Minimal Taylor Clones

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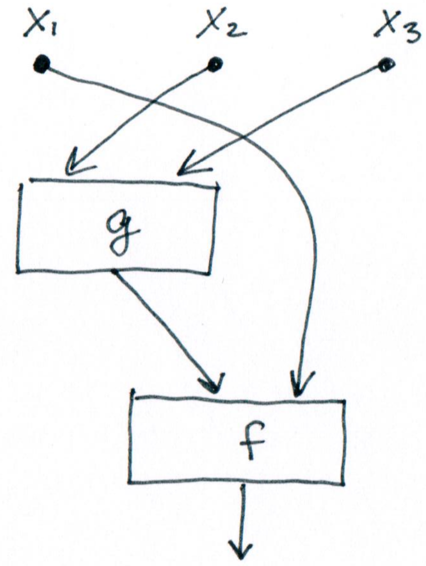
CLONES

Clone on A: set of operations on A ($A^n \to A$) closed under forming term operations

e.g. if $f, g \in \mathcal{C}$ binary, then $h \in \mathcal{C}$, where
$h(x_1, x_2, x_3) = f(g(x_2, x_3), x_1)$



How clones show up

- $\mathrm{Clo}(A; \overbrace{f_1, f_2, \ldots}^{\text{basic operations / gates}})$
  = clone generated by $f_1, f_2, \ldots$
  – all operations expressible using gates $f_1, \ldots$

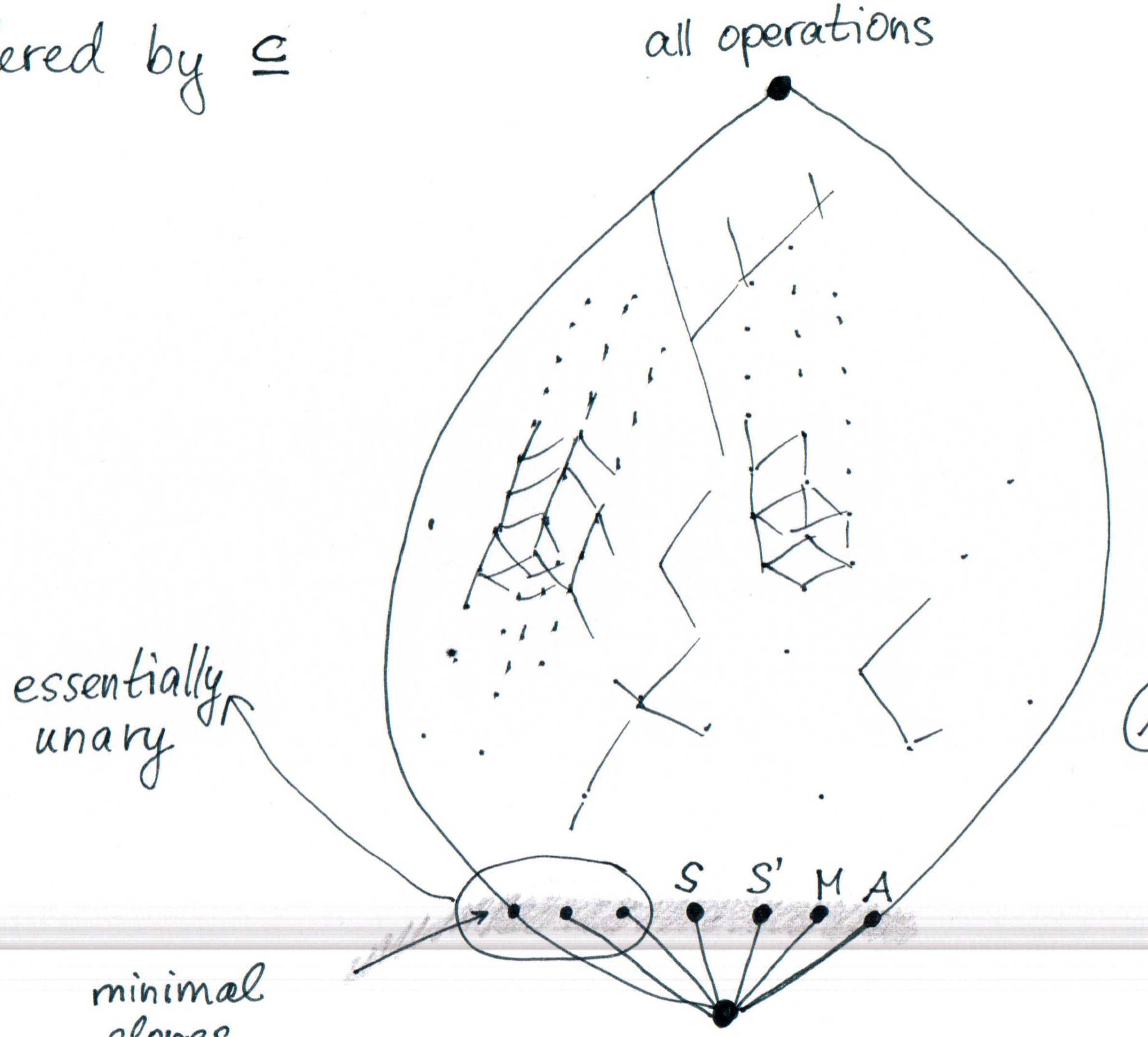
- $\mathrm{Pol}(A; \underbrace{R_1, R_2, \ldots}_{\text{relations}})$, i.e. $R_i \subseteq A^{k_i}$
  = all polymorphisms = compatible operations, "symmetries"

