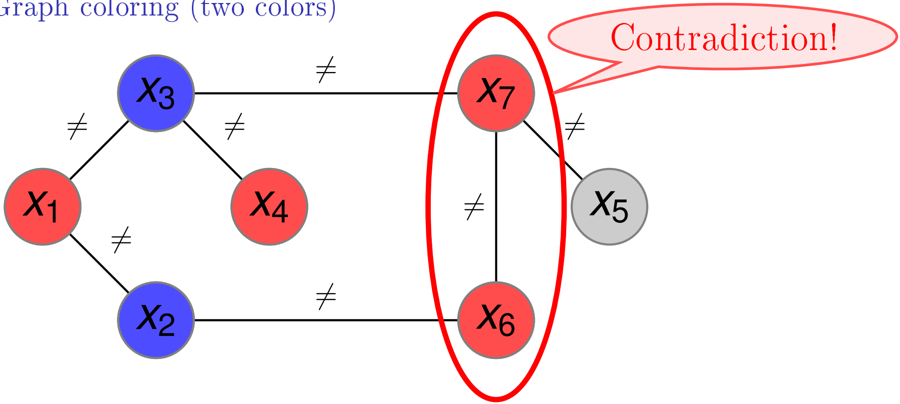

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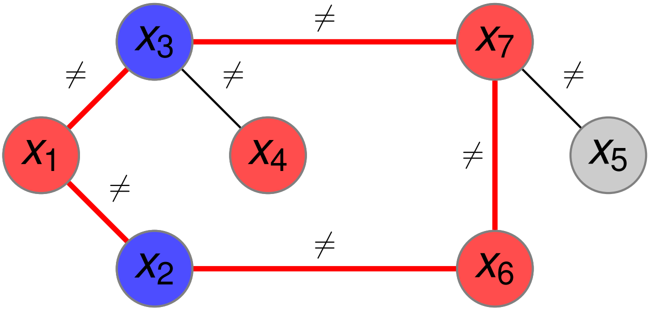

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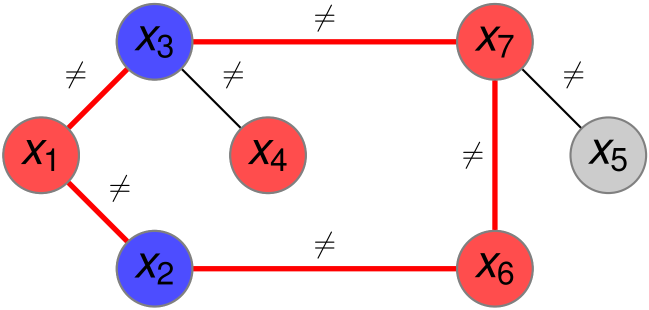

- Either we can color every vertex,
- or we can find an odd cycle.

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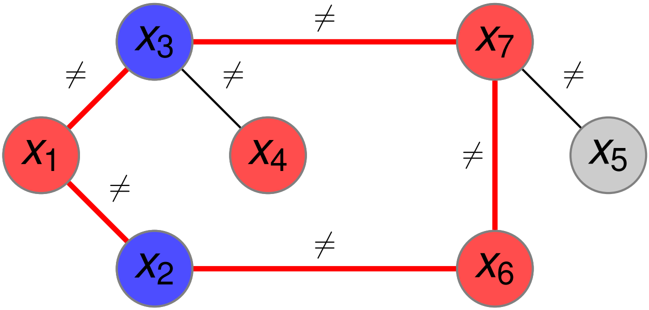

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Local consistency check solves the problem.

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System of linear equations in a finite field

$$\begin{cases} x_1 + x_2 + 2x_3 = 0 \mod 3 \\ x_1 + 2x_3 + x_5 = 0 \mod 3 \\ 2x_2 + x_4 + x_5 = 0 \mod 3 \\ x_1 + x_3 + 2x_5 = 1 \mod 3 \end{cases}$$

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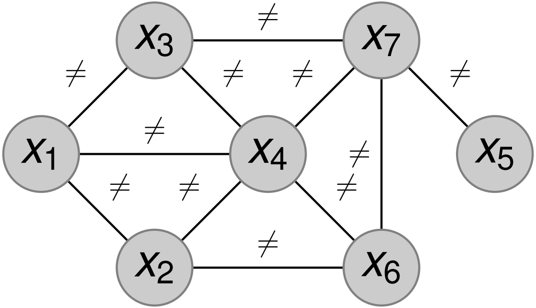

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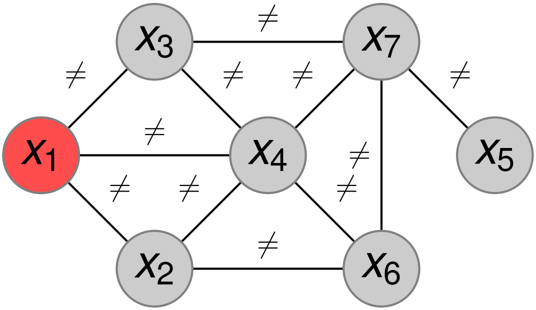

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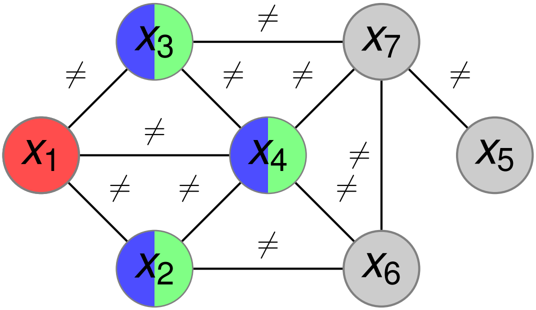
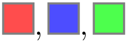


Domain $D = \{$$\}$
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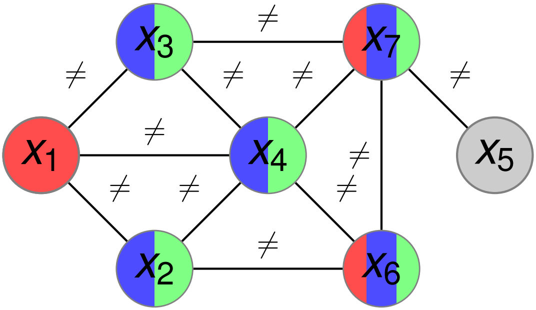

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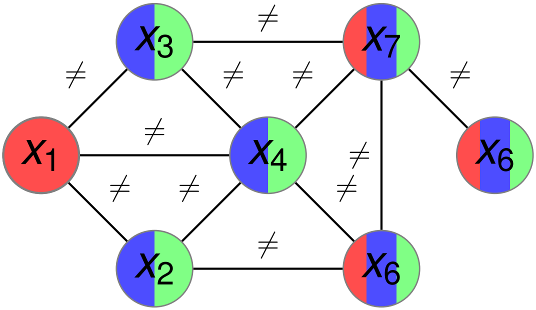

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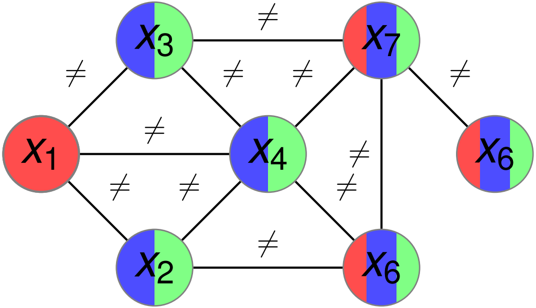

- Local consistency check doesn't give a contradiction.

Domain $D = \{$■, ■, ■$\}$
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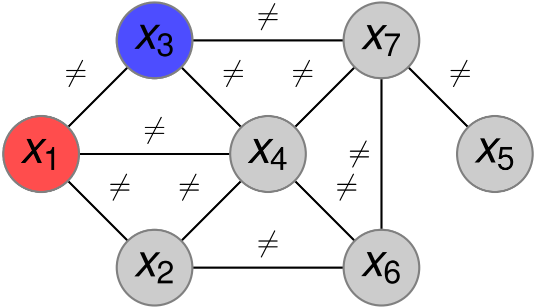

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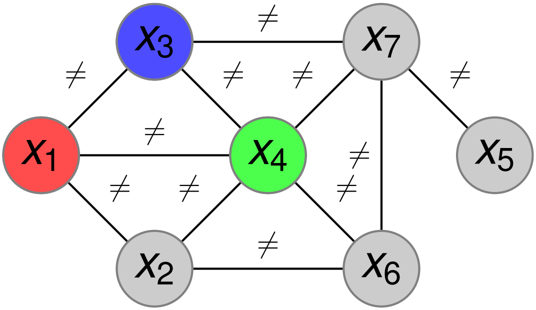

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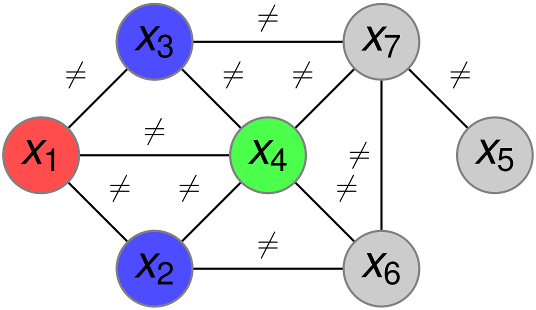

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Domain $D = \{$ red, blue, green $\}$
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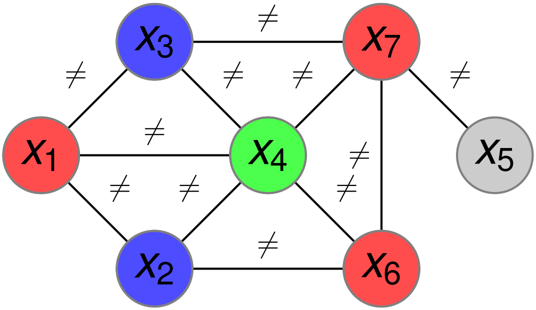
Graph coloring (three colors)



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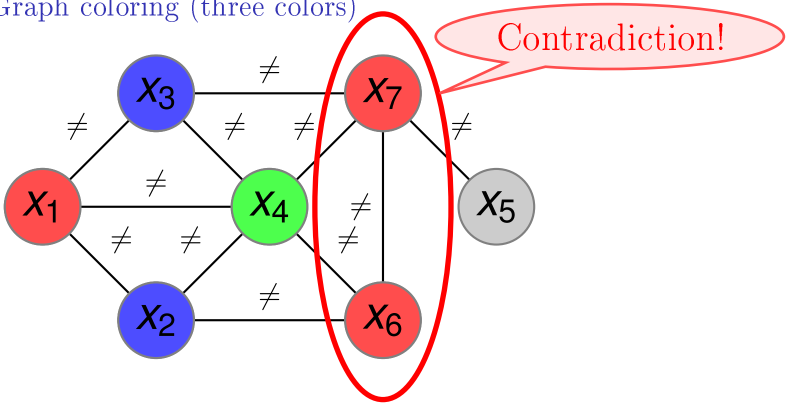
Graph coloring (three colors)



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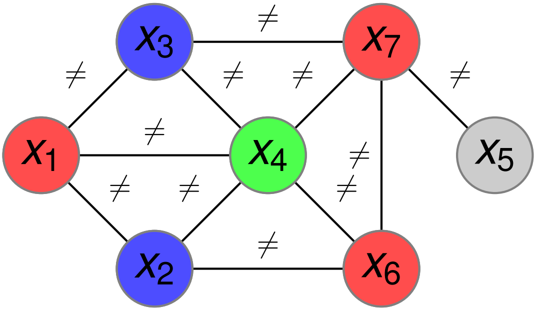

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- The instance has no solutions.

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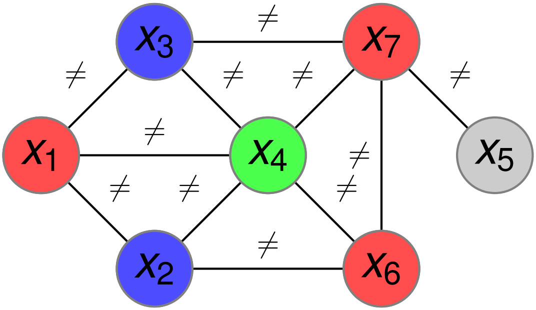

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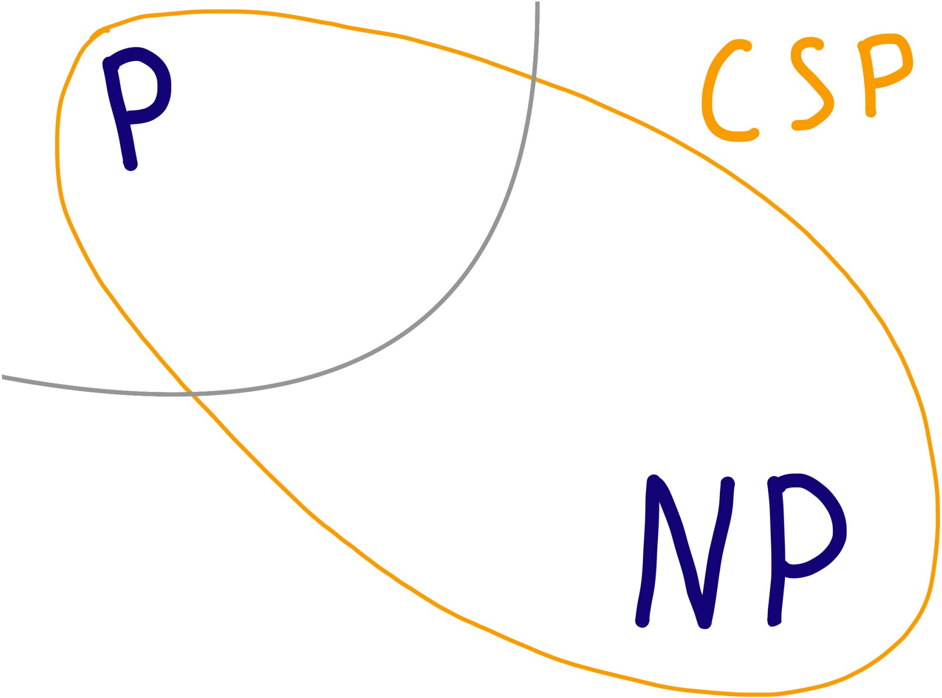
P
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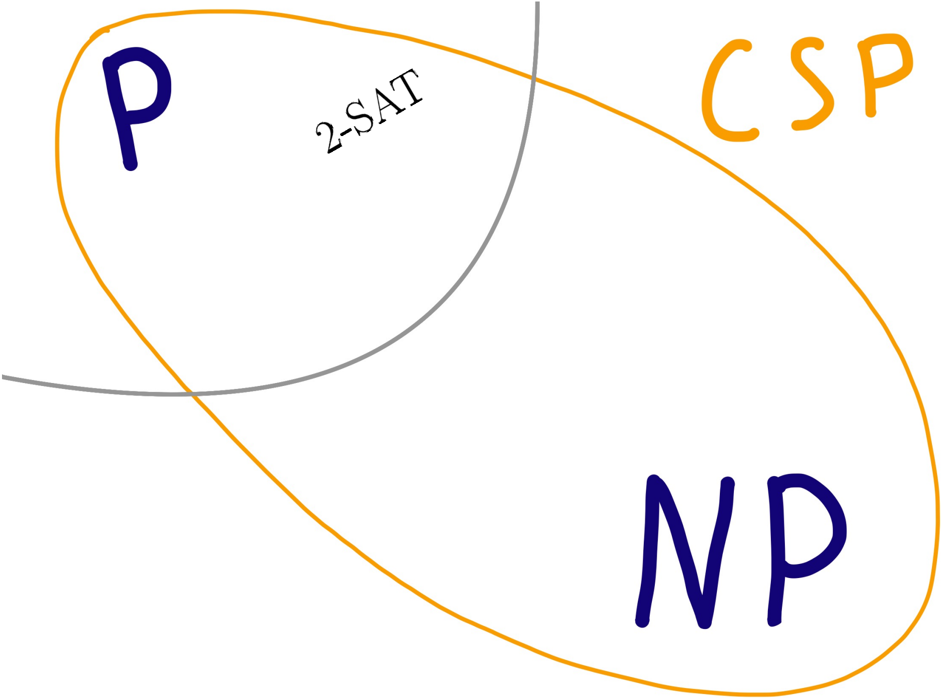
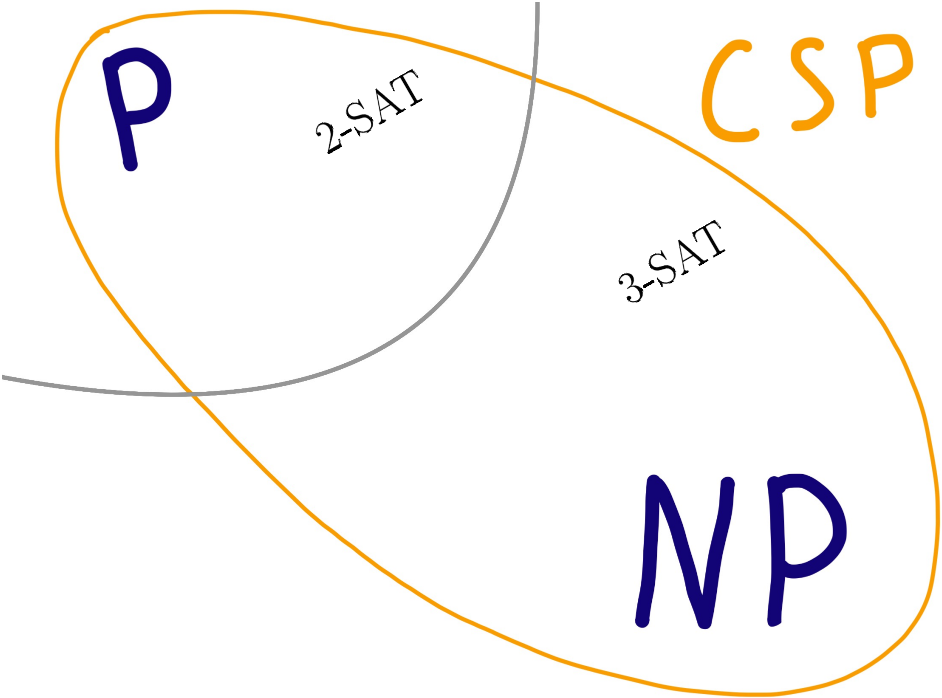

