Fixed-Template Promise Model Checking Problems

Kristina Asimi[1]
joint work with L. Barto[1] and S. Butti[2]

[1] Department of Algebra, Faculty of Mathematics and Physics, Charles University, Czechia
[2] Department of Information and Communication Technologies, Universitat Pompeu Fabra, Spain

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Outline

Model Checking Problem

Model checking problem :
We define the model checking problem over a logic $\mathcal{L}$ to have

- Input : a structure $\mathbb{A}$ (model), a sentence ϕ of $\mathcal{L}$
- Question : does $\mathbb{A} \models \phi$

First-order model checking problem parameterized by the model :
For any $\mathcal{L} \subseteq \{\exists, \forall, \wedge, \vee, =, \neq, \neg\}$ we define the problem $\mathcal{L}\text{-MC}(\mathbb{A})$ to have

- Input : a sentence ϕ of $\mathcal{L}$-**FO**
- Output : yes if $\mathbb{A} \models \phi$, no otherwise

$\mathcal{L}\text{-MC}(\mathbb{A})$	Complexity
$\{\exists, \wedge\}\text{-MC}(\mathbb{A})$ (CSP)	P or NP-complete
$\{\exists, \forall, \wedge\}\text{-MC}(\mathbb{A})$ (QCSP)	?
$\{\exists, \wedge, \vee\}\text{-MC}(\mathbb{A})$	L or NP-complete
$\{\exists, \forall, \wedge, \vee\}\text{-MC}(\mathbb{A})$	L, NP-complete, coNP-complete, PSPACE-complete

Figure – Known complexity results for $\mathcal{L}\text{-MC}(\mathbb{A})$.

Promise Model Checking Problem

$$\left.\begin{array}{l} \mathbb{A} = (A; R_1^{\mathbb{A}}, R_2^{\mathbb{A}}, \ldots, R_n^{\mathbb{A}}) \\ \mathbb{B} = (B; R_1^{\mathbb{B}}, R_2^{\mathbb{B}}, \ldots, R_n^{\mathbb{B}}) \end{array}\right\} \text{similar relational structures}$$

Definition

A pair of similar structures $(\mathbb{A}, \mathbb{B})$ is called an **$\mathcal{L}$-PMC template** if $\mathbb{A} \models \phi$ implies $\mathbb{B} \models \phi$ for every $\mathcal{L}$-sentence ϕ in the signature of $\mathbb{A}$ and $\mathbb{B}$.
Given an $\mathcal{L}$-PMC template $(\mathbb{A}, \mathbb{B})$, the **$\mathcal{L}$-Promise Model Checking Problem over $(\mathbb{A}, \mathbb{B})$**, denoted $\mathcal{L}$-PMC$(\mathbb{A}, \mathbb{B})$, is the following problem.
Input : an $\mathcal{L}$-sentence ϕ in the signature of $\mathbb{A}$ and $\mathbb{B}$;
Output : yes if $\mathbb{A} \models \phi$; no if $\mathbb{B} \not\models \phi$.

$\mathcal{L}$-$\mathrm{PMC}(\mathbb{A}, \mathbb{B})$	**Condition**	**Complexity**
$\{\exists, \forall, \wedge\}$-$\mathrm{PMC}(\mathbb{A}, \mathbb{B})$		L/NP-complete
$\{\exists, \forall, \wedge, \vee\}$-$\mathrm{PMC}(\mathbb{A}, \mathbb{B})$	AE-smuhom	L
	A-smuhom and E-smuhom	$\mathrm{NP} \cap \mathrm{coNP}$
	A-smuhom, no E-smuhom	NP-complete
	E-smuhom, no A-smuhom	coNP-complete
	no A-smuhom, no E-smuhom	NP-hard and coNP-hard

Figure – Complexity results for $\mathcal{L}$-$\mathrm{PMC}(\mathbb{A}, \mathbb{B})$.

Preliminaries

Let $\mathbb{A}$ and $\mathbb{B}$ be two similar relational structures.