Theorem Every clone on <u>finite</u> A is of this form
[Geiger 60s, Bodnarchuk et al. 60s]

CLONES ON $\{0,1\}$

We know all of them [Post 40s]

ordered by $\subseteq$



$Proj_2$ = projections $(x_1, \dots, x_n) \mapsto x_i$

(S) $Clo(\{0,1\}; \min)$

(S') $Clo(\{0,1\}; \max)$

(M) $Clo(\{0,1\}; \text{3-majority})$
$= Pol(\{0,1\}; \leq, \neq)$
monotone, self-dual

(A) $Clo(\{0,1\}; x+y+z \bmod 2)$
$= Pol(\{0,1\}; \text{affine subspaces of } GF(2)^n)$

CONSTRAINT SATISFACTION PROBLEMS

$\mathrm{CSP}(A; R_1, R_2, \dots, R_k)$ (A finite)

INPUT: conjunction of atomic formulas (= constraints)

e.g. $R_1(x, y, z) \wedge R_2(z, x) \wedge R_1(u, u, y)$

QUESTION: satisfiable?

Examples: 3-SAT, 3-COLORING, solving system of equations over...

Theorem: Computational complexity depends only on $\mathcal{C} = \mathrm{Pol}(A; R_1, R_2, \dots, R_k)$ [Jeavons et al. 90s]

- the bigger $\mathcal{C}$, the easier CSP

Theorem: Can restrict to idempotent $\mathcal{C}$: $\forall f \in \mathcal{C}\ \forall x \in A\ f(x, x, \dots, x) = x$ [Bulatov et al 00s]