P
2-SAT
CSP
NP


P
2-SAT
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3-SAT
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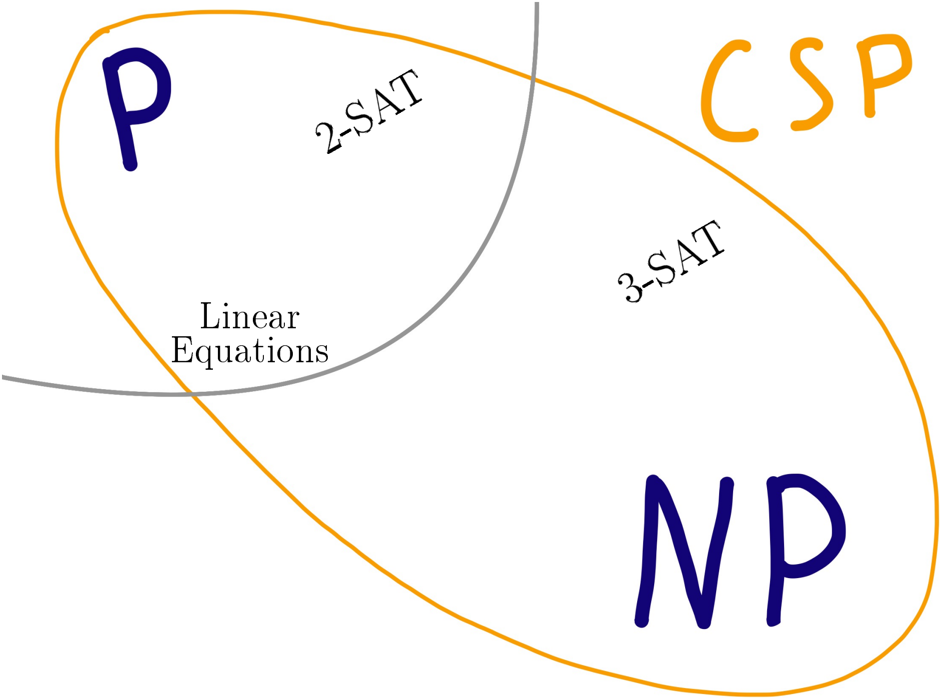
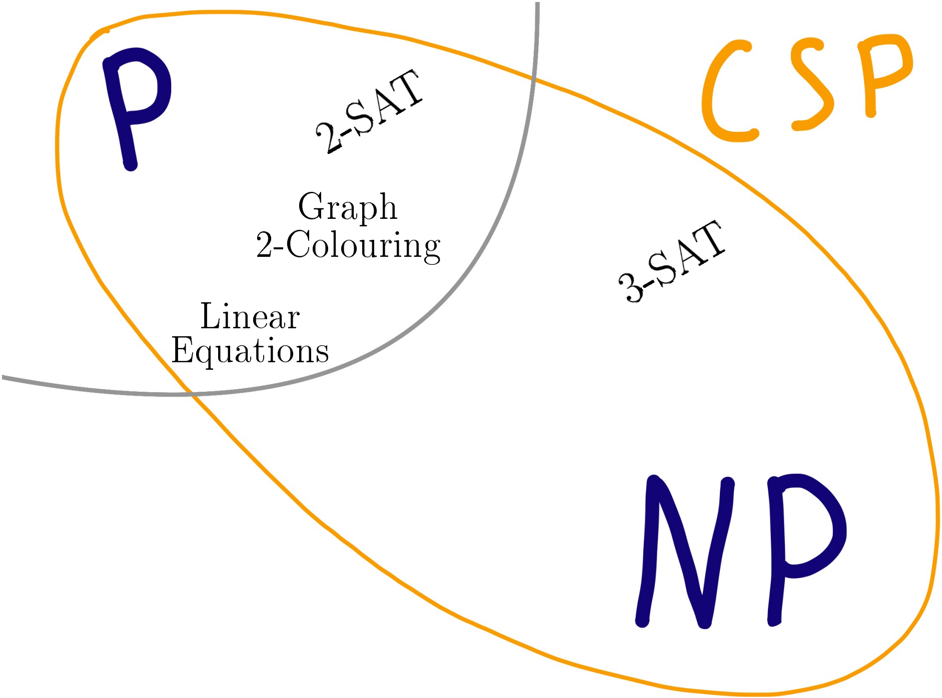
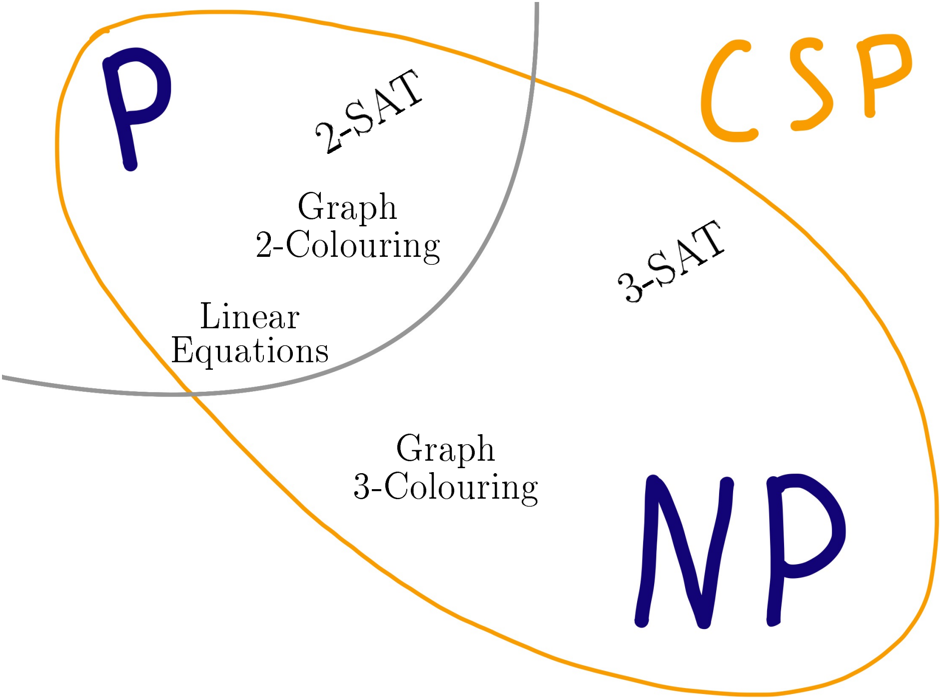

P
2-SAT
CSP
3-SAT
Linear Equations
NP


P
CSP
2-SAT
Graph 2-Colouring
3-SAT
Linear Equations
NP



CSP
P
2-SAT
Graph
2-Colouring
Linear
Equations
3-SAT
Graph
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NP

Reduction from one language to another

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CSP(Γ)

Given: a sentence

$$\exists x_1 \ldots \exists x_n \, R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether it holds.

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Theorem [Bodnarchuk, Kaluzhnin, Kotov, Romov, Geiger, 1969]

Γ_2 pp-defines Γ_1 IFF every operation preserving Γ_2 preserves Γ_1

Polymorphisms

Polymorphisms

An operation f preserves a relation R,
(equivalently, f is a polymorphism of R)
if for all $\begin{pmatrix} a_1^1 \\ \vdots \\ a_1^s \end{pmatrix}, \ldots, \begin{pmatrix} a_n^1 \\ \vdots \\ a_n^s \end{pmatrix} \in R$,

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or, equivalently, f is monotone.

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CSP Dichotomy Theorem [Bulatov, Zhuk, 2017]

CSP(Γ) is solvable in polynomial time if there is a WNU operation preserving Γ; it is NP-complete otherwise.

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A weak near unanimity operation (WNU) is an operation f satisfying $f(x, \ldots, x, y) = f(x, \ldots, x, y, x) = \cdots = f(y, x, \ldots, x)$.

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Examples: $x \vee y, x \wedge y, xy \vee xz \vee yz, x + y + z, 0, \min(x, y), \dots$

Hardness part

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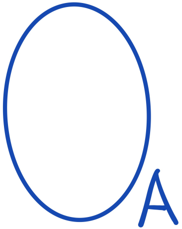

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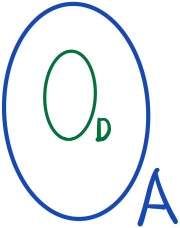

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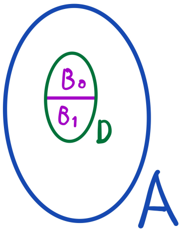

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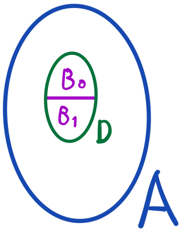

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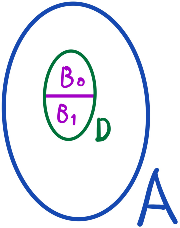

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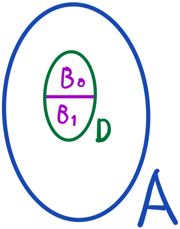

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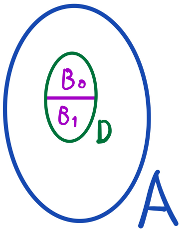

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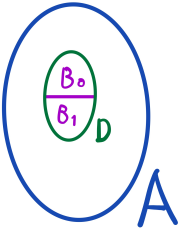

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- CSP(NAE3) is log-space reducible to CSP(Γ)
- CSP(Γ) is NP-complete.
- CSP(Δ) is log-space reducible to CSP(Γ) for any finite constraint language Δ.

Tractable part

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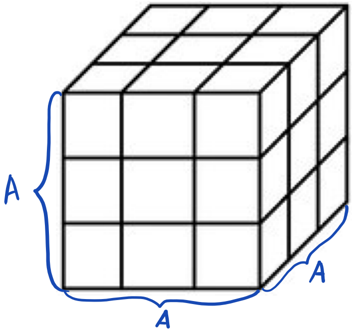

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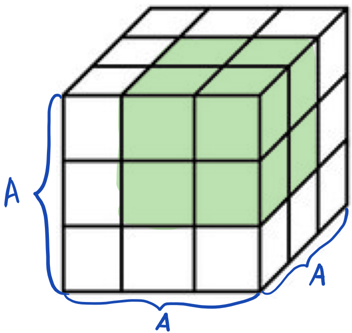

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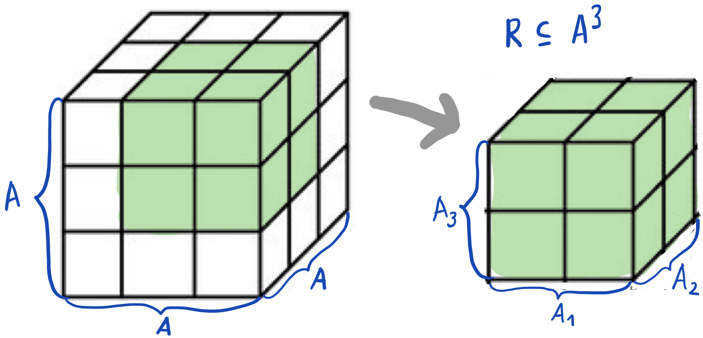

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Toy Algorithm

1. If $x = c$ then substitute and simplify.

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2. $(x = a \vee y = b)$ and $(y = c \vee z = d)$, where $b \neq c$, produce $(x = a \vee z = d)$

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3. $(x = a \vee x = a)$ produces $x = a$.
4. Take all linear equations L, solve them if $L \Rightarrow (x_i = c)$ produce $x_i = c$.

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5. Repeat steps 1-4.

Tractable part

$$(x_1 + \cdots + x_m = c) \vee (x_{m+1} = d_{m+1}) \vee \cdots \vee (x_n = d_n)$$

Toy Algorithm

1. If $x = c$ then substitute and simplify.
2. $(x = a \vee y = b)$ and $(y = c \vee z = d)$, where $b \neq c$, produce $(x = a \vee z = d)$
3. $(x = a \vee x = a)$ produces $x = a$.
4. Take all linear equations L, solve them if $L \Rightarrow (x_i = c)$ produce $x_i = c$.
5. Repeat steps 1-4.

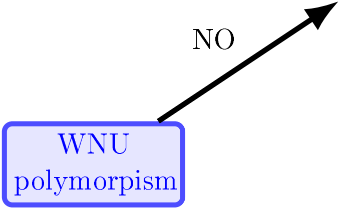
- Real algorithm is much harder.