- A function $f : A \to B$ is called a **homomorphism** from $\mathbb{A}$ to $\mathbb{B}$ if $f(\mathbf{a}) \in R^{\mathbb{B}}$ for any $\mathbf{a} \in R^{\mathbb{A}}$, where $f(\mathbf{a})$ is computed component-wise.
- A **multi-valued function** f from A to B is a mapping from A to $\mathcal{P}_{\neq\emptyset}B$.
- It is called **surjective** if for every $b \in B$, there exists $a \in A$ such that $b \in f(a)$.
- A multi-valued function f from A to B is called a **multi-homomorphism** from $\mathbb{A}$ to $\mathbb{B}$ if for any R in the signature and any $\mathbf{a} \in R^{\mathbb{A}}$, we have $f(\mathbf{a}) \subseteq R^{\mathbb{B}}$.
- $\mathrm{MuHom}(\mathbb{A}, \mathbb{B})$ - the set of all multi-homomorphisms from $\mathbb{A}$ to $\mathbb{B}$
  $\mathrm{SMuHom}(\mathbb{A}, \mathbb{B})$ - the set of all surjective multi-homomorphisms from $\mathbb{A}$ to $\mathbb{B}$

We say that a relation $S \subseteq A^n$ is **$\mathcal{L}$-definable** from $\mathbb{A}$ if there exists an $\mathcal{L}$-formula $\psi(v_1, \ldots, v_n)$ such that, for all $(a_1, \ldots, a_n) \in A^n$, we have $(a_1, \ldots, a_n) \in S$ if and only if $\mathbb{A} \models \psi(a_1, \ldots, a_n)$.

Definition

Assume $\neg \notin \mathcal{L}$ and let $(\mathbb{A}, \mathbb{B})$ be a pair of similar structures. We say that a pair of relations (S, T), where $S \subseteq A^n$ and $T \subseteq B^n$, is **promise-$\mathcal{L}$-definable** (or **p-$\mathcal{L}$-definable**) from $(\mathbb{A}, \mathbb{B})$ if there exist relations S' and T' and an $\mathcal{L}$-formula $\psi(v_1, \ldots, v_n)$ such that $S \subseteq S'$, $T' \subseteq T$, $\psi(v_1, \ldots, v_n)$ defines S' in $\mathbb{A}$, and $\psi(v_1, \ldots, v_n)$ defines T' in $\mathbb{B}$.
We say that an $\mathcal{L}$-PMC template $(\mathbb{C}, \mathbb{D})$ is p-$\mathcal{L}$-definable from $(\mathbb{A}, \mathbb{B})$ (the signatures can differ) if $(Q^{\mathbb{C}}, Q^{\mathbb{D}})$ is p-$\mathcal{L}$-definable from $(\mathbb{A}, \mathbb{B})$ for each relation symbol Q in the signature of $\mathbb{C}$ and $\mathbb{D}$.

Theorem

Assume $\neg \notin \mathcal{L}$. If $(\mathbb{A}, \mathbb{B})$ and $(\mathbb{C}, \mathbb{D})$ are $\mathcal{L}$-PMC templates such that $(\mathbb{C}, \mathbb{D})$ is p-$\mathcal{L}$-definable from $(\mathbb{A}, \mathbb{B})$, then $\mathcal{L}\text{-}\mathrm{PMC}(\mathbb{C}, \mathbb{D}) \leq \mathcal{L}\text{-}\mathrm{PMC}(\mathbb{A}, \mathbb{B})$.

$\{\exists, \wedge, \vee\}$-PMC

A pair $(\mathbb{A}, \mathbb{B})$ of similar structures is an $\{\exists, \wedge, \vee\}$-PMC template if and only if there exists a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

Theorem

Let $(\mathbb{A}, \mathbb{B})$ *and* $(\mathbb{C}, \mathbb{D})$ *be* $\{\exists, \wedge, \vee\}$-PMC *templates such that* $A = C$ *and* $B = D$. *Then* $(\mathbb{C}, \mathbb{D})$ *is p-*$\{\exists, \wedge, \vee\}$*-definable from* $(\mathbb{A}, \mathbb{B})$ *if and only if* $\mathrm{MuHom}(\mathbb{A}, \mathbb{B}) \subseteq \mathrm{MuHom}(\mathbb{C}, \mathbb{D})$. *Moreover, in such a case,* $\{\exists, \wedge, \vee\}\text{-PMC}(\mathbb{C}, \mathbb{D}) \leq \{\exists, \wedge, \vee\}\text{-PMC}(\mathbb{A}, \mathbb{B})$.

Theorem

Let $(\mathbb{A}, \mathbb{B})$ *be an* $\{\exists, \wedge, \vee\}$-PMC *template. If there is a constant homomorphism from* $\mathbb{A}$ *to* $\mathbb{B}$, *then* $\{\exists, \wedge, \vee\}\text{-PMC}(\mathbb{A}, \mathbb{B})$ *is in* L, *otherwise* $\{\exists, \wedge, \vee\}\text{-PMC}(\mathbb{A}, \mathbb{B})$ *is* NP*-complete.*

$\{\exists, \forall, \wedge, \vee\}$-PMC

A pair $(\mathbb{A}, \mathbb{B})$ of similar structures is an $\{\exists, \forall, \wedge, \vee\}\text{-PMC}$ template if and only if there exists a surjective multi-homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