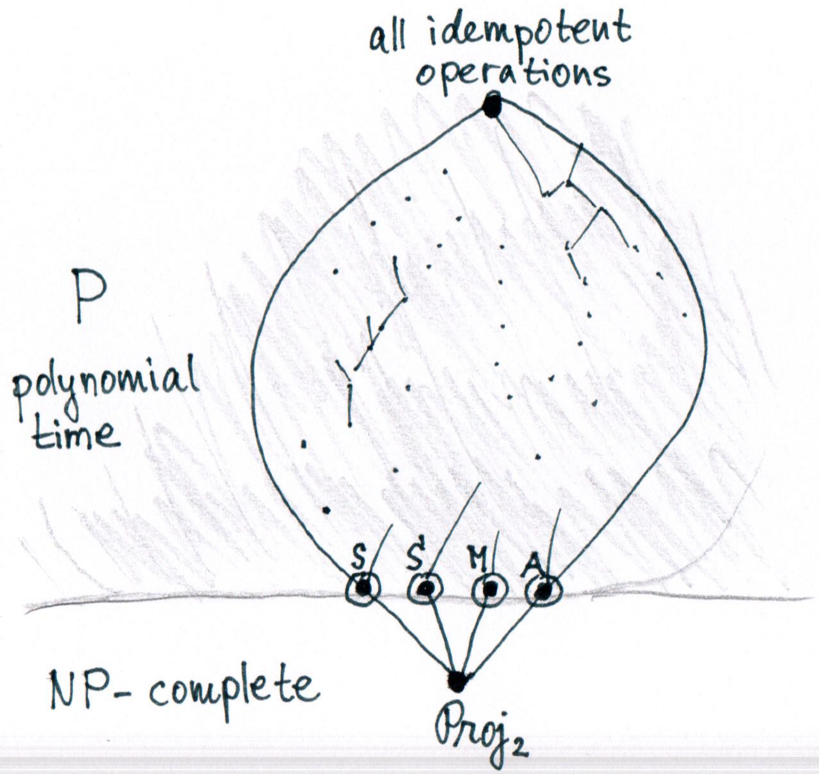
= clones with trivial unary part

- we restrict to idempotent clones

CSP DICHOTOMY THEOREM

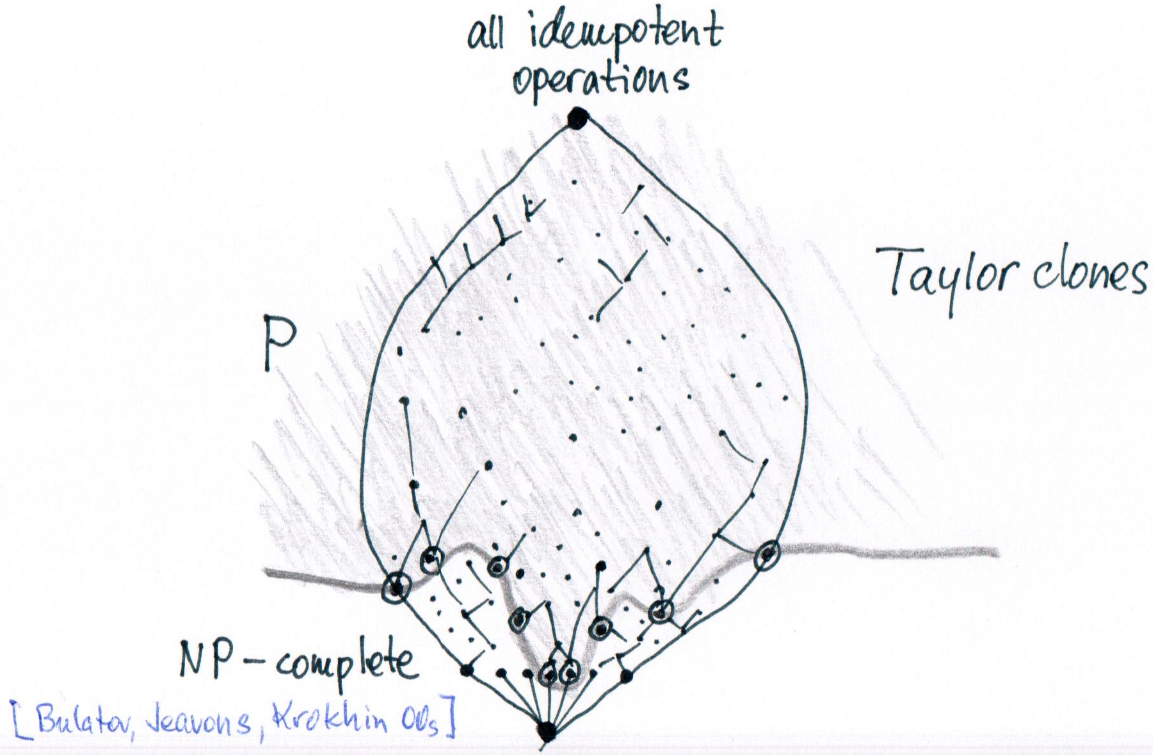
$A = \{0,1\}$

[Schaefer 70s]



$|A| > 2$

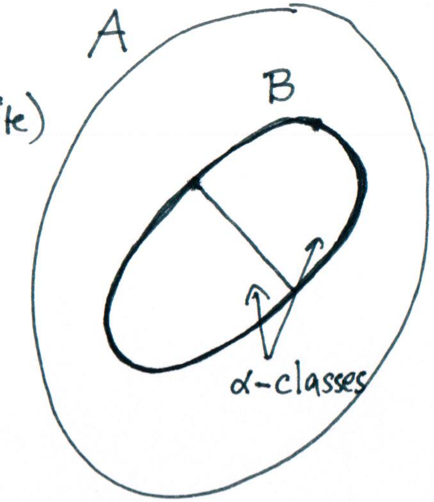
[Bulatov 17, Zhuk 17]



- "hardest" CSPs in P
  → explicit list of CSPs in P

- minimal Taylor clones

TAYLOR CLONES



Taylor clone: No factor is Proj_2 (+ idempotent, finite)

Minimal Taylor clone: Minimal (wrt $\leq$) among Taylor clones on A

B **subuniverse of $\mathcal{C}$** = invariant subset $B \subseteq A$
can define clone $\mathcal{C} \restriction B$ on B