To sum up

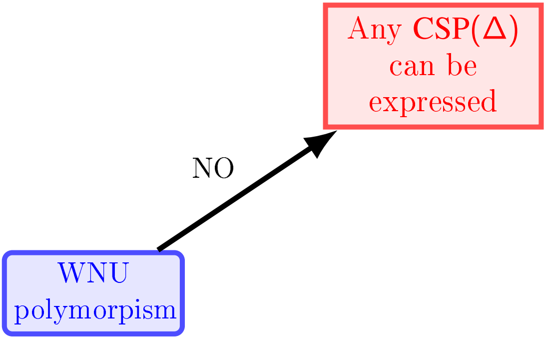
To sum up

WNU
polymorpism

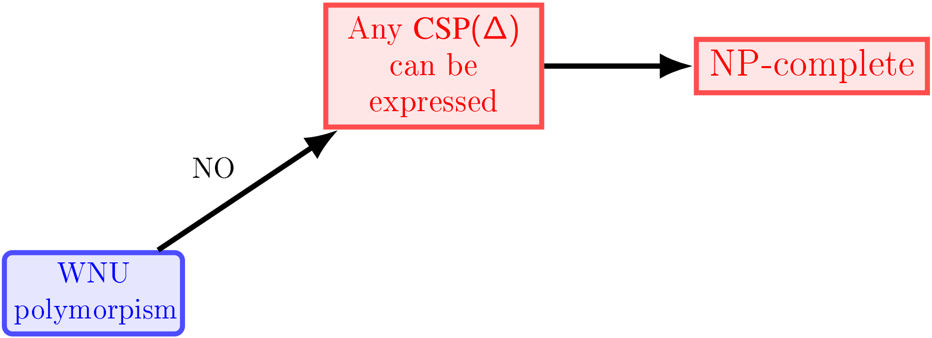
To sum up



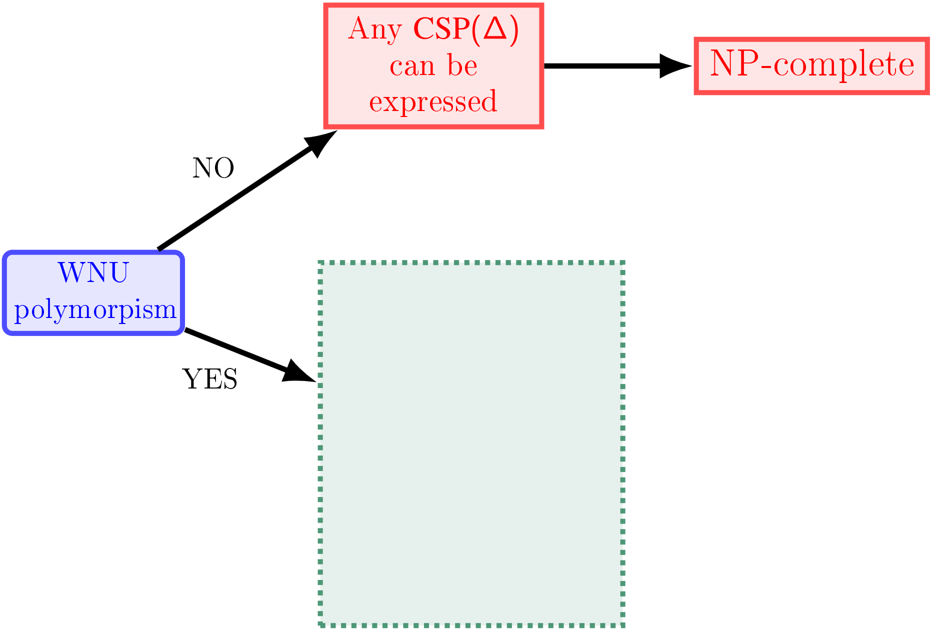
To sum up



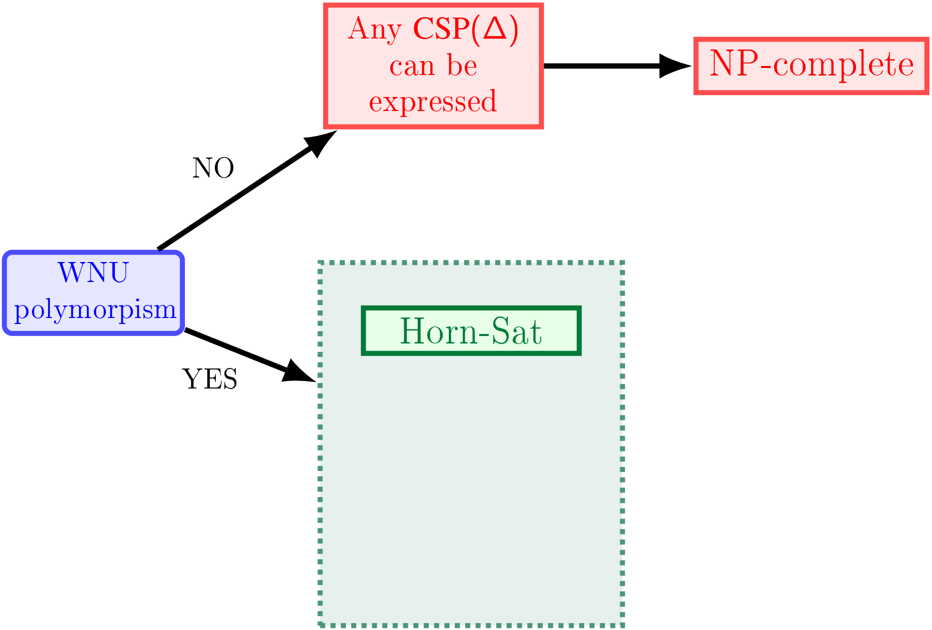
To sum up



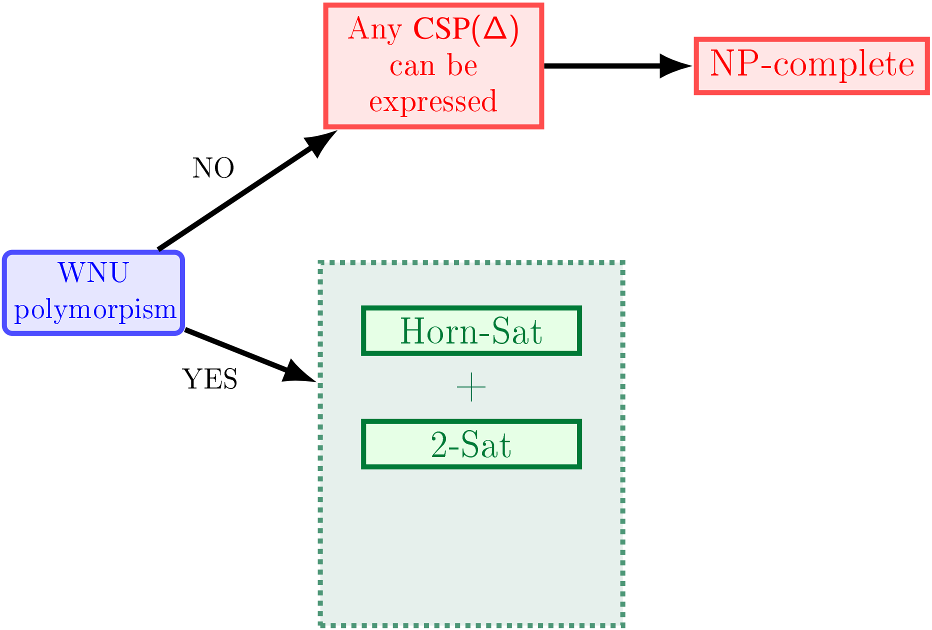
To sum up


Any $\mathbf{CSP}(\Delta)$ can be expressed
NP-complete
NO
WNU polymorpism
YES

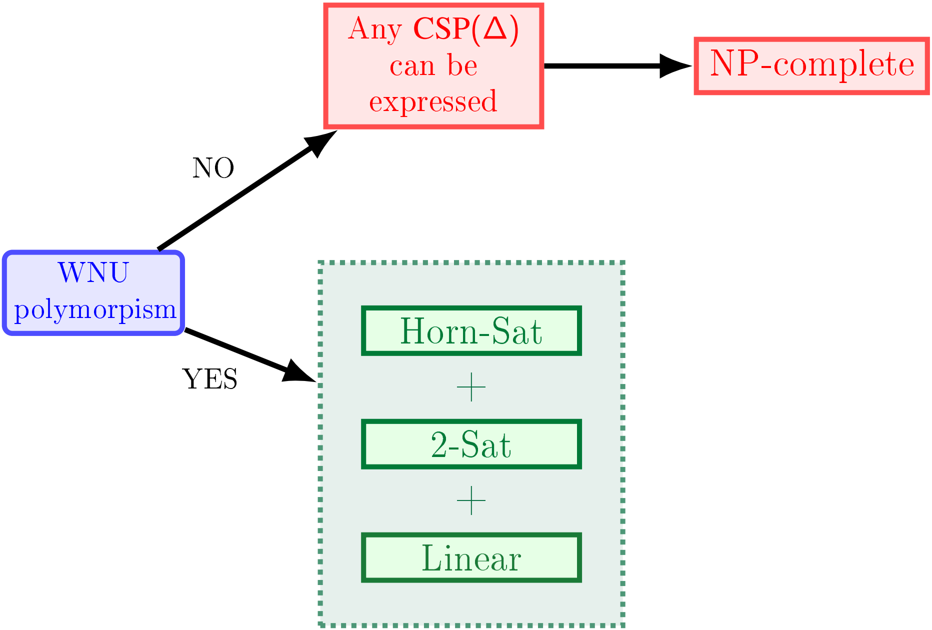
To sum up


Any CSP(Δ) can be expressed
NP-complete
NO
WNU polymorpism
YES
Horn-Sat

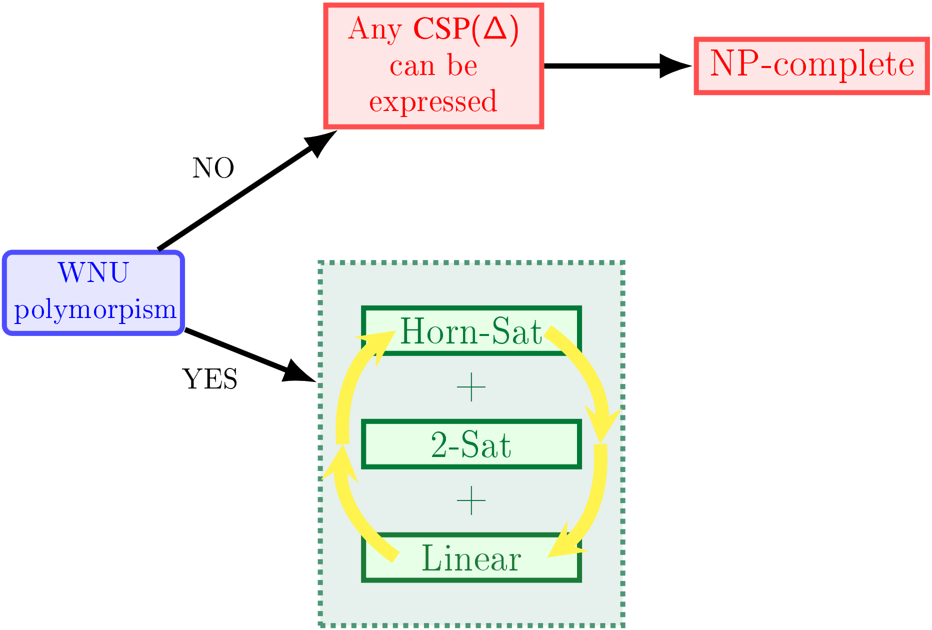
To sum up


Any **CSP**(Δ) can be expressed
NP-complete
NO
WNU polymorpism
YES
Horn-Sat
+
2-Sat

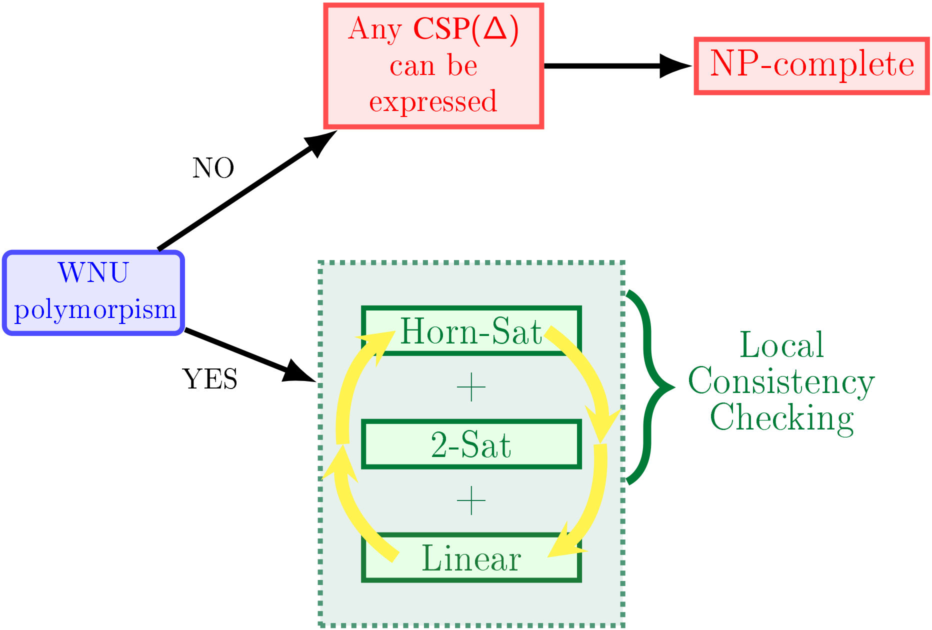
To sum up


Any CSP(Δ) can be expressed
NP-complete
NO
WNU polymorpism
YES
Horn-Sat
+
2-Sat
+
Linear

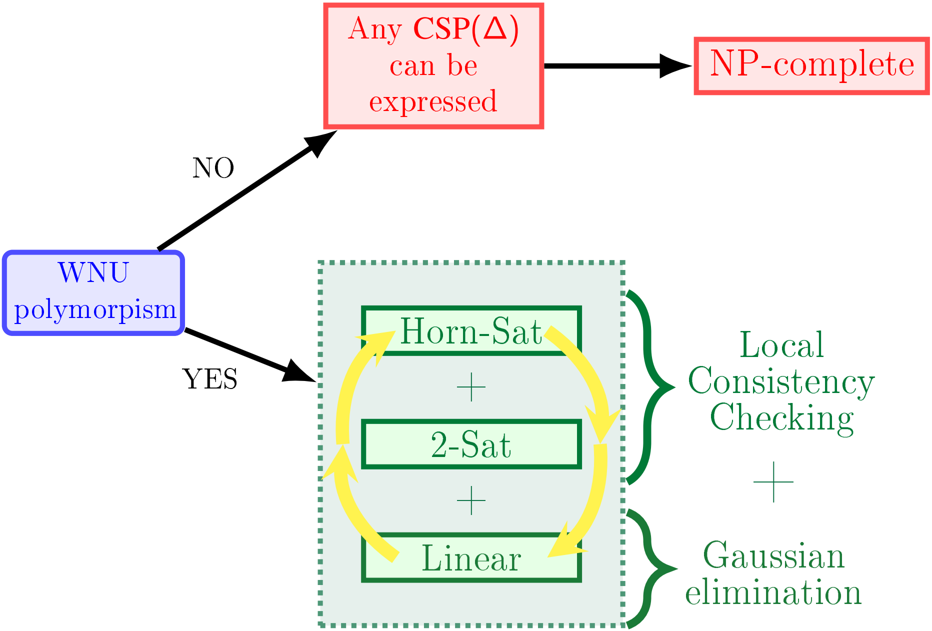
To sum up


Any CSP(Δ) can be expressed
NP-complete
NO
WNU polymorpism
YES
Horn-Sat
+
2-Sat
+
Linear

To sum up


Any **CSP**(Δ) can be expressed
NP-complete
NO
WNU polymorpism
YES
Horn-Sat
+
2-Sat
+
Linear
Local Consistency Checking

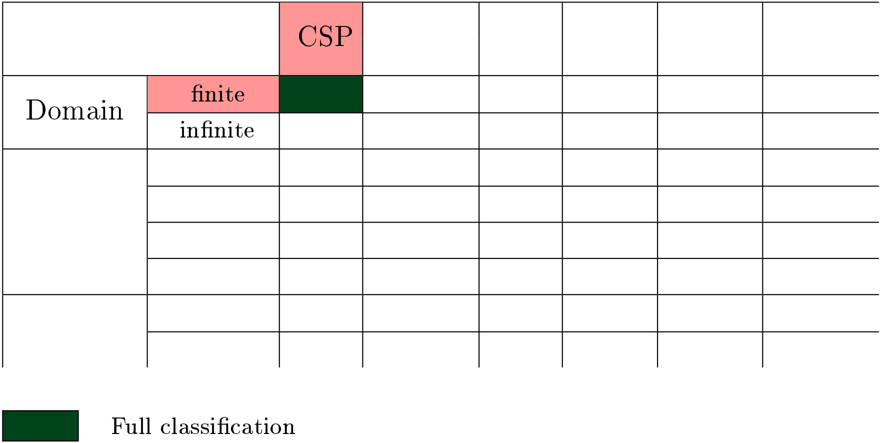
To sum up



Any CSP(Δ) can be expressed
NP-complete
NO
WNU polymorpism
YES
Horn-Sat
+
2-Sat
+
Linear
Local Consistency Checking
+
Gaussian elimination

		CSP					
Domain	finite						
	infinite						

		CSP					
Domain	finite						
	infinite						


CSP
Domain
finite
infinite
Full classification

		CSP					
Domain	finite						
	infinite						

Full classification

Infinite Domain CSP

Infinite Domain CSP

Γ is a set of relations on $\mathbb{Q}$.

Infinite Domain CSP

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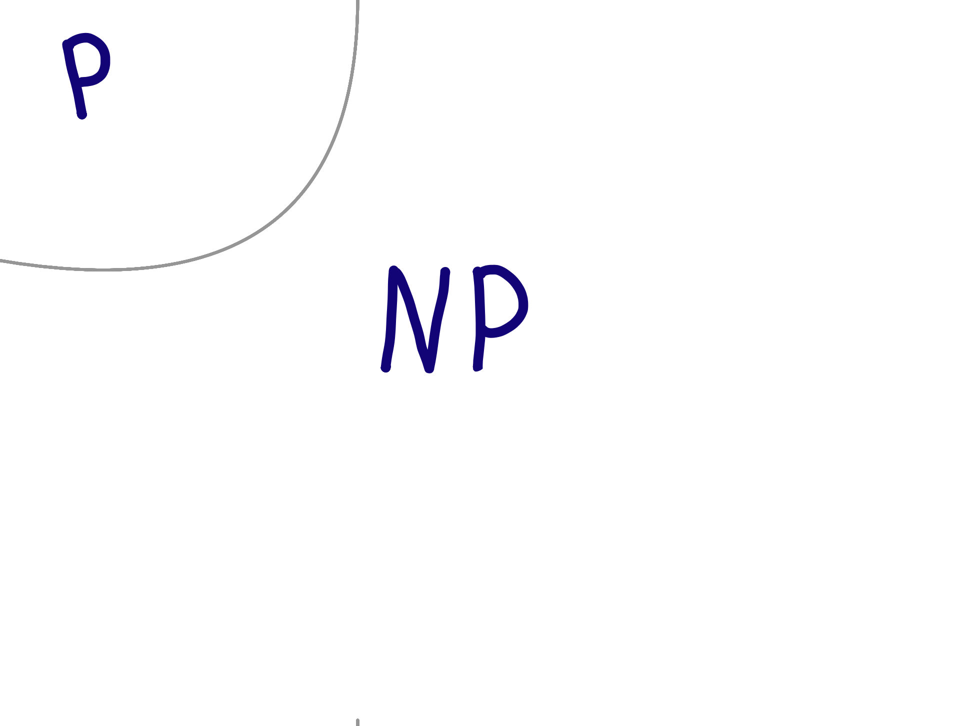
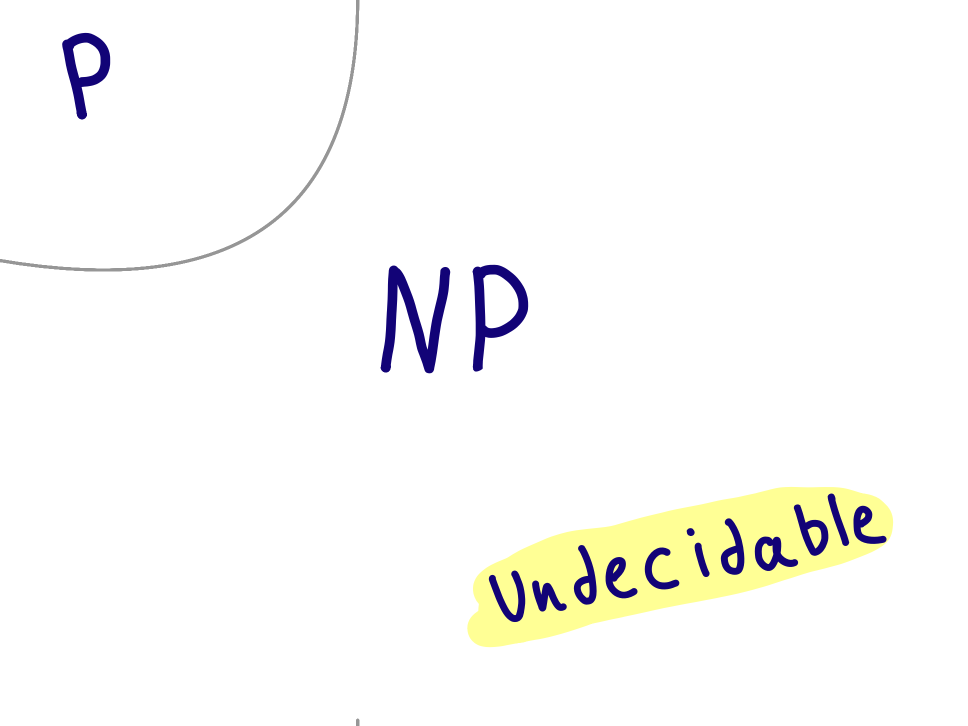
CSP(Γ)

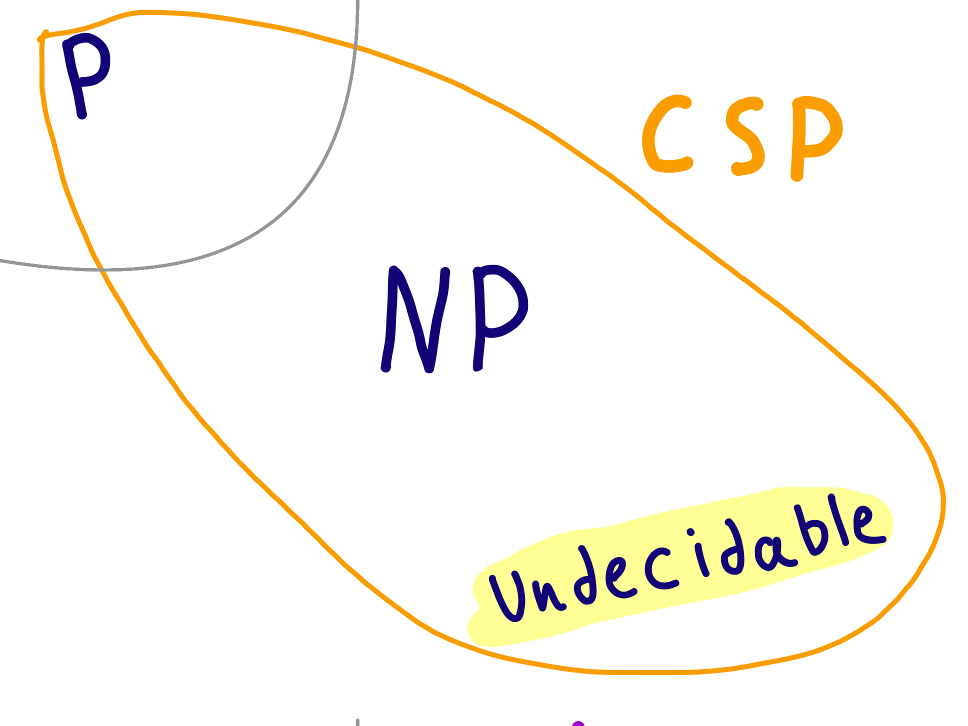
Given: a conjunction of relations, i.e. a formula

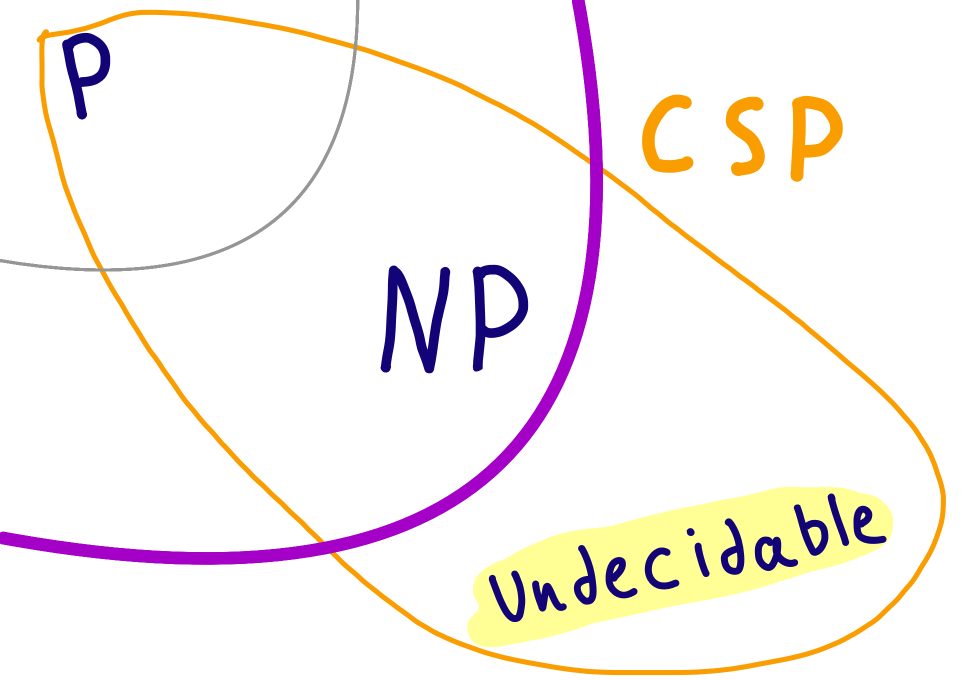
$$R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.


P
NP


P
NP
Undecidable


P
CSP
NP
Undecidable


P
NP
CSP
Undecidable

Infinite Domain CSP

Γ is a set of relations on $\mathbb{Q}$.

CSP(Γ)

Given: a conjunction of relations, i.e. a formula

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Decide: whether the formula is satisfiable.

Examples

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Examples

1. CSP($\{x < y\}$)

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Examples

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CSP instances:

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Examples

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CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$,

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Examples

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CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$, has a solution

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CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$, has a solution
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_1$,

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Examples

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CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$, has a solution
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_1$, has no solutions

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Examples

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$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$, has a solution
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_1$, has no solutions
The instance has a solution IFF there is no oriented cycle.

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Examples

1. CSP($\{x < y\}$) is in P.

CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$, has a solution
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_1$, has no solutions
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Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$)

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where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

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2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.