Theorem

Let $(\mathbb{A}, \mathbb{B})$ *and* $(\mathbb{C}, \mathbb{D})$ *be* $\{\exists, \forall, \wedge, \vee\}\text{-PMC}$ *templates such that* $A = C$ *and* $B = D$. *Then* $(\mathbb{C}, \mathbb{D})$ *is p-*$\{\exists, \forall, \wedge, \vee\}$*-definable from* $(\mathbb{A}, \mathbb{B})$ *if and only if* $\mathrm{SMuHom}(\mathbb{A}, \mathbb{B}) \subseteq \mathrm{SMuHom}(\mathbb{C}, \mathbb{D})$. *Moreover, in such a case,* $\{\exists, \forall, \wedge, \vee\}\text{-PMC}(\mathbb{C}, \mathbb{D}) \leq \{\exists, \forall, \wedge, \vee\}\text{-PMC}(\mathbb{A}, \mathbb{B})$.

Let f be a surjective multi-homomorphism from $\mathbb{A}$ to $\mathbb{B}$. We say that :

- f is an $\mathsf{A}\text{-smuhom}$ if there exists $a^* \in A$ such that $f(a^*) = B$.
- f is an $\mathsf{E}\text{-smuhom}$ if $f^{-1}(b^*) = A$ for some $b^* \in B$.
- f is an $\mathsf{AE}\text{-smuhom}$ if it is simultaneously an $\mathsf{A}\text{-smuhom}$ and an $\mathsf{E}\text{-smuhom}$.

Theorem

Let $(\mathbb{A}, \mathbb{B})$ *be an* $\{\exists, \forall, \wedge, \vee\}$-PMC *template. Then the following holds.*

1. *If* $(\mathbb{A}, \mathbb{B})$ *admits an* A-smuhom*, then* $\{\exists, \forall, \wedge, \vee\}$-PMC$(\mathbb{A}, \mathbb{B})$ *is in* NP*.*
2. *If* $(\mathbb{A}, \mathbb{B})$ *admits an* E-smuhom*, then* $\{\exists, \forall, \wedge, \vee\}$-PMC$(\mathbb{A}, \mathbb{B})$ *is in* coNP*.*
3. *If* $(\mathbb{A}, \mathbb{B})$ *admits an* AE-smuhom*, then* $\{\exists, \forall, \wedge, \vee\}$-PMC$(\mathbb{A}, \mathbb{B})$ *is in* L*.*

Theorem

Let $(\mathbb{A}, \mathbb{B})$ *be an* $\{\exists, \forall, \wedge, \vee\}$-PMC *template.*

1. *If there is no* E-smuhom *from* $\mathbb{A}$ *to* $\mathbb{B}$*, then* $\{\exists, \forall, \wedge, \vee\}$-PMC$(\mathbb{A}, \mathbb{B})$ *is* NP*-hard.*
2. *If there is no* A-smuhom *from* $\mathbb{A}$ *to* $\mathbb{B}$*, then* $\{\exists, \forall, \wedge, \vee\}$-PMC$(\mathbb{A}, \mathbb{B})$ *is* coNP*-hard.*

Open problems

Examples of templates that admit both an A-smuhom and an E-smuhom, but no AE-smuhom :

$$\mathbb{A} = ([3];\ \{(1,2,3)\}), \quad \mathbb{B} = ([3];\ \{1,2,3\} \times \{2\} \times \{3\}\ \cup\ \{1,2\} \times \{2\} \times \{2,3\})$$
$$\mathbb{A} = ([3];\ \{12\},\ \{13\}), \quad \mathbb{B} = ([3];\ \{12,22,32\},\ \{12,13,22,23,33\})$$

Is $\{\exists, \forall, \wedge, \vee\}$-PMC$(\mathbb{A}, \mathbb{B})$ in L ?

Examples of templates that admit neither an A-smuhom nor an E-smuhom :

$$\mathbb{A} = ([3];\ \{(1,2,3)\}), \quad \mathbb{B} = ([3];\ \{2,3\} \times \{1,3\} \times \{1,2\})$$
$$\mathbb{A} = ([3];\ \{(1,2,3)\}), \quad \mathbb{B} = ([3]; \{1,2\} \times \{1,2\} \times \{3\}\ \cup\ \{1,3\} \times \{2\} \times \{2\})$$
$$\mathbb{A} = ([4];\ \{12,34\}), \quad \mathbb{B} = ([4];\ \{12,13,14,23,24,34,32\})$$

Is $\{\exists, \forall, \wedge, \vee\}$-PMC$(\mathbb{A}, \mathbb{B})$ PSPACE-complete ?

Thank you for your attention !