α **congruence of $\mathcal{C}$** = invariant equivalence relation on A
can define clone $\mathcal{C}/\alpha$ on A/α

factor of $\mathcal{C}$ = $\mathcal{C} \restriction B / \alpha$ where B subuniverse of $\mathcal{C}$
α congruence of $\mathcal{C} \restriction B$

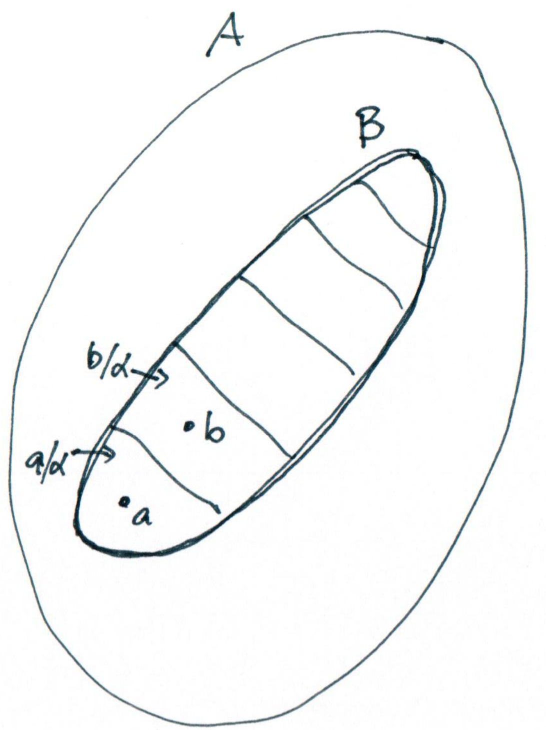
Theorem $\mathcal{C}$ is Taylor $\Leftrightarrow$ $\mathcal{C}$ contains a p-ary cyclic operation (p prime $> |A|$)
[Barto, Kozik '10s]

$\forall x_1, x_2, \ldots \in A$
$f(x_1, x_2, \ldots, x_p) = f(x_2, \ldots, x_p, x_1)$

Tools for Taylor clones: classic + Bulatov's theory, Zhuk's theory, absorption theory

BULATOV

$\mathcal{C}$ clone on A $\longmapsto$ directed graph on A



$a \longrightarrow b$ if $\exists$ factor $\mathcal{C}{\restriction}B/\alpha$ such that $a, b \in B$, $a/\alpha \neq b/\alpha$ and

$a \xrightarrow{\text{semilattice}} b$ $\exists$ binary $s \in \mathcal{C}$ such that s on $\{a/\alpha, b/\alpha\}$ is max on $\{0,1\}$ (i.e. $s(a/\alpha, b/\alpha) = b/\alpha = s(b/\alpha, a/\alpha)$)

$a \xrightarrow{\text{majority}} b$ $\exists$ ternary $m \in \mathcal{C}$ such that m on $\{a/\alpha, b/\alpha\}$ is majority on $\{0,1\}$

$a \xrightarrow{\text{affine}} b$ $\mathcal{C}{\restriction}B/\alpha$ is affine, i.e. $\mathrm{Clo}(\mathcal{C}{\restriction}B/\alpha + \text{constants}) = \mathrm{Clo}(M)$ for some module M

Fundamental theorem $\mathcal{C}$ Taylor $\Rightarrow$ this directed graph is connected

ZHUK

Fundamental theorem $\mathcal{C}$ Taylor $\Rightarrow$ one of the following

- $\mathcal{C}$ has a nontrivial 2-absorbing subuniverse e.g. $\{1\}$ in $\mathrm{Clo}(\{0,1\}; \max)$

  i.e. $\exists$ binary $s \in \mathcal{C}$ $s(B,A) \cup s(A,B) \subseteq B$

- $\mathcal{C}$ —— " —— 3-absorbing subuniverse (+ extra prop.) e.g. $\{0\}$, $\{1\}$ in $\mathrm{Clo}(\{0,1\}, \text{majority})$

  i.e. $\exists$ ternary $m \in \mathcal{C}$ $m(B,B,A) \cup m(B,A,B) \cup m(A,B,B) \subseteq B$

- $\mathcal{C}$ has a proper congruence α such that $\mathcal{C}/\alpha$ is affine e.g. $=$ in $\mathrm{Clo}(\{0,1\}, x+y+z)$