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where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.
3. CSP($\{x = y < z \vee x = z < y \vee y = z < x\}$)

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where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

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2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.
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4. CSP($\{x = y < z \vee x = z < y\}$)

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where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.
3. CSP($\{x = y < z \vee x = z < y \vee y = z < x\}$) is in P.
4. CSP($\{x = y < z \vee x = z < y\}$) is NP-complete.
5. CSP($\{x = y < z \vee x = z < y \vee y = z < x, x = y + 1\}$)

Infinite Domain CSP

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CSP(Γ)

Given: a conjunction of relations, i.e. a formula

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where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.
3. CSP($\{x = y < z \vee x = z < y \vee y = z < x\}$) is in P.
4. CSP($\{x = y < z \vee x = z < y\}$) is NP-complete.
5. CSP($\{x = y < z \vee x = z < y \vee y = z < x, x = y + 1\}$) is NP-complete.

Infinite Domain CSP

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where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.
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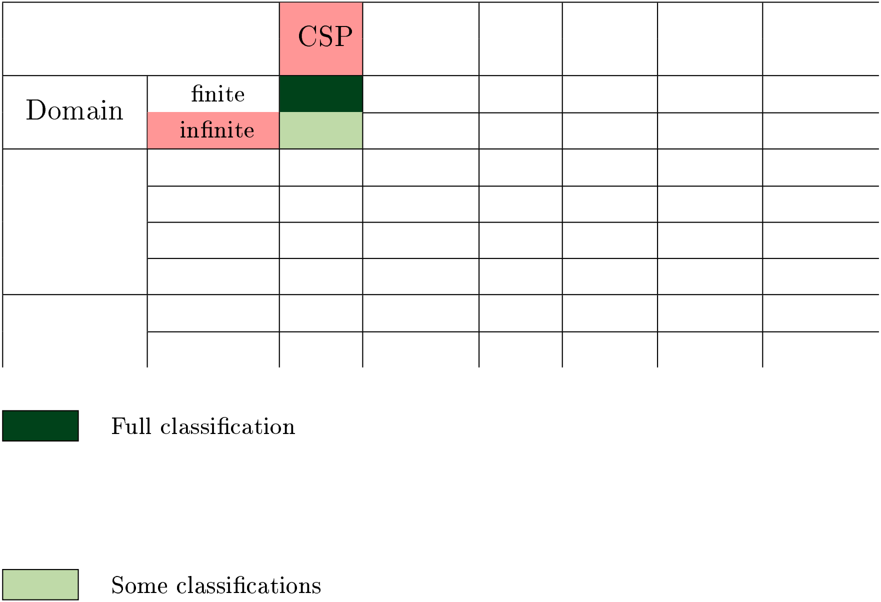
Classification for temporal constraint languages [Bodirsky, Kára, 2008]

A full classification of the complexity for constraint languages admitting a first-order definition in($\mathbb{Q}; <$) (P vs NP-complete).

		CSP					
Domain	finite						
	infinite						

Full classification


CSP
Domain
finite
infinite
Full classification
Some classifications

		CSP	Quantified CSP				
Domain	finite						
	infinite						

Full classification

Some classifications

Quantified Constraint Satisfaction Problem

Γ is a set of relations on a finite set A.

CSP(Γ)

Given: a sentence

$$\exists x_1 \ldots \exists x_n \; R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.

Decide: whether it holds.

Quantified Constraint Satisfaction Problem

Γ is a set of relations on a finite set A.

QCSP(Γ)

Given: a sentence

$$\exists y_1 \forall x_1 \dots \exists y_t \forall x_t (R_1(\dots) \wedge \dots \wedge R_s(\dots)),$$

where $R_1, \dots, R_s \in \Gamma$.
Decide: whether it holds.

Quantified Constraint Satisfaction Problem

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Given: a sentence

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Decide: whether it holds.

Examples:
$A = \{0, 1, 2\}, \Gamma = \{x \neq y\}$.

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Given: a sentence

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where $R_1, \ldots, R_s \in \Gamma$.

Decide: whether it holds.

Examples:

$A = \{0, 1, 2\}, \Gamma = \{x \neq y\}$. QCSP instances:

$\forall x \exists y_1 \exists y_2 (x \neq y_1 \wedge x \neq y_2 \wedge y_1 \neq y_2),$

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QCSP(Γ)

Given: a sentence

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where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether it holds.

Examples:

$A = \{0, 1, 2\}, \Gamma = \{x \neq y\}$. QCSP instances:

$\forall x \exists y_1 \exists y_2 (x \neq y_1 \wedge x \neq y_2 \wedge y_1 \neq y_2)$, true

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Given: a sentence

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where $R_1, \ldots, R_s \in \Gamma$.
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Examples:

$A = \{0, 1, 2\}, \Gamma = \{x \neq y\}$. QCSP instances:

$\forall x \exists y_1 \exists y_2 (x \neq y_1 \wedge x \neq y_2 \wedge y_1 \neq y_2)$, true

$\forall x_1 \forall x_2 \forall x_3 \exists y (x_1 \neq y \wedge x_2 \neq y \wedge x_3 \neq y)$,

Quantified Constraint Satisfaction Problem

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Given: a sentence

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$\forall x \exists y_1 \exists y_2 (x \neq y_1 \wedge x \neq y_2 \wedge y_1 \neq y_2)$, true

$\forall x_1 \forall x_2 \forall x_3 \exists y (x_1 \neq y \wedge x_2 \neq y \wedge x_3 \neq y)$, false

Quantified Constraint Satisfaction Problem

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QCSP(Γ)

Given: a sentence

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where $R_1, \dots, R_s \in \Gamma$.
Decide: whether it holds.

Examples:

$A = \{0, 1, 2\}, \Gamma = \{x \neq y\}$. QCSP instances:

$\forall x \exists y_1 \exists y_2 (x \neq y_1 \wedge x \neq y_2 \wedge y_1 \neq y_2)$, true

$\forall x_1 \forall x_2 \forall x_3 \exists y (x_1 \neq y \wedge x_2 \neq y \wedge x_3 \neq y)$, false

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Quantified Constraint Satisfaction Problem

Γ is a set of relations on a finite set A.

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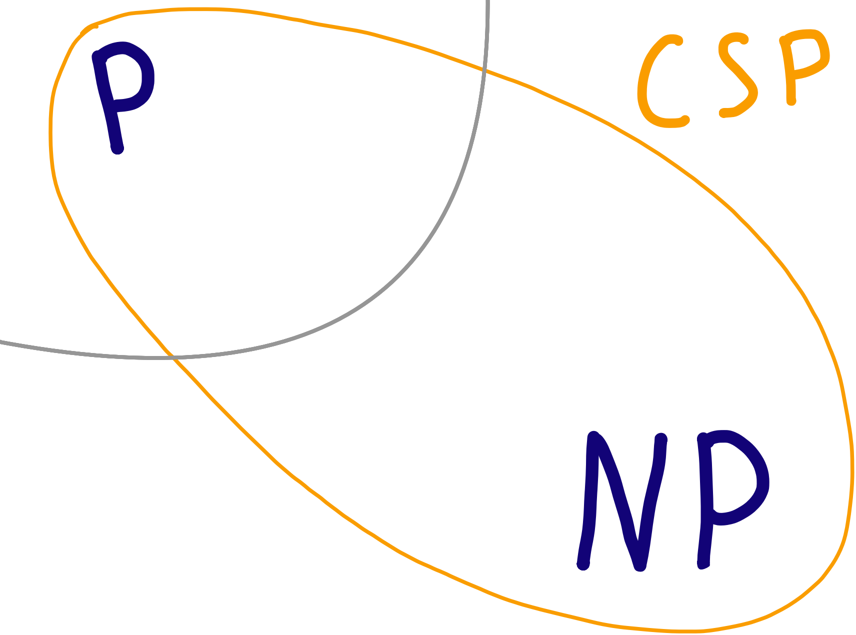
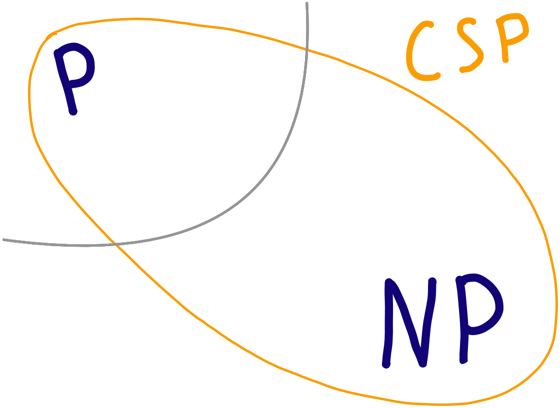
Question

What is the complexity of QCSP(Γ) for different Γ?


P
CSP
NP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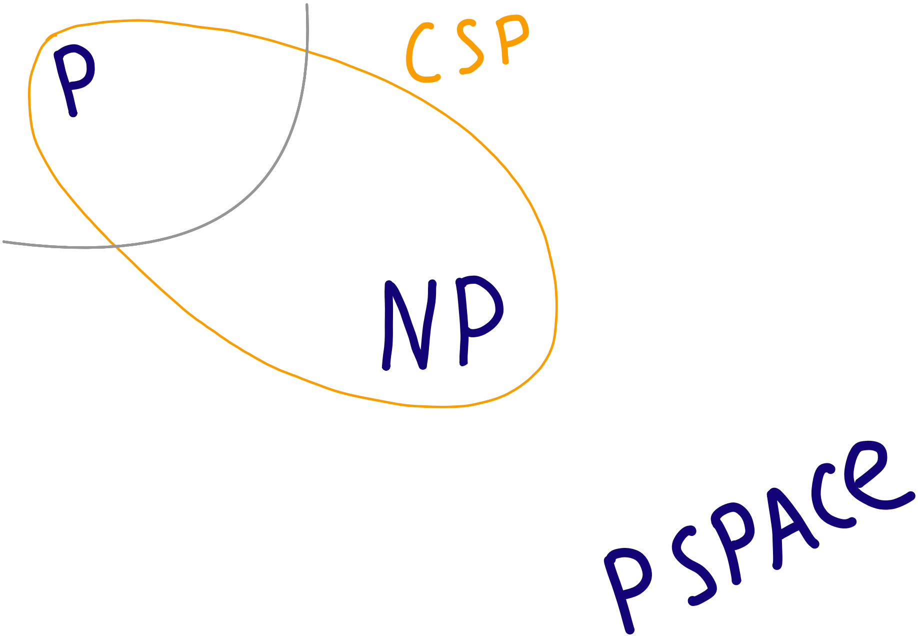
P
CSP
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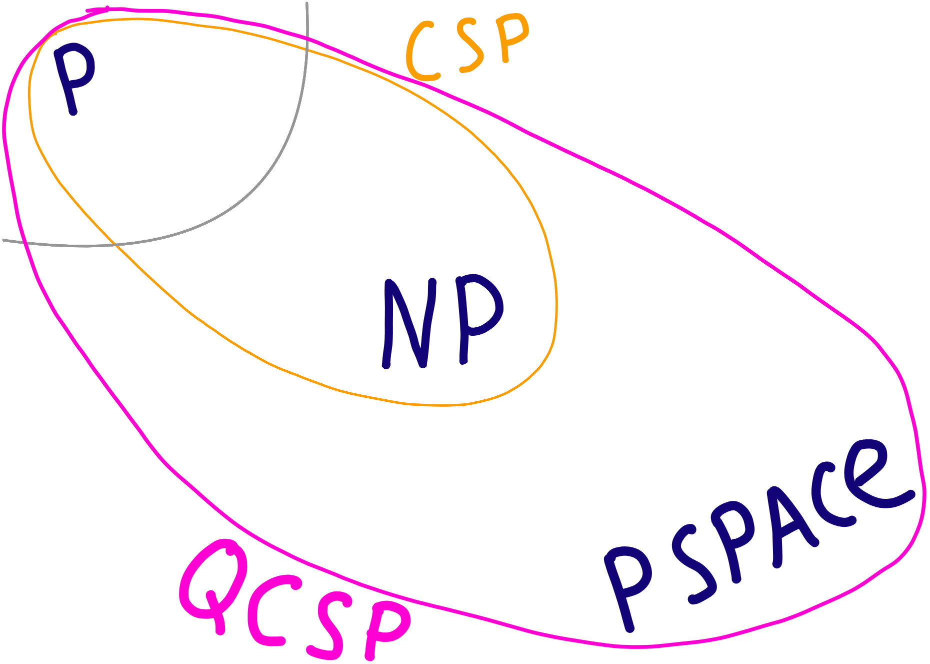
PSPACE


CSP
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QCSP Complexity Classes

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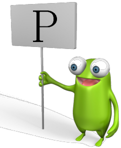


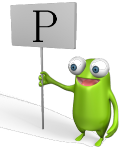
QCSP Complexity Classes

- If Γ contains all predicates then $\mathrm{QCSP}(\Gamma)$ is PSPACE-complete.
- If Γ consists of linear equations in a finite field then $\mathrm{QCSP}(\Gamma)$ is in P.

Theorem [Schaefer 1978 + Creignou et al. 2001/ Dalmau 1997.]

Suppose Γ is a constraint language on $\{0, 1\}$. Then

- $\mathrm{QCSP}(\Gamma)$ is in P if Γ is preserved by an idempotent WNU operation,
- $\mathrm{QCSP}(\Gamma)$ is PSPACE-complete otherwise.



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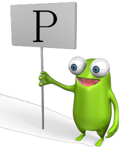
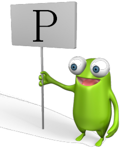


QCSP Complexity Classes

- Put $A' = A \cup \{*\}$, Γ' is Γ extended to A'. Then QCSP(Γ') is equivalent to CSP(Γ).



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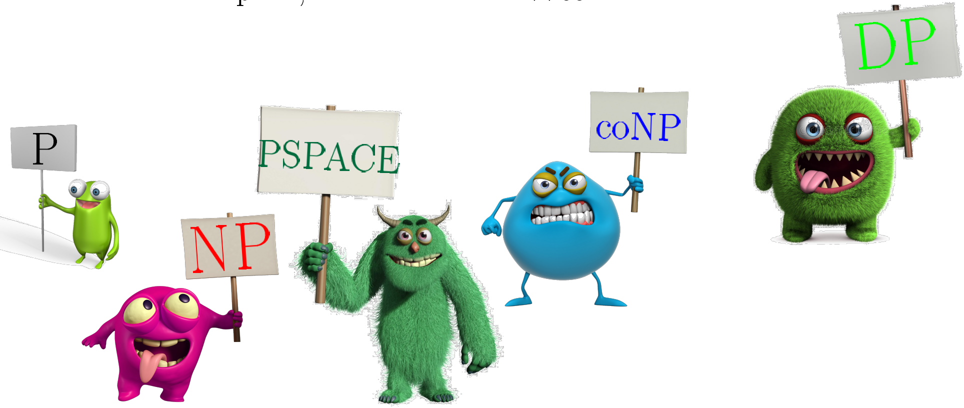

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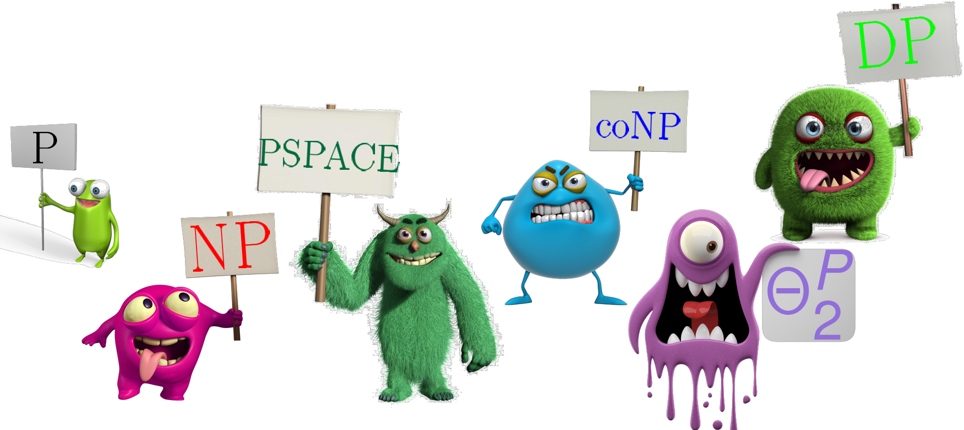
- there exists Γ on a 10-element domain such that QCSP(Γ) is Θ_2^P-complete.

QCSP Complexity Classes



QCSP Complexity Classes

Theorem [Zhuk, Martin, 2019]

Suppose Γ is a constraint language on $\{0, 1, 2\}$ containing $\{x = a \mid a \in \{0, 1, 2\}\}$. Then QCSP($\Gamma$) is

- in P, or
- NP-complete, or
- coNP-complete, or
- PSPACE-complete.

