Symmetric Promise Constraint Satisfaction Problems

Beyond the Boolean Case

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Examples of PCSPs

PCSPs can express classic CSPs, e.g. 3SAT, and more. For example, the problem of finding an l-coloring of a k-colorable graph when $k \leq l$ is $\mathrm{PCSP}(\mathbf{K}_k, \mathbf{K}_l)$.

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CSPs are known to have a hardness dichotomy – either NP-complete or in P (Bulatov, Zhuk '17). No such dichotomy is currently known for PCSPs. There is a dichotomy theorem over Boolean *symmetric* templates (Brakensiek, Guruswami '18, Ficak et al. '19), i.e., templates for which the relations are invariant under permutations.

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1in3 is $\{0, 1\}$ with the single relation $\{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$.
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E.g., **1in3** becomes $\to$ and **NAE**, the relation for Not-All-Equal 3SAT, becomes $\leftrightarrows$.

Diagrams of Three Element Symmetric Structures

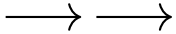
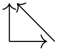
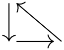
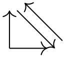
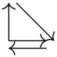
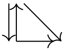
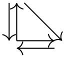
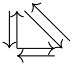
Diagram						
Structure **B**	**1in3**	**NAE**	$\mathbf{D}_1$	$\mathbf{D}_2$	$\mathbf{T}_1$	$\mathbf{T}_2$

Diagram					
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The Hierarchy of Three Element Symmetric Structures

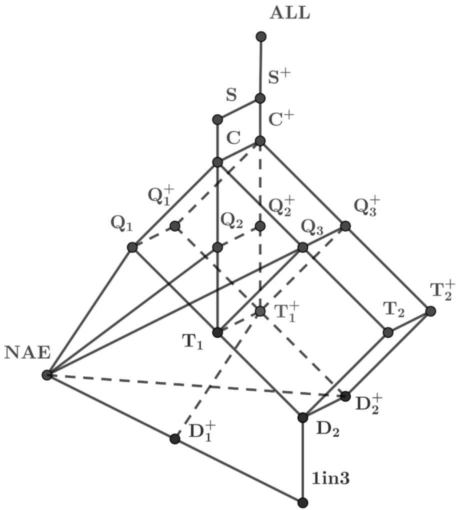

Figure: The templates **B** ordered by the relation $\mathbf{B} \leq \mathbf{B}'$ if $\mathbf{B} \to \mathbf{B}'$.

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By combining this hierarchy with known hardness criteria (e.g. Brandts, Wrochna, Živný '20) and sufficient tractability conditions (e.g. Brakensiek, Guruswami '20), we were able to classify all but one case:

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Let $(\mathbf{1in3}, \mathbf{B})$ be a PCSP template, where **B** *has domain-size three.*

- *If* $\mathbf{NAE} \to \mathbf{B}$ *or* $\mathbf{T}_2 \to \mathbf{B}$*, then* $\mathrm{PCSP}(\mathbf{1in3}, \mathbf{B})$ *is in P.*

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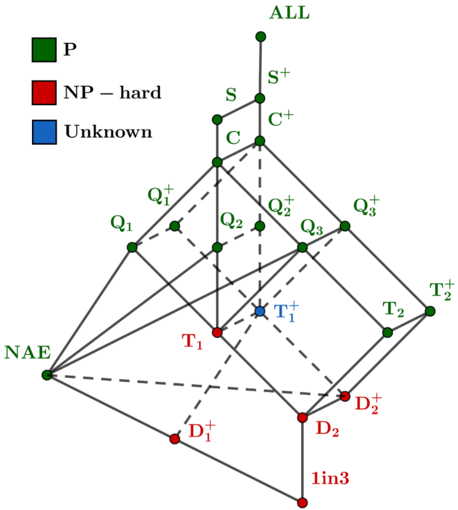
- *If $\mathbf{B} \to \mathbf{T}_1$ or $\mathbf{B} \to \mathbf{D}_1^+$ or $\mathbf{B} \to \mathbf{D}_2^+$, then $\mathrm{PCSP}(\mathbf{1in3}, \mathbf{B})$ is NP-hard.*

The Hierarchy of the Results



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If true, there is a unique source of hardness for our templates.

Larger Domains

When $|B| = 4$, our conjecture leaves the interval between $\check{\mathbf{C}}$ and $\check{\mathbf{C}}^+$, where $\check{\mathbf{C}}$ is given by the relation of the permutations of $(0, 0, 1)$, $(1, 1, 2)$, $(2, 2, 3)$, $(3, 3, 0)$.

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A negative answer would also be valuable – it would need a P algorithm that has not yet been used for PCSPs!

Thank you for your time!