- $\mathcal{C}$ —— " —— $\mathcal{C}/\alpha$ is polynomially complete e.g. $=$ in $\mathrm{Clo}(\{\text{rock, paper, scissors}\}; \text{winner})$

  i.e. $\mathrm{Clo}(\mathcal{C}/\alpha + \text{constants}) = \text{all operations}$

RESULTS

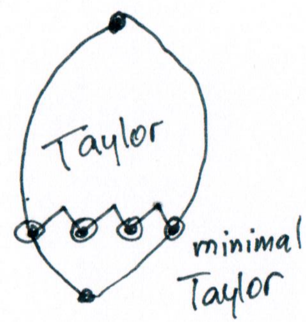
L. Barto, Z. Brady, A. Bulatov, M. Kozik, D. Zhuk: Minimal Taylor Algebras as a Common Framework for the Three Algebraic Approaches to the CSP, LICS'21, arXiv

Taylor clones

- connections absorption ⟷ Zhuk
- simple theorem that implies both "Fundamental theorems"

Minimal Taylor clones

- "exist": every Taylor clone has a minimal Taylor clone as a subclone
- concepts get simpler and stronger
- surprising connections Bulatov ⟷ Zhuk



Follow up work

- all minimal Taylor clones on $\{0,1,2\}$ found (24 up to renaming elements) [Brady]

  ⟶ explicit list of CSPs in P for $|A|=3$

RESULTS: EDGES

$\mathcal{C}$ minimal Taylor

Theorem Witness $\mathcal{D} = \mathcal{C}\restriction_B/\alpha$ for $a \to b$ can be chosen so that

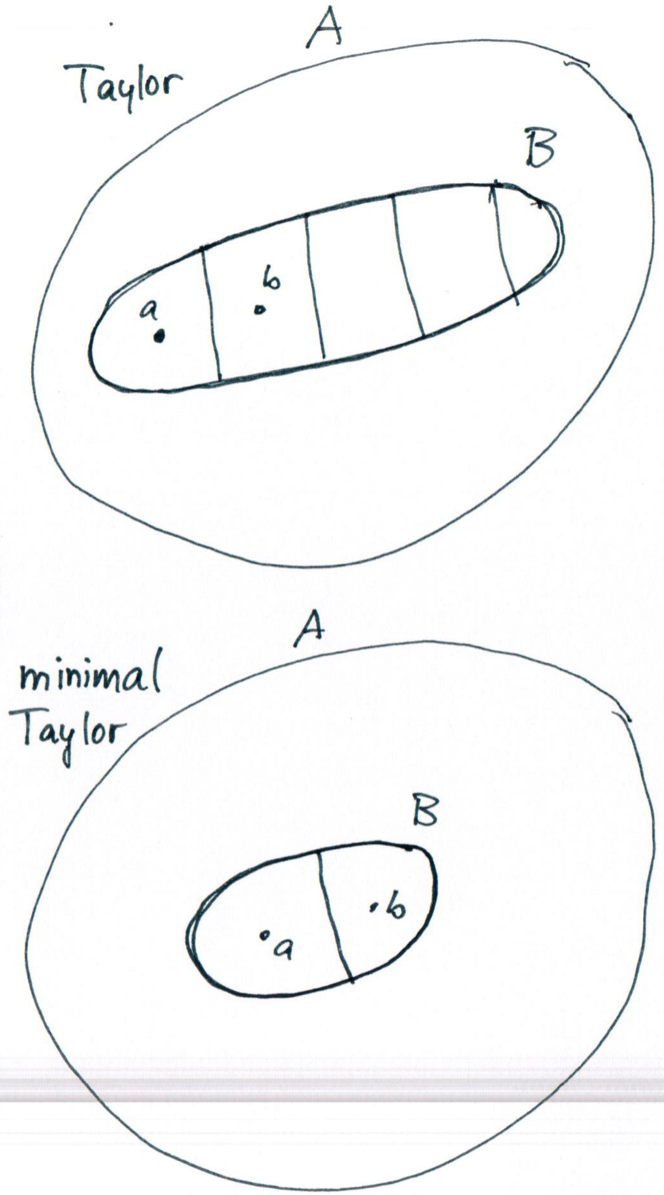
$a \xrightarrow{\text{semilattice}} b$ $\quad \mathcal{D}$ is $\mathrm{Clo}(\{0,1\}; \max)$
(up to renaming elements)

$a \xrightarrow{\text{majority}} b$ $\quad \mathcal{D}$ is $\mathrm{Clo}(\{0,1\}; \text{majority})$