QCSP Complexity Classes

Theorem [Zhuk, 2021]

QCSP(Γ)

- is either PSpace-complete,
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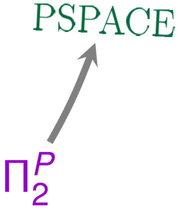

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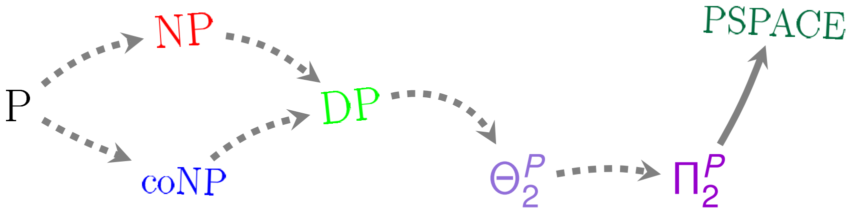

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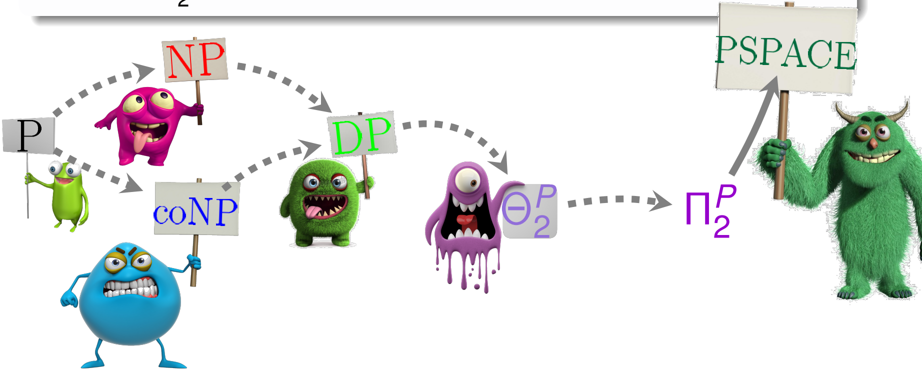

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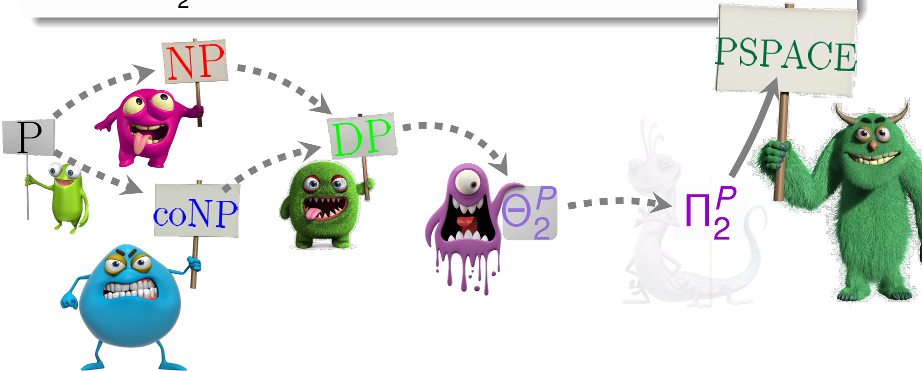

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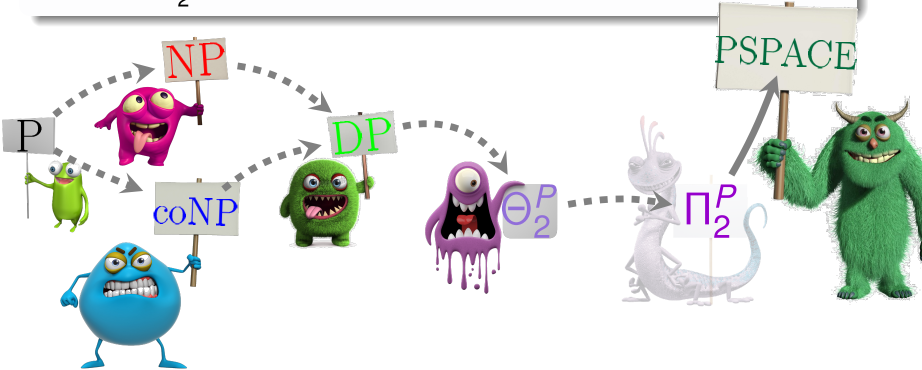

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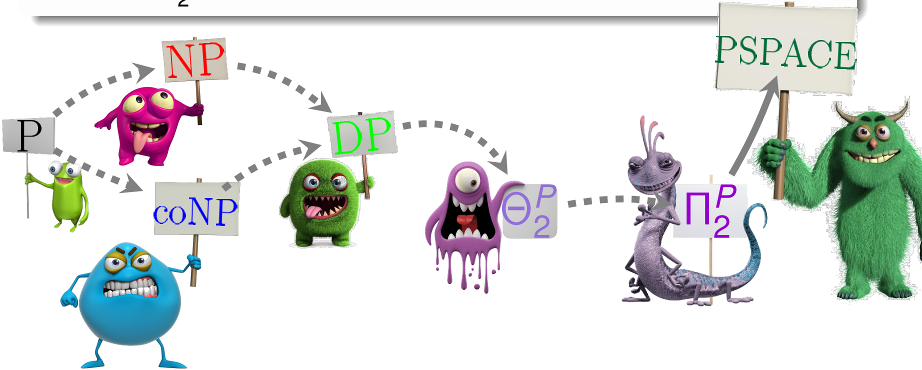

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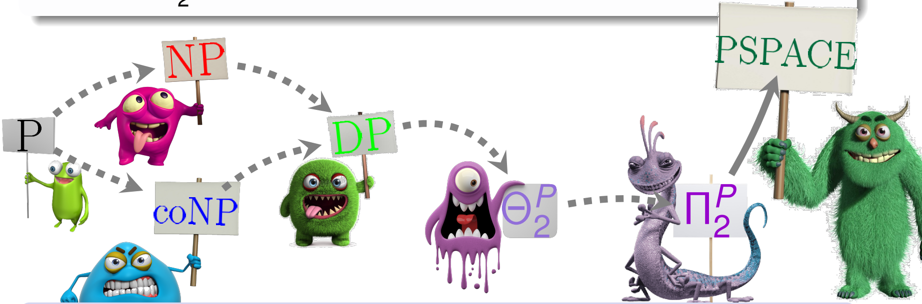

Lemma [Zhuk, 2021]

There exists Γ on a 6-element set such that QCSP(Γ) is Π_2^P-complete.

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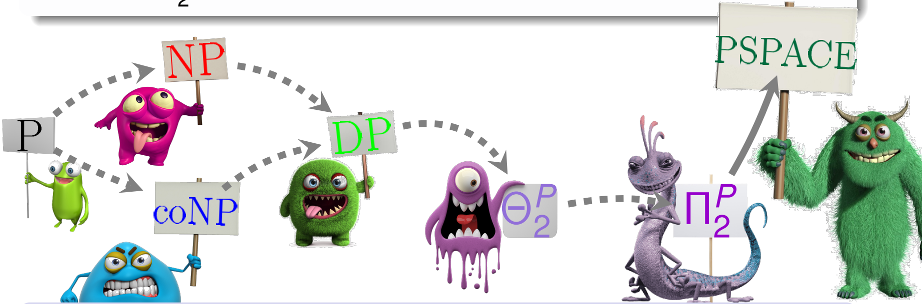

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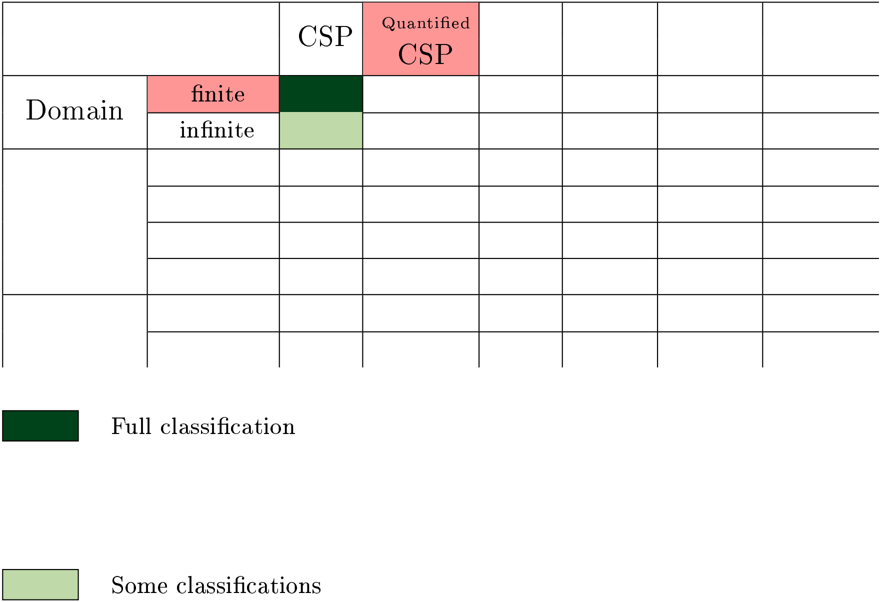
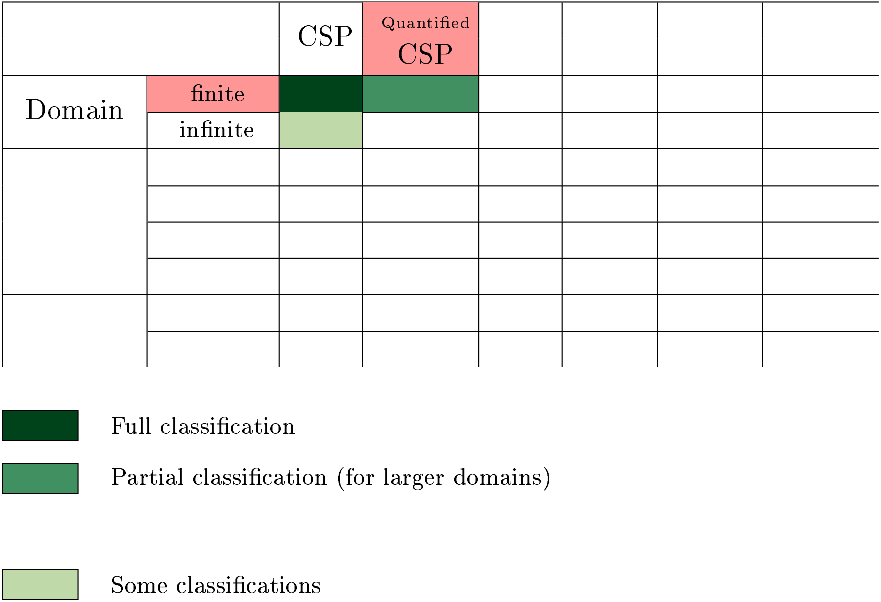
Are there any other complexity classes?


CSP
Quantified CSP
Domain
finite
infinite
Full classification
Some classifications



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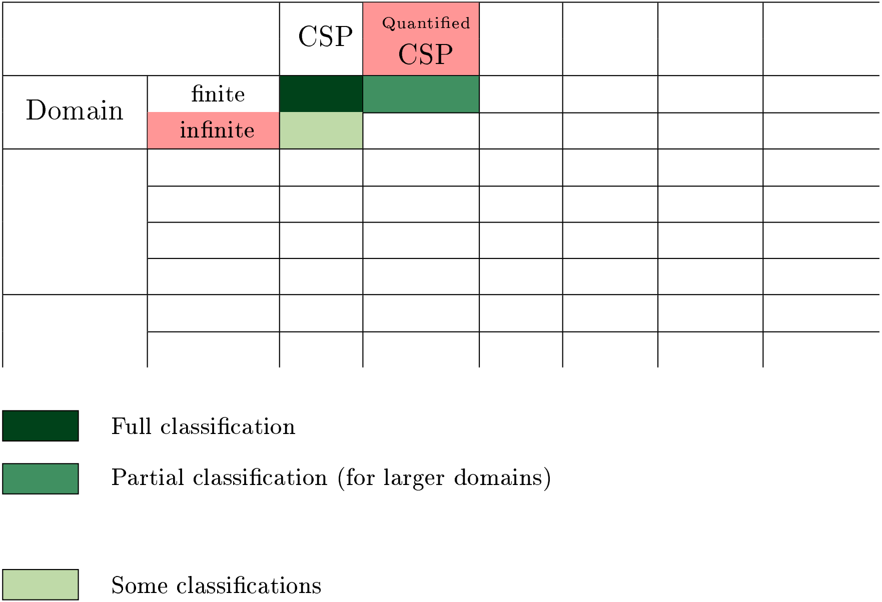
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Infinite Domain QCSP

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Γ is a set of relations on $\mathbb{Q}$.

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Given: a sentence $\exists y_1 \forall x_1 \dots \exists y_t \forall x_t (R_1(\dots) \wedge \cdots \wedge R_s(\dots))$, where $R_1, \dots, R_s \in \Gamma$.
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Classification for equality constraints [Bodirsky, Chen, 2010 + Zhuk, Martin, 2021]

A full classification of the complexity for constraint languages whose relations are boolean combinations of equalities. (P, NP-complete, PSPACE-complete)

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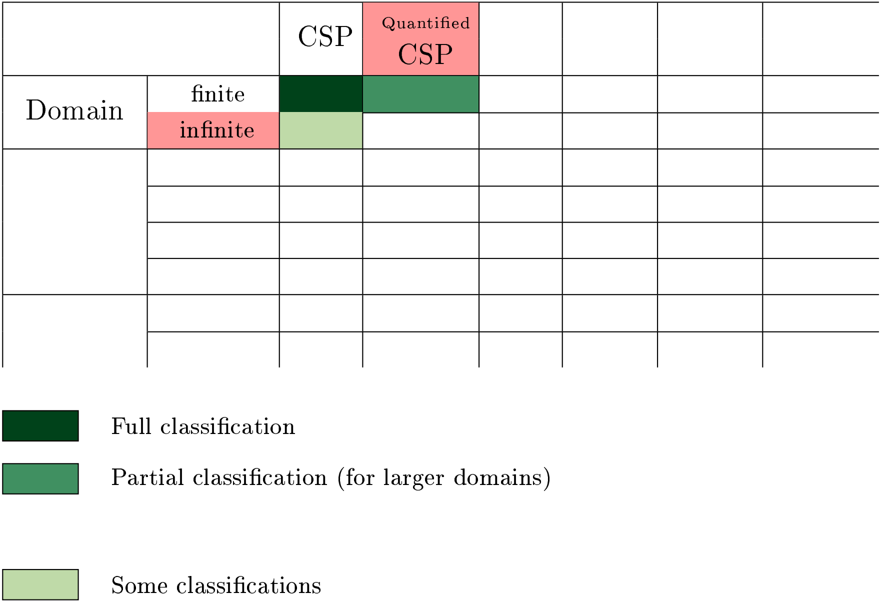
What is the complexity of QCSP($\{x = y \rightarrow z > t\}$)?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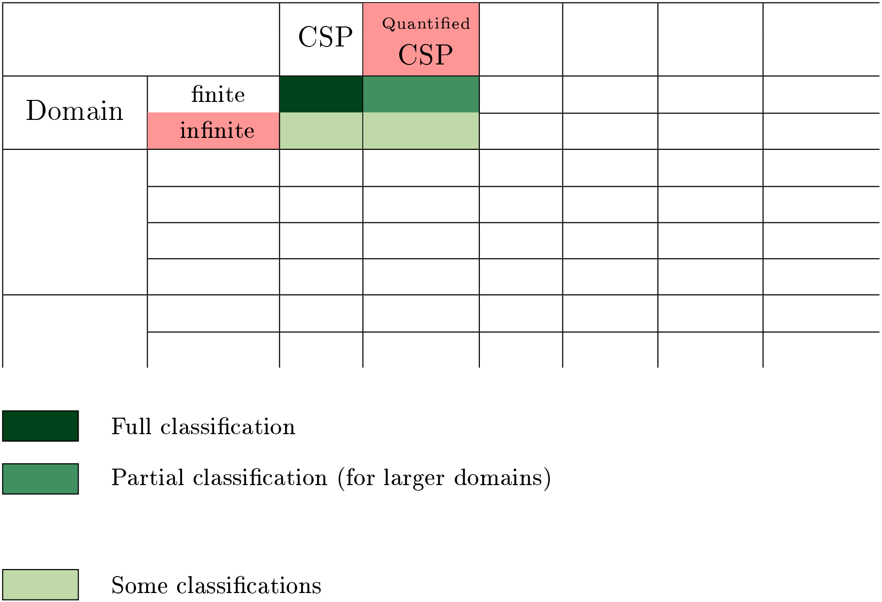
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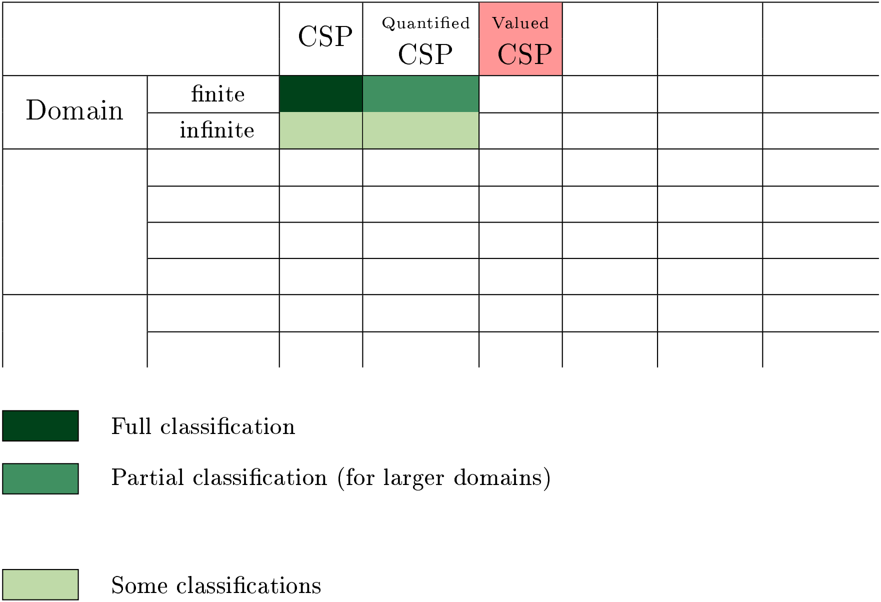
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Valued Constraint Satisfaction Problem

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Γ is a set of cost functions on a finite set A, i.e. mappings $A^n \to \mathbb{Q} \cup \{\infty\}$.

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Decide: whether $f_1(\dots) + f_2(\dots) + \dots + f_s(\dots) < T$ is satisfiable.

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$$A = \{0, 1\}, f(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases}.$$

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- $f(x_1, x_2) + f(x_1, x_3) + f(x_2, x_3) < 2$ is an instance VCSP($\{f\}$)

Valued Constraint Satisfaction Problem

Γ is a set of cost functions on a finite set A, i.e. mappings $A^n \to \mathbb{Q} \cup \{\infty\}$.

VCSP(Γ)

Given: a threshold T and a sum $f_1(\dots) + f_2(\dots) + \dots + f_s(\dots)$, where $f_1, \dots, f_s \in \Gamma$.
Decide: whether $f_1(\dots) + f_2(\dots) + \dots + f_s(\dots) < T$ is satisfiable.

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Complexity classification
[Kolmogorov, Krokhin, Rolínek, 2015+Bulatov, Zhuk, 2017]

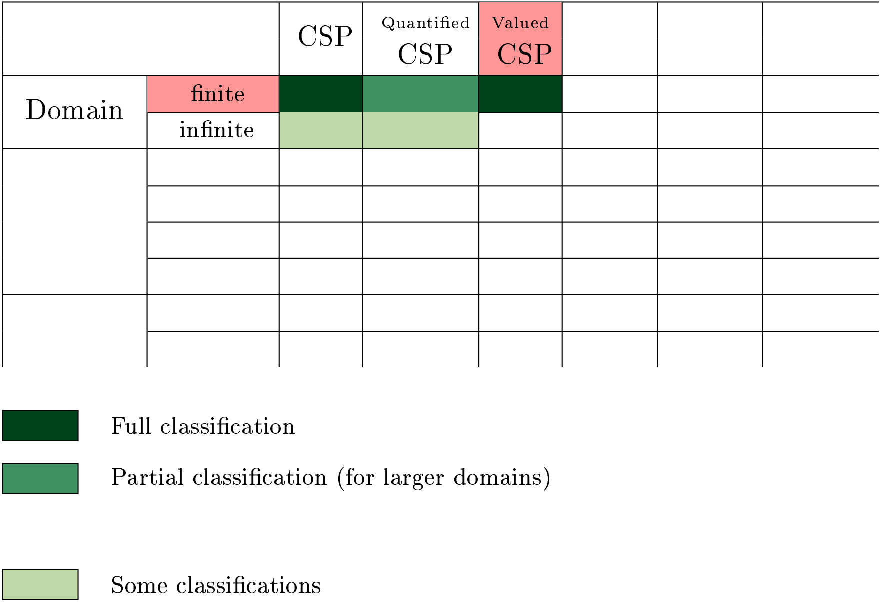
A full classification of the complexity for any finite set of cost functions Γ (P vs NP-complete).