$a \xrightarrow{\text{affine}} b$ $\quad \mathcal{D}$ is $\mathrm{Clo}(G; x-y+z)$

abelian group

- more information about "minimal" edges
  e.g. unique type



RESULTS: ABSORPTION

$\mathcal{C}$ minimal Taylor

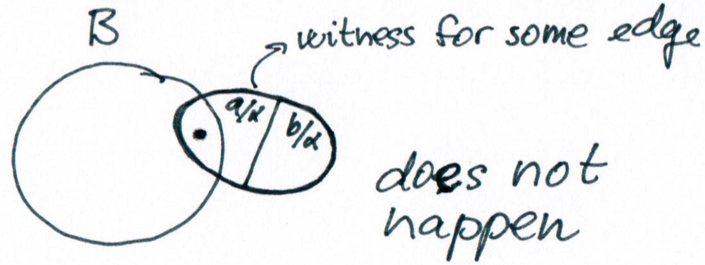
Theorem TFAE for $B \subseteq A$

- B is a 2-absorbing <u>subset</u> of $\mathcal{C}$
  i.e $\exists s \in Clo(\mathcal{C})$ $s(B,A) \cup s(A,B) \subseteq B$
- —— " —————— <u>subuniverse</u> of $\mathcal{C}$

"$\mathcal{C}$ acts on B as max on $\{0,1\}$"

- $\forall t \in \mathcal{C}$ that depends on 1st coordinate
  $t(B, A, A, \dots, A) \subseteq B$
- B is stable under edges

i.e.



- further strong properties
  e.g. $\exists$unique minimal 2-absorbing subuniverse
- similar (a bit weaker) theorem for 3-absorption

RESULTS: UNIFIED WITNESSES

$\mathcal{C}$ minimal Taylor

Theorem $\exists$ ternary $f \in \mathcal{C}$ "witnessing all edges and 2,3-absorptions"

- if $a \xrightarrow{\text{semilat.}} b$ then $f(x,y,z) = \max(x,y,z)$ on $\mathcal{C} \restriction B/\alpha$ from definition of edge
- if $a \xrightarrow{\text{major}} b$ then $f(x,y,z) = \text{majority}(x,y,z)$ — " —
- if $a \xrightarrow{\text{affine}} b$ then $f(x,y,z) = x - y + z$ — " —
- if B is 3-absorbing, then $f(B,B,A) \cup \dots \subseteq B$
- if B is 2-absorbing, then $f(B,A,A) \cup \dots \subseteq B$

In fact $f(x,y,z) = t(x,x,\dots,x,y,z,z,\dots z)$ for a cyclic $t \in \mathcal{C}$

Moreover, any such f generates $\mathcal{C}$.

$\Rightarrow$ minimal Taylor clone is generated by a single ternary operation!

RESULTS: OMITTING EDGE TYPES

$\mathbf{C}$ minimal Taylor

Theorem TFAE

- no affine or semilattice edges (i.e. only majority edges)
- $\mathbf{C}$ has a majority operation $m(xxy) = m(xyx) = m(yxx) = x$
- $\mathbf{C}$ has a near unanimity operation $n(xx..xy) = n(x...xyx) = ... n(yx..x) = x$

Theorem TFAE

- no semilattice or majority edges (only affine edges)
- no $\mathbf{C} \restriction B$ has a nontrivial absorbing subuniverse
- $\mathbf{C}$ has a Mal'cev operation $p(yxx) = y = p(xxy)$

- similar theorems for avoiding 1 type of edges

SUMMARY

Minimal Taylor clones are

- much nicer than general Taylor clones
- sufficiently general for some purposes (e.g. CSP)

LONG TERM AIMS

- simplify the proofs of the dichotomy theorem for CSPs
- find all ((minimal) Taylor) clones
- create one coherent theory incorporating Bulatov, Zhuk, absorption + classic theories (TCT, commutator theory)

SPECIFIC QUESTIONS

- many (e.g. see the paper)
- the most embarrassing: $\mathcal{C}$ minimal Taylor clone on A, $a, b \in A$, $a \neq b$. Is always $a \to b$ or $b \to a$?

THANK YOU

Appendix:

SIMPLE THEOREM ABOUT RELATIONS

Theorem Let R be an irredundant subdirect proper relation on A. Then

- R pp-defines a binary irredundant subdirect proper relation on A, or
- $\exists$ ternary strongly functional $R_1, \ldots, R_n$ such that the set $\{R_1, \ldots, R_n\}$ is inter-pp-definable with R

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redundant some binary projection is a graph of a bijection

strongly functional graph of a quasigroup operation