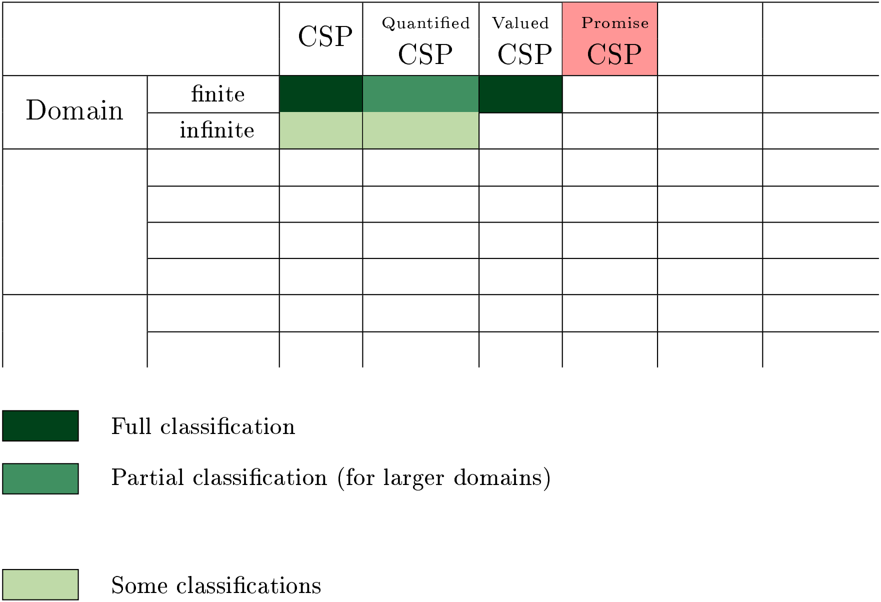
		CSP	Quantified CSP	Valued CSP			
Domain	finite						
	infinite						

Full classification

Partial classification (for larger domains)

Some classifications


CSP
Quantified CSP
Valued CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications


CSP
Quantified CSP
Valued CSP
Promise CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications

Promise Constraint Satisfaction Problem

There are two versions of each relation (weak and strong) in Γ

Promise Constraint Satisfaction Problem

There are two versions of each relation (weak and strong) in Γ

PCSP(Γ)

Given a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$,
Distinguish between two cases:

- Strong version is satisfied
- Weak version is not satisfied

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- Strong version is $1\mathrm{IN}3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

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Distinguish between two cases:

- Strong version is satisfied
- Weak version is not satisfied

Example 1

- Strong version is $\mathrm{1IN3} = \{(1,0,0),(0,1,0),(0,0,1)\}$
- Weak version is $\mathrm{NAE3} = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$

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- CSP($\{\text{NAE}3\}$) and CSP($\{1\text{IN}3\}$) are NP-hard, but PCSP($\{1\text{IN}3, \text{NAE}3\}$) is in P

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- Weak version is $\text{NAE3} = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$
- CSP({NAE3}) and CSP({1IN3}) are NP-hard, but PCSP({1IN3, NAE3}) is in P (promise helps).

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Theorem [Ficak, Kozik, Olsák, Stankiewicz, 2019])

A classification of the complexity of PCSP(Γ) for Γ consising of symmetric relations on $\{0,1\}$.

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There are two versions of each relation (weak and strong) in Γ

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Example 2 ((K, L)-colorability)

Given a graph G.

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Example 2 ((K, L)-colorability)

Given a graph G. Distinguish between two cases

- the graph is K-colorable;
- the graph is not even L-colorable;

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Open questions

- What is the complexity of $(3, 6)$-colorability?

Promise Constraint Satisfaction Problem

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Given a graph G. Distinguish between two cases

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Open questions

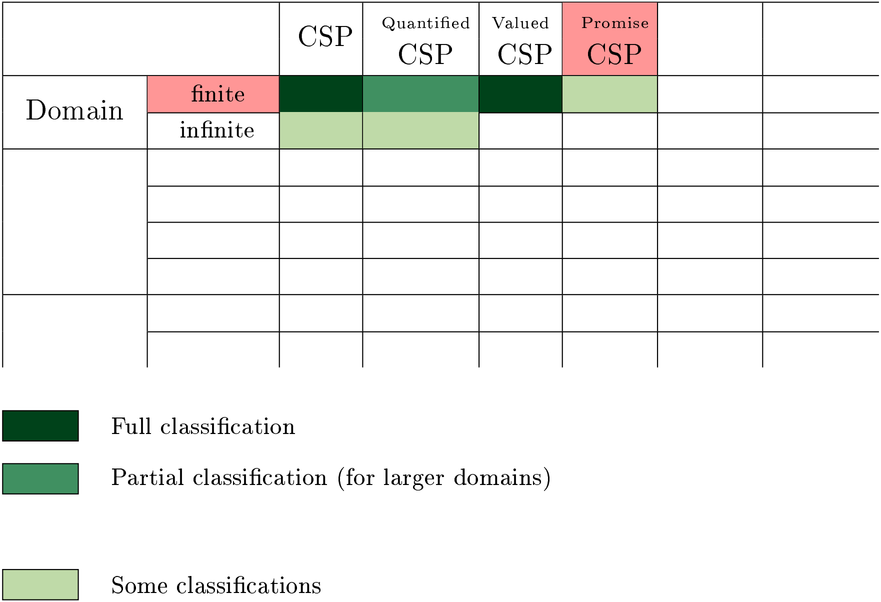
- What is the complexity of $(3, 6)$-colorability?
- What is the complexity of $(3, 1000000000)$-colorability?

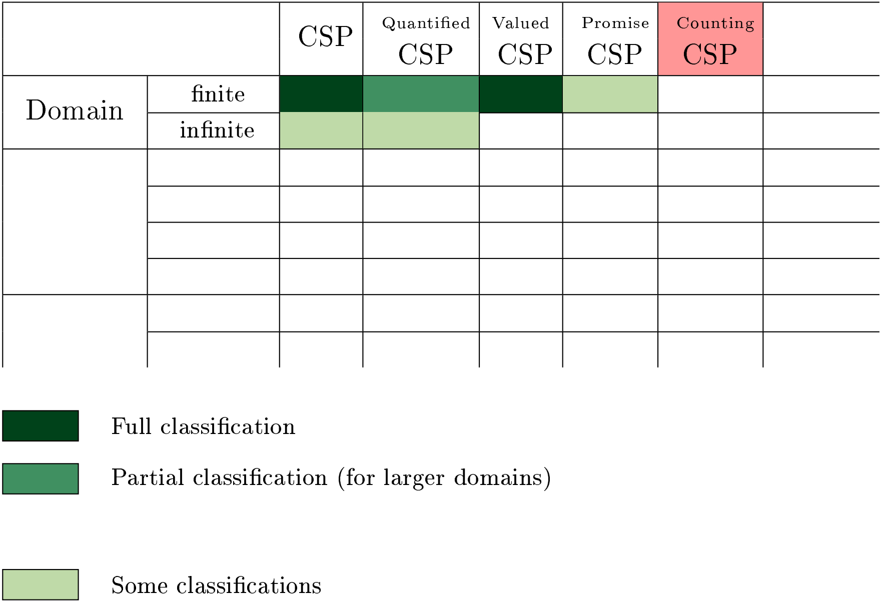
		CSP	Quantified CSP	Valued CSP	Promise CSP		
Domain	finite						
	infinite						

Full classification

Partial classification (for larger domains)

Some classifications


CSP
Quantified CSP
Valued CSP
Promise CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications


CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications

Counting Constraint Satisfaction Problem

Γ is a set of relations on a finite set A.

Counting-CSP(Γ)

Given: a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$,
where $R_1, \ldots, R_s \in \Gamma$.
Find the number of solutions.

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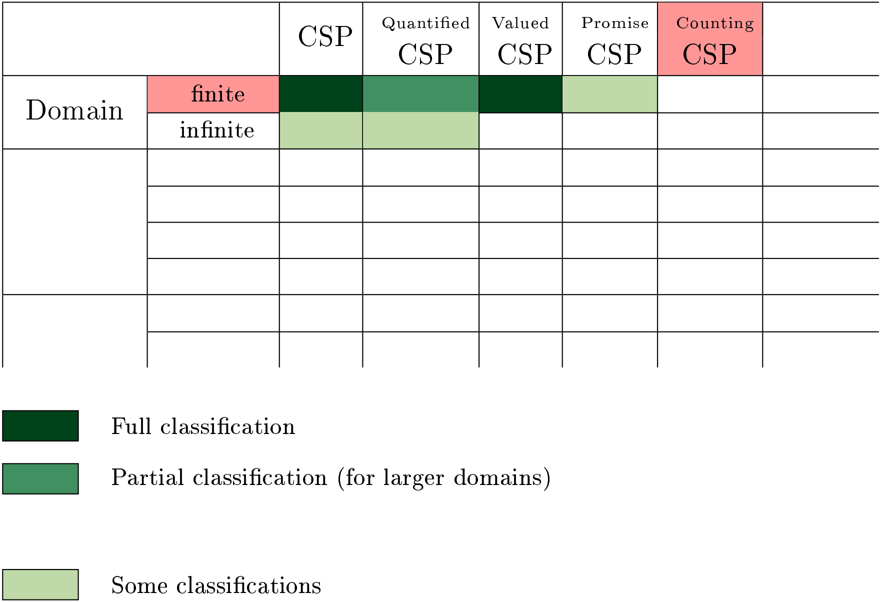
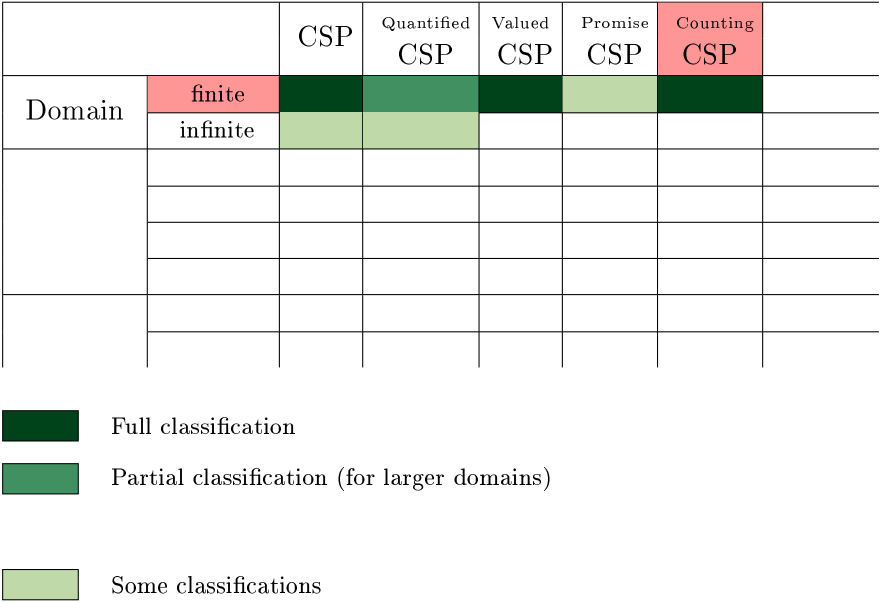
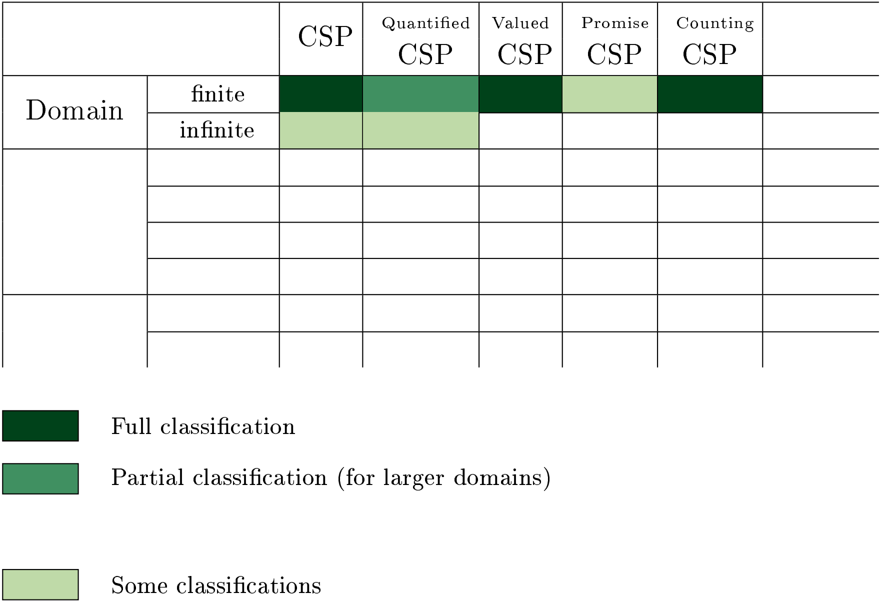
Find the number of solutions.

Theorem [Bulatov, 2008]

A classification of the complexity of Counting-CSP(Γ) for every Γ.


CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications


CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications


CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Full classification
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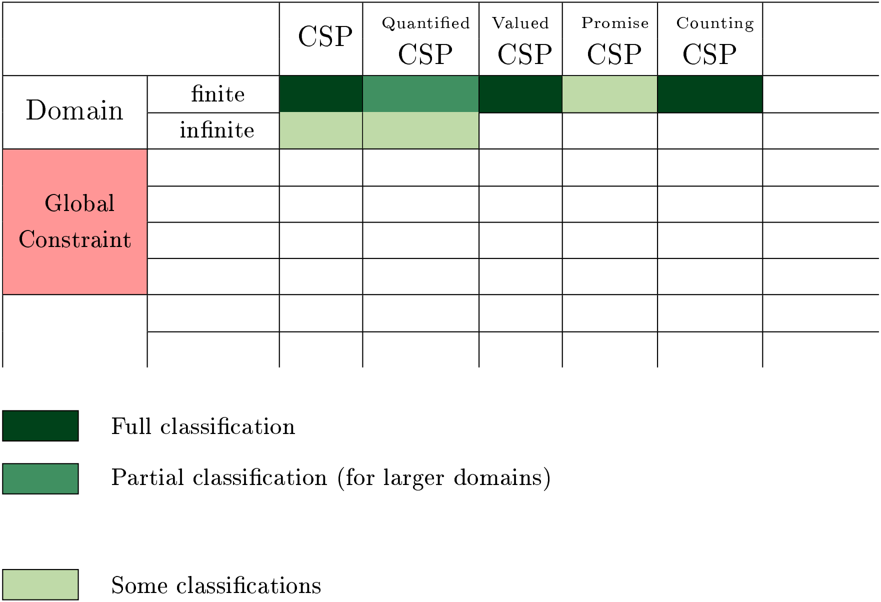
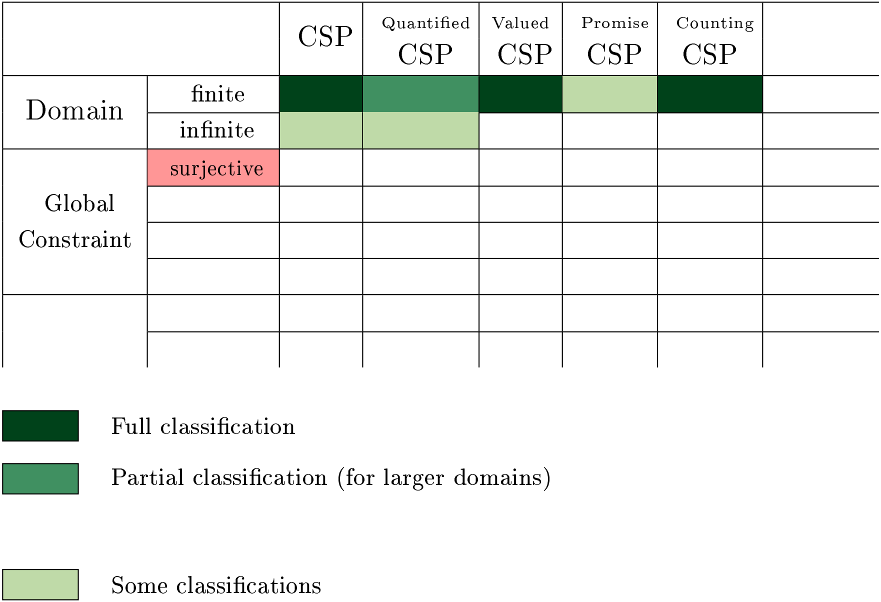
Some classifications



CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Global Constraint
Full classification
Partial classification (for larger domains)
Some classifications


CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Global Constraint
surjective
Full classification
Partial classification (for larger domains)
Some classifications

Surjective Constraint Satisfaction Problem

Surjective Constraint Satisfaction Problem

Γ is a set of relations on A.

SCSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula has a surjective solution, that is, a solution such that $\{x_1, \ldots, x_n\} = A$.

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Example

$A = \{0, 1, 2\}, \Gamma = \{x \leq y\}$.

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Example

$A = \{0, 1, 2\}, \Gamma = \{x \leq y\}$. Surjective CSP instances:

$x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_4$,

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Example

$A=\{0,1,2\},\Gamma=\{x\le y\}$. Surjective CSP instances:
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Question

What is the complexity of the SCSP(Γ)?

Surjective Graph Homomorphism Problem

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Let H be a finite graph.

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SurjHom(H):

Given: a graph G.

Decide: whether there exists a surjective homomorphism from G to H.

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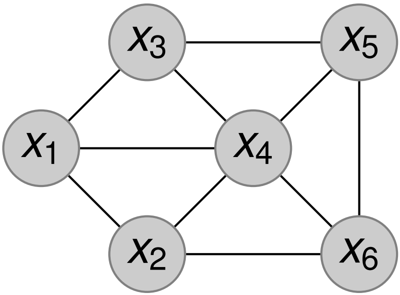
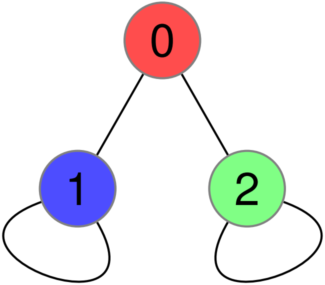
Graph G



Graph H



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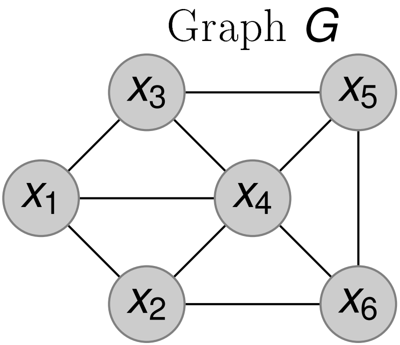

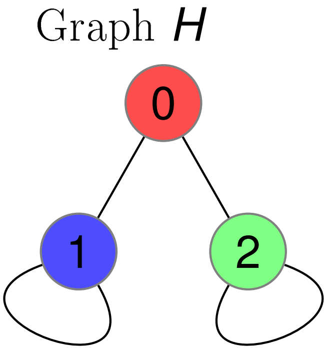

SurjHom(H) is equivalent to SCSP($\{x + y \neq 0 \mod 3\}$).

History

History

- The complexity was described for a two-element domain [Creignou, N., and Hébrard, 1997].

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- The No-Rainbow Problem is NP-complete [Zhuk, 2020] SCSP($\{(a, b, c) \mid \{a, b, c\} \neq \{0, 1, 2\}\}$).

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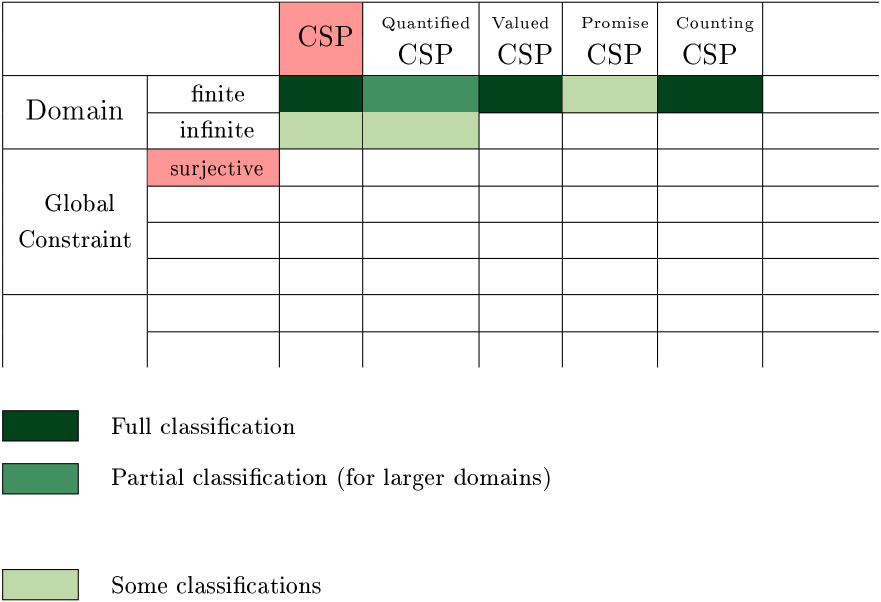
- The complexity of $\mathbf{SurjHom}(H)$ was described for all graphs of size 4 other than $\mathcal{C}_4^{ref}$[S. Dantas, Figueiredo, Gravier, Klein, 2005]
- the complexity of $\mathbf{SurjHom}(H)$ was described for partially reflexive forests [Golovach, Paulusma, Song, 2011]
- $\mathbf{SurjHom}(\mathcal{C}_4^{ref})$ is NP-complete [Martin, Paulusma, Vikas, 2011]
- $\mathbf{SurjHom}(\mathcal{C}_6)$ is NP-complete [Vikas, 2017]
- The No-Rainbow Problem is NP-complete [Zhuk, 2020] $\mathbf{SCSP}(\{(a, b, c) \mid \{a, b, c\} \neq \{0, 1, 2\}\})$.
- The complexity cannot be described in terms of polymorphisms [Zhuk, 2020]



CSP
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Global Constraint
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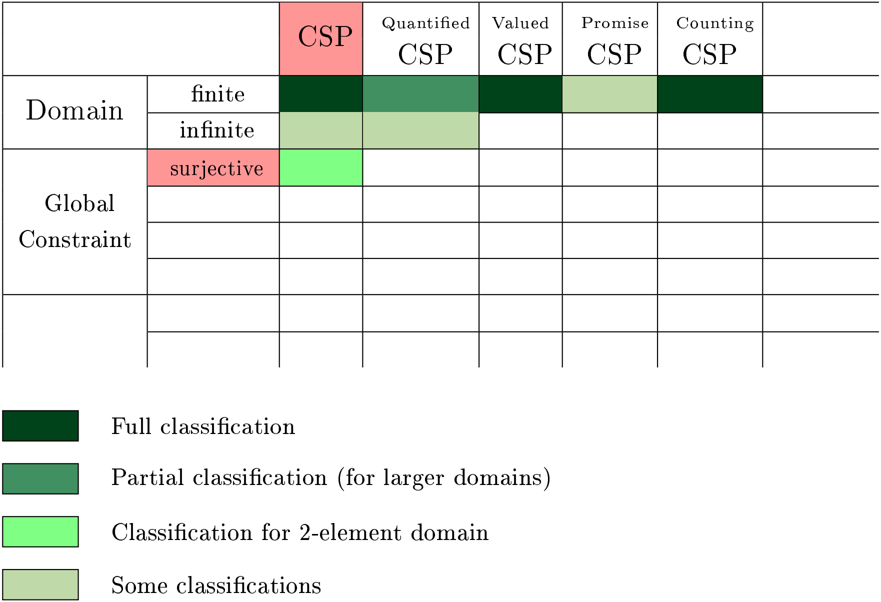
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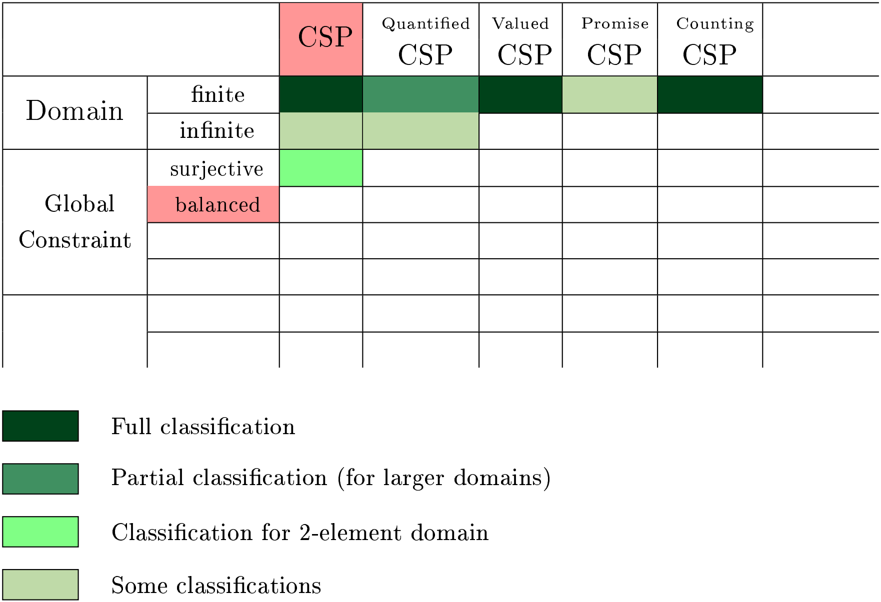
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Balanced CSP

Γ is a set of relations on a finite set A.

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Balanced-CSP(Γ)

Given: a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$, where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether it has a balanced solution, i.e., a solution with equal number of every element.

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Balanced-CSP(=) on $\{0, 1\}$

Given an instance $x_{i_1} = x_{j_1} \wedge \cdots \wedge x_{i_s} = x_{j_s}$. Decide whether it has a solution with equal number of 0 and 1.

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Balanced-CSP($=$) on $\{0, 1\}$ solvable in polynomial time

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Balanced-CSP($\leq$) on $\{0,1\}$ NP-complete

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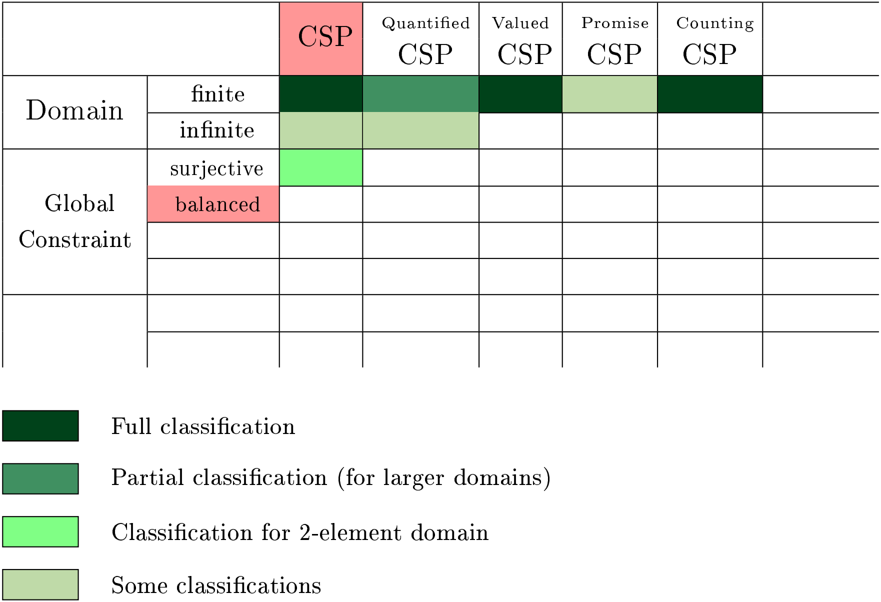
Theorem [Creignou, H. Schnoor, I. Schnoor, 2008]

A classification of the complexity of Balanced-CSP(Γ) and Cardinality-CSP(Γ) for each Γ on $\{0,1\}$.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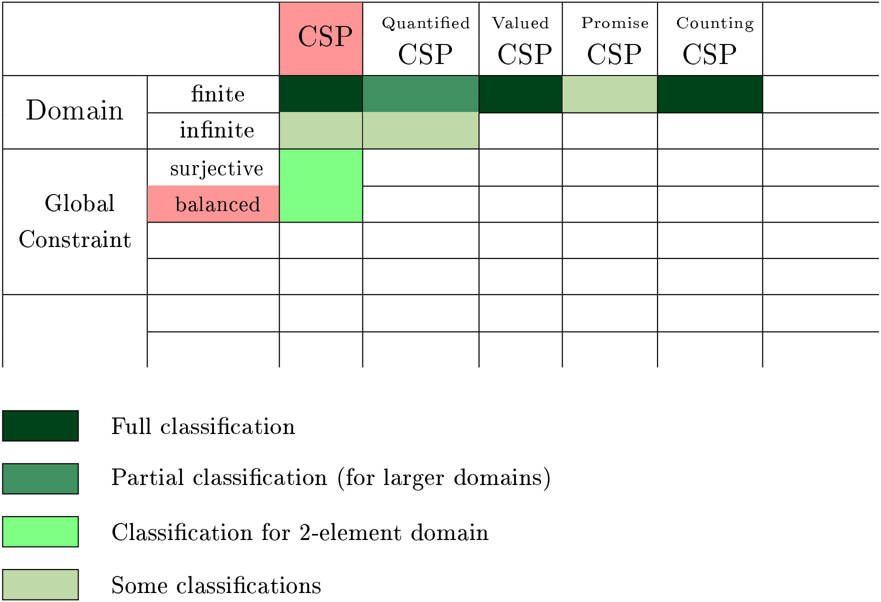
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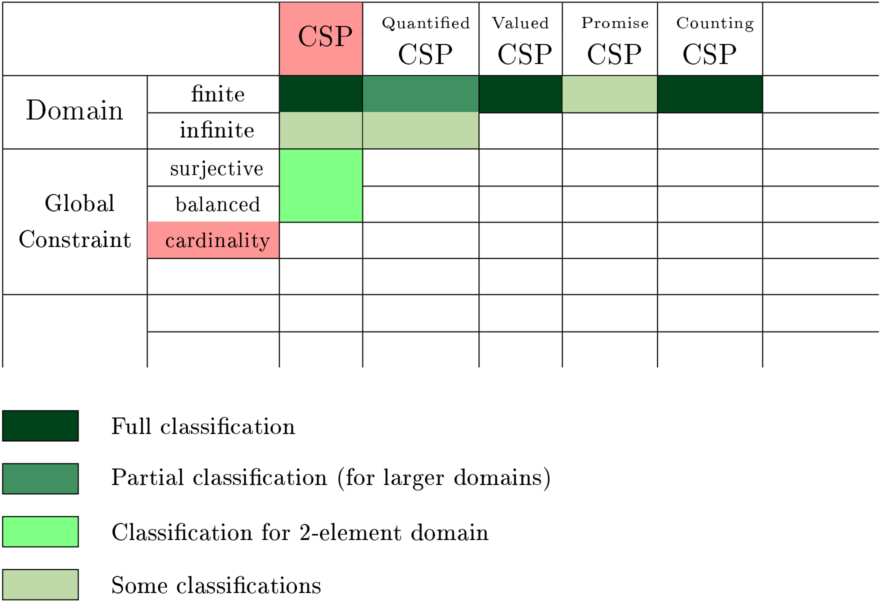

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Global cardinality constraint

Γ is a set of relations on a finite set A.

Cardinality-CSP(Γ)

Given: a mapping $\pi\colon A \to \mathbb{N}$ and a formula $R_1(x_{i_{1,1}}, \dots, x_{i_{1,n_1}}) \wedge \dots \wedge R_s(x_{i_{s,1}}, \dots, x_{i_{s,n_s}})$, where $R_1, \dots, R_s \in \Gamma$.
Decide: whether it has a solution containing each element $a \in A$ exactly $\pi(a)$ times.

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Decide: whether it has a solution containing each element $a \in A$ exactly $\pi(a)$ times.

Cardinality-CSP(Linear Equations in $\mathbb{Z}_2$)

Given a system of linear equations in $\mathbb{Z}_2$ and $k \in \mathbb{N}$.
Decide whether there exists a solution with exactly k 1s.

Global cardinality constraint

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Cardinality-CSP(Linear Equations in $\mathbb{Z}_2$) NP-complete

Given a system of linear equations in $\mathbb{Z}_2$ and $k \in \mathbb{N}$.
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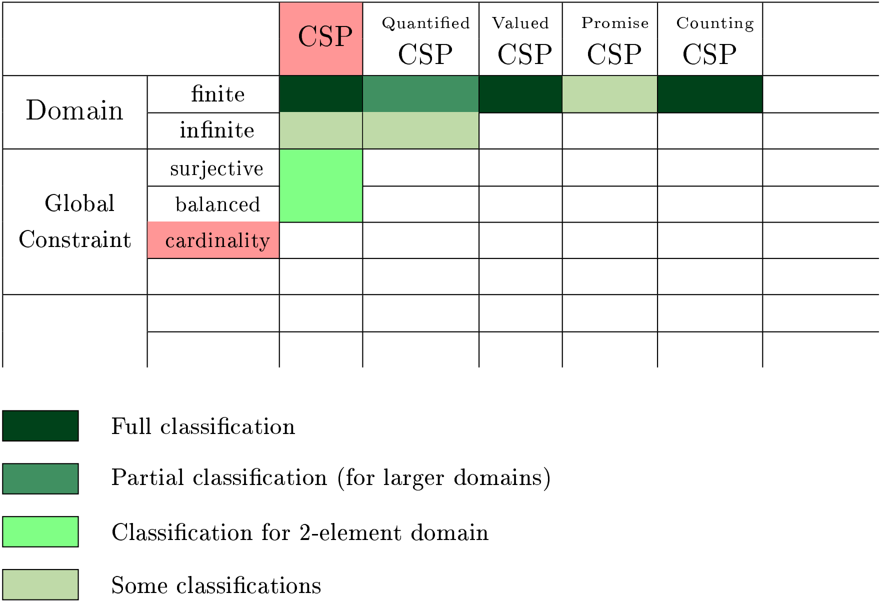
Decide whether there exists a solution with exactly k 1s.

Theorem [Bulatov, Marx, 2009]

A classification of the complexity of Cardinality-CSP(Γ) for each Γ.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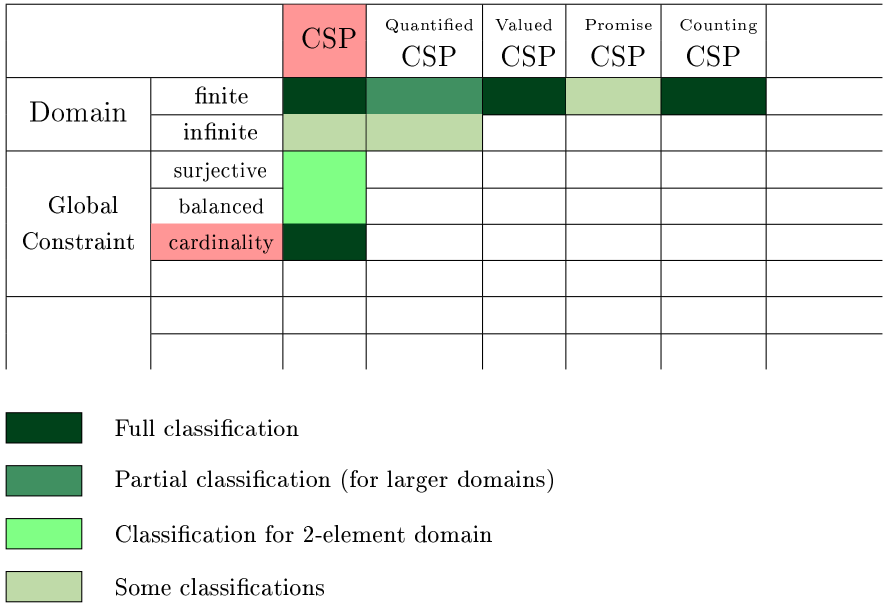
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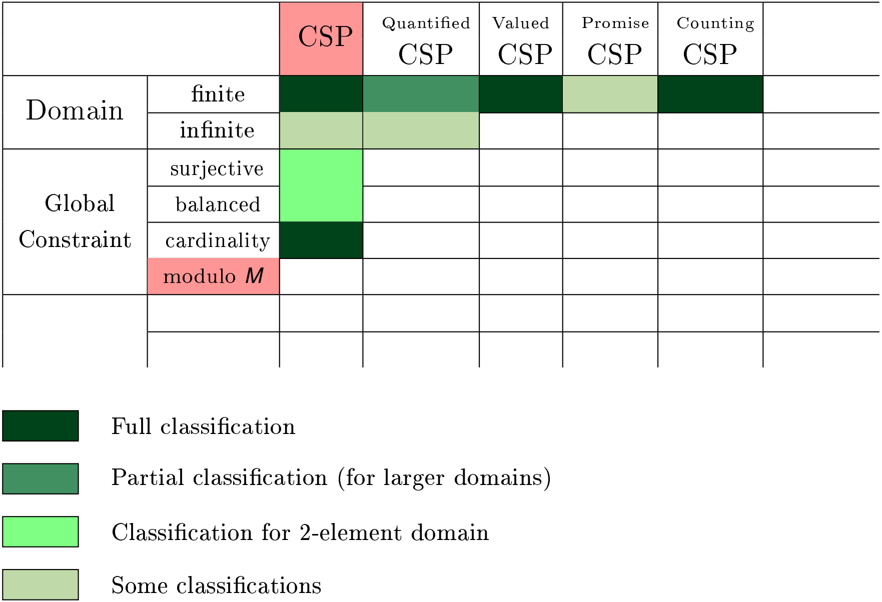
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Global modular constraint

Mod_M-CSP(Γ)

Given: a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$, where $R_1, \ldots, R_s \in \Gamma$.

Decide: whether it has a solution satisfying $x_1 + \cdots + x_n = 0 \mod M$.

Global modular constraint

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- If Γ consists of linear equations on $\{0, 1\}$ and $M = 25$ then Mod_M-CSP(Γ) is tractable

Global modular constraint

Mod_M-CSP(Γ)

Given: a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$, where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether it has a solution satisfying $x_1 + \cdots + x_n = 0 \mod M$.

- If Γ consists of linear equations on $\{0, 1\}$ and $M = 25$ then Mod_M-CSP(Γ) is tractable
- If Γ consists of linear equations on $\{0, 1\}$ and $M = 15$ then Mod_M-CSP(Γ) is not tractable

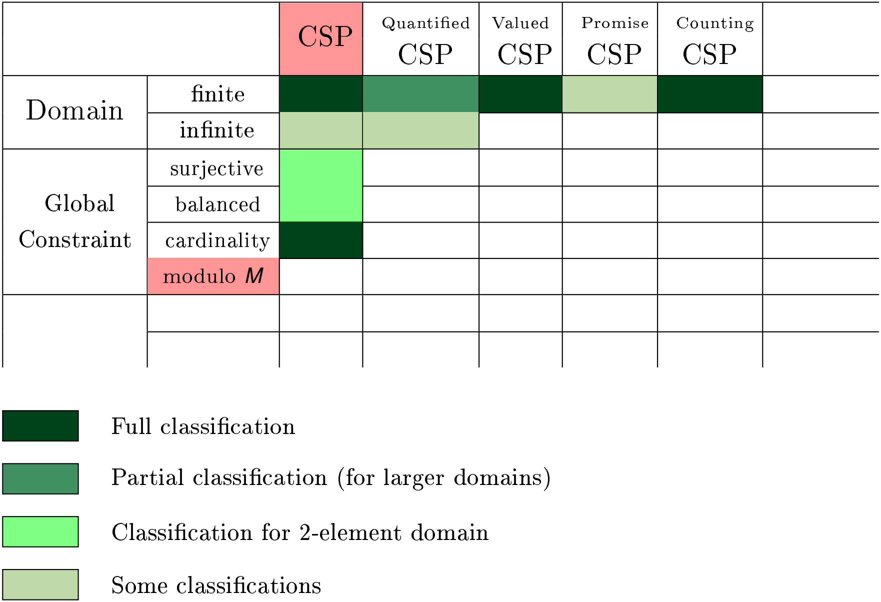
Global modular constraint

Mod_M-CSP(Γ)

Given: a formula $R_1(x_{i_{1,1}}, \dots, x_{i_{1,n_1}}) \wedge \dots \wedge R_s(x_{i_{s,1}}, \dots, x_{i_{s,n_s}})$, where $R_1, \dots, R_s \in \Gamma$.
Decide: whether it has a solution satisfying $x_1 + \dots + x_n = 0 \mod M$.

- If Γ consists of linear equations on $\{0, 1\}$ and $M = 25$ then Mod_M-CSP(Γ) is tractable
- If Γ consists of linear equations on $\{0, 1\}$ and $M = 15$ then Mod_M-CSP(Γ) is not tractable
- If Γ consists of linear equations on $\{0, 1\}$ and $M = 24$ then the complexity of Mod_M-CSP(Γ) is not known.

$$\begin{cases} x_1 + x_2 + x_3 = 0 & \mod 2 \\ x_1 + x_3 + x_5 = 0 & \mod 2 \\ x_2 + x_4 + x_5 = 1 & \mod 2 \\ x_2 + x_3 + x_5 = 0 & \mod 24 \end{cases}$$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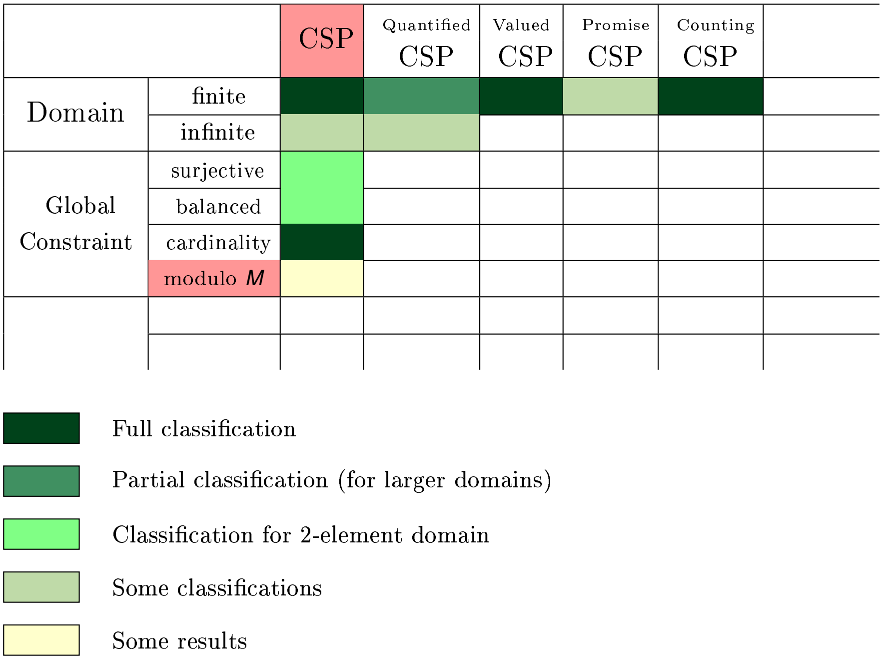
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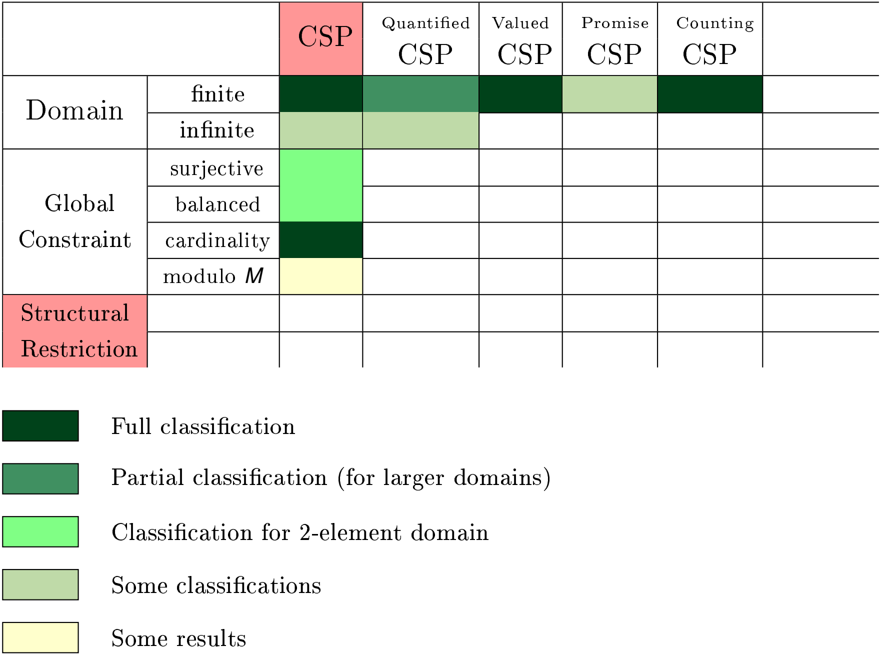
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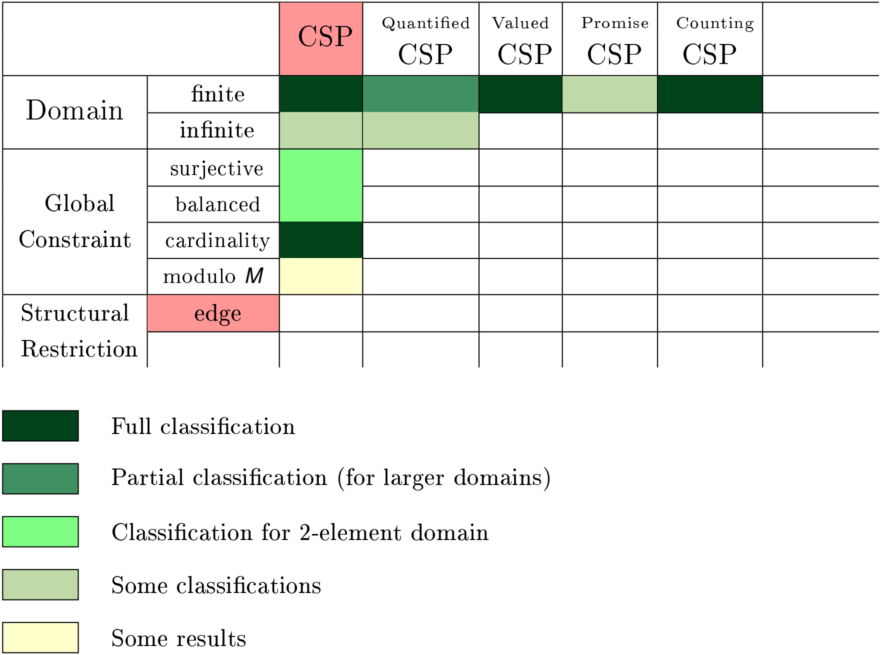
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Edge Constraint Satisfaction Problem

Γ is a set of relations on a finite set A.

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Edge-CSP(Γ)

Given: a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$, where $R_1, \ldots, R_s \in \Gamma$ and every variable appears exactly twice.
Decide: whether it has a solution.

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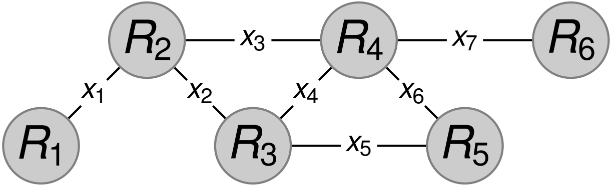

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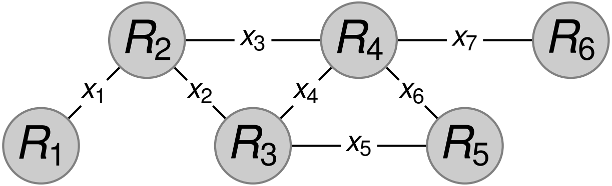

- Edge-CSP($\{$1IN2, 1IN3, 1IN4, $\ldots\}$) is equivalent to the Perfect Matching Problem.

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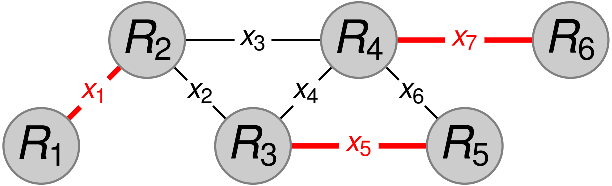

- Edge-CSP($\{1\text{IN}2, 1\text{IN}3, 1\text{IN}4, \dots\}$) is equivalent to the Perfect Matching Problem.

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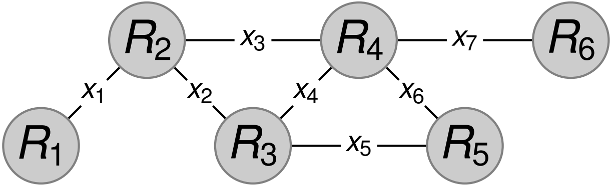
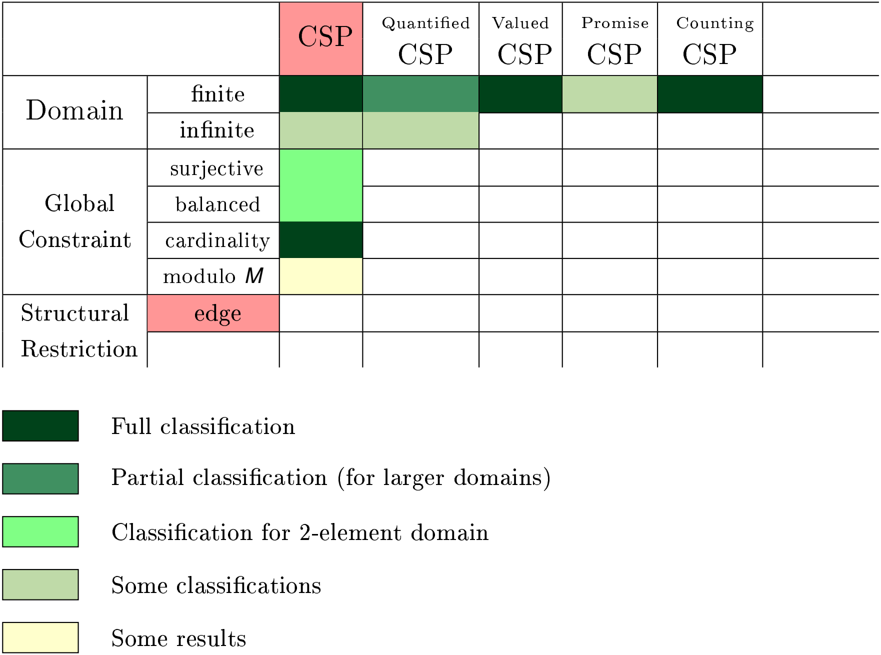


- Edge-CSP($\{1IN2, 1IN3, 1IN4, \dots\}$) is equivalent to the Perfect Matching Problem.

Theorem [Kazda, Kolmogorov, Rolinek, 2018]

A classification of the complexity for planar Edge-CSP(Γ) for every Γ on $\{0, 1\}$.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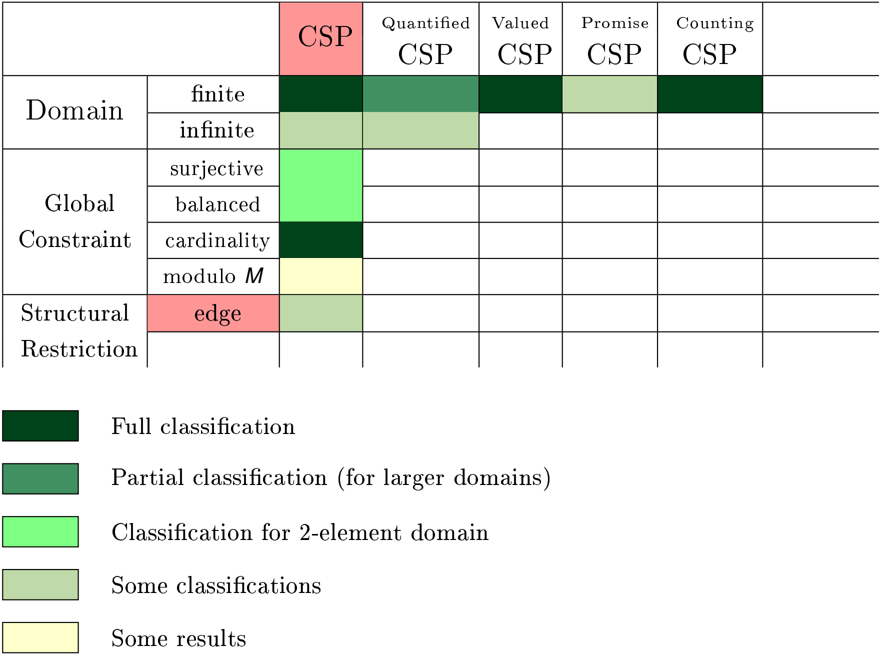
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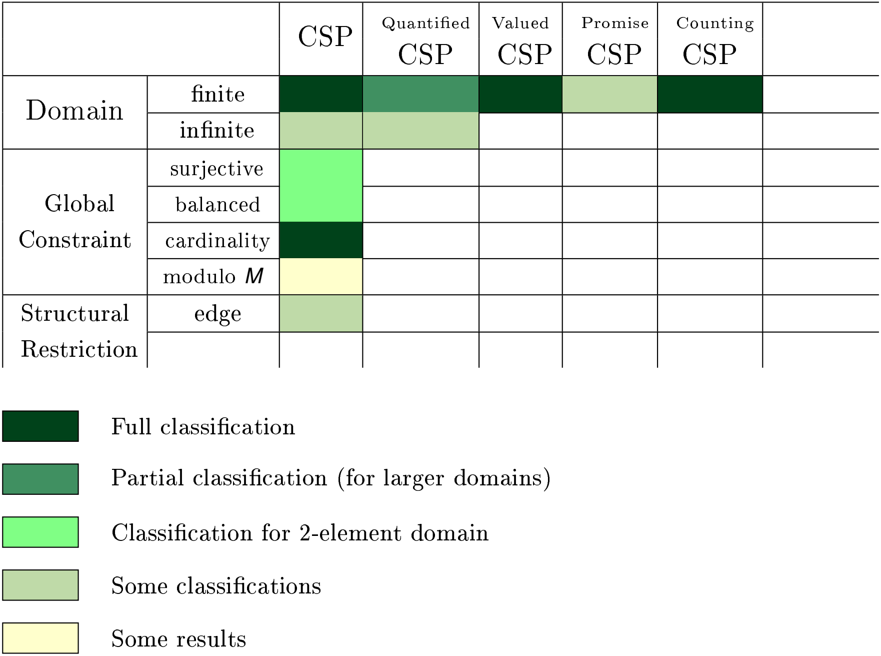
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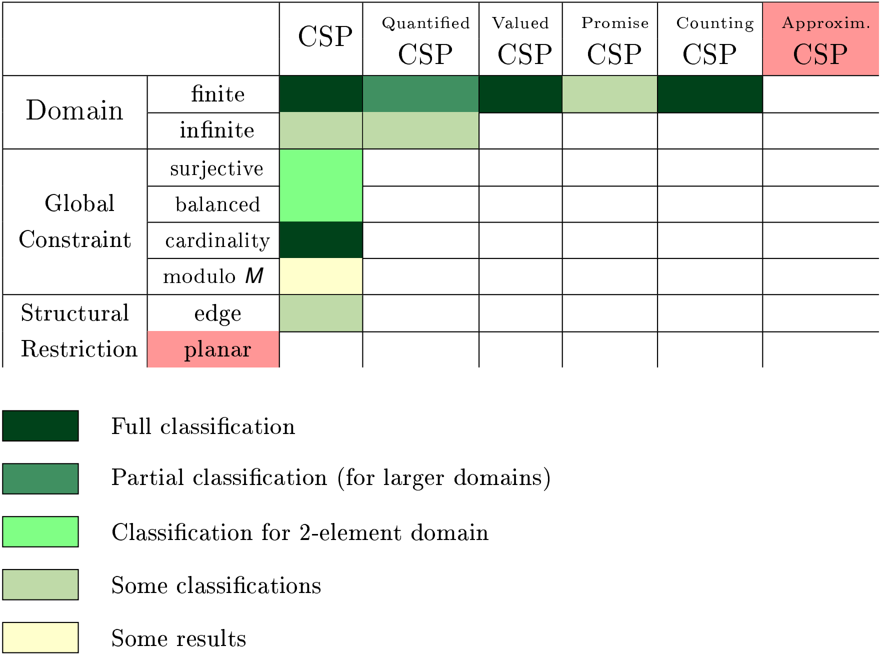
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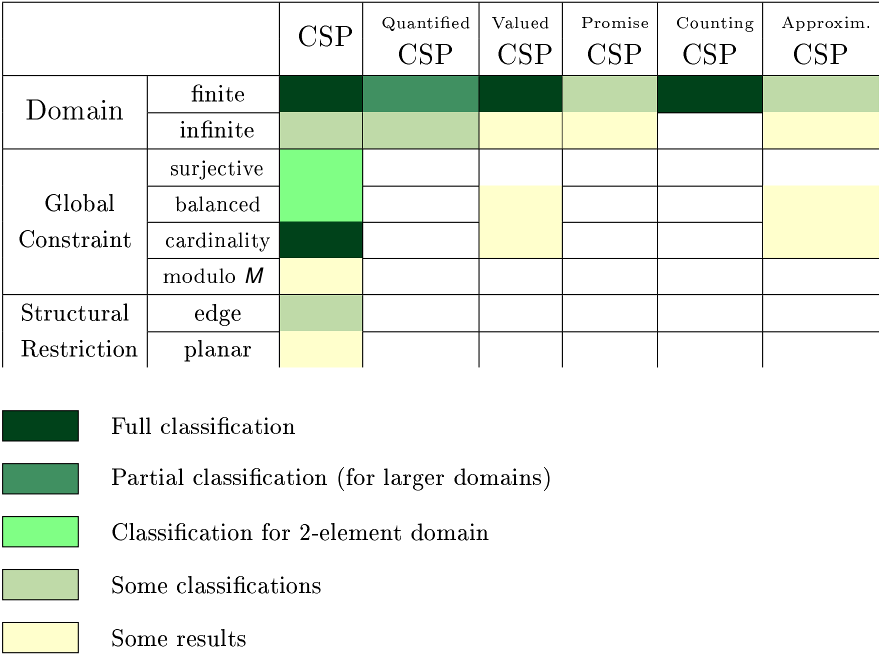
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