Infinity: Relevant or Irrelevant?

Kristina Asimi

Department of Algebra, Charles University, Prague

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CSP - Constraint Satisfaction Problem

- $$\left.\begin{array}{l}\mathcal{E} = (E; S_1, S_2, \dots, S_n) \\ \mathcal{D} = (D; R_1, R_2, \dots, R_n)\end{array}\right\}$$ similar relational structures
- $h : E \to D$ is a homomorphism from $\mathcal{E}$ to $\mathcal{D}$ if
  $(a_1, a_2, \dots, a_k) \in S_i \Rightarrow (h(a_1), h(a_2), \dots, h(a_k)) \in R_i$
- $\mathrm{CSP}(\mathcal{D})$
  Search: Given $\mathcal{E}$ find $\mathcal{E} \to \mathcal{D}$.

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Examples:

- 1-in-3-SAT (NP-complete)
  $\text{1-in-3} = (\{0,1\};\{(1,0,0),(0,1,0),(0,0,1)\})$
- NAE-3-SAT (NP-complete)
  $\text{NAE-3} = (\{0,1\};\{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\})$

Theorem ([Bulatov '17];[Zhuk '17])

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- $\mathrm{PCSP}(\mathcal{A}, \mathcal{B})$
- $\mathcal{A}, \mathcal{B}$ - relational structures, $\mathcal{A} \to \mathcal{B}$
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  - 5-coloring of a 3-colorable graph (NP-complete)
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PCSP dichotomy?

- Dichotomy for PCSP?
- Yes for symmetric Boolean PCSPs (allowing negations)
- allows negations:
  $\mathcal{A} = (\{0,1\}; \neq, \dots)$
  $\mathcal{B} = (\{0,1\}; \neq, \dots)$
  where $\neq = \{(0,1),(1,0)\}$

Theorem (Brakensiek, Guruswami '17)

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Classification

j-in-k $= \{(x_1, x_2, \dots, x_k) : \Sigma x_i = j\}$
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c) $(\text{j-in-k}, \text{NAE-k})$
then $\mathrm{PCSP}((\{0,1\}; P, \neq), (\{0,1\}; Q, \neq))$ *is tractable.*

All tractable cases: relaxations and modifications of the cases above

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PCSP(1-in-3, NAE-3)

- If $\mathcal{A} \to \mathcal{C} \to \mathcal{B}$, then
  CSP($\mathcal{C}$) is in P $\Rightarrow$ PCSP($\mathcal{A}, \mathcal{B}$) is in P
- $(\mathcal{A}, \mathcal{B}) = (\text{1-in-3}, \text{NAE-3})$
  - $\exists$ such a $\mathcal{C}$ infinite
  - necessarily so [Barto]
- If $\mathcal{C}$ has to be infinite, we say that the PCSP is infinitary.

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Infinity is relevant

c) $(\text{j-in-k}, \text{NAE-k})$

- j - odd, k - odd: infinitary
- j - even, k - even, $j \neq \frac{k}{2}$: infinitary
- j - odd, k - even: reducible to finite CSP
- j - even, k - odd: probably infinitary
  ($j = 2$, $k = 5$: infinitary)

Preliminaries

Definition

Let $\mathcal{C}$ be a CSP template. $c : C^n \to C$ is a *polymorphism* of $\mathcal{C}$ if for each relation R in $\mathcal{C}$

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \in R, \ldots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \in R \Rightarrow \begin{bmatrix} c(a_{11}, \ldots, a_{1n}) \\ \vdots \\ c(a_{m1}, \ldots, a_{mn}) \end{bmatrix} \in R.$$

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$c : C^n \to C$ is cyclic if

$$c(a_1, a_2, \ldots, a_n) = c(a_2, \ldots, a_n, a_1)$$

Theorem (Barto, Kozik '12)

Let $\mathcal{C}$ be a finite CSP template. If $\mathrm{CSP}(\mathcal{C})$ is not NP-complete, then $\mathcal{C}$ has a cyclic polymorphism of arity p for every prime number $p > |C|$.

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$\mathrm{PCSP}(\text{2-in-5}, \text{NAE-5})$ that allows negations is infinitary

Theorem

Let $\mathcal{C} = (C; R, N)$ be a finite relational structure such that $(\{0,1\}; \text{2-in-5}, \neq) \to \mathcal{C} \to (\{0,1\}; \text{NAE-5}, \neq)$. Then $\mathrm{CSP}(\mathcal{C})$ is NP-complete.

- assume $\mathrm{CSP}(\mathcal{C})$ is not NP-complete
- $f : (\{0,1\}; \text{2-in-5}, \neq) \to \mathcal{C}$
  $g : \mathcal{C} \to (\{0,1\}; \text{NAE-5}, \neq)$
- gf is a homomorphism
  $gf(1,1,0,0,0) \in \text{NAE-5}$, $gf(0,1) \in \neq$
- $f(0) \neq f(1)$
  We rename the elements of C so that $\{0,1\} \subseteq C$ and f is the inclusion
- $\text{2-in-5} \subseteq R$; $(a,b,c,d,e) \in R \Rightarrow (g(a), g(b), g(c), g(d), g(e)) \in \text{NAE-5}$
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$\mathrm{PCSP}(\text{2-in-5}, \text{NAE-5})$ that allows negations is infinitary

Theorem

Let $\mathcal{C} = (C; R, N)$ *be a finite relational structure such that* $(\{0,1\}; \text{2-in-5}, \neq) \to \mathcal{C} \to (\{0,1\}; \text{NAE-5}, \neq)$*. Then* $\mathrm{CSP}(\mathcal{C})$ *is NP-complete.*

- assume $\mathrm{CSP}(\mathcal{C})$ is not NP-complete
- $f : (\{0,1\}; \text{2-in-5}, \neq) \to \mathcal{C}$
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- $\text{2-in-5} \subseteq R$; $(a,b,c,d,e) \in R \Rightarrow (g(a), g(b), g(c), g(d), g(e)) \in \text{NAE-5}$
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- $\mathcal{C}$ has a cyclic polymorphism of arity p for every prime number $p > |C|$
- For every cyclic polymorphism c $g(c_0) = g(c_1) = \cdots = g(c_{\lfloor \frac{2}{5}n \rfloor}) \neq g(c_{\lfloor \frac{2}{5}n \rfloor + 1}) = \cdots = g(c_n)$, where $n = 5l + 1 \geq 11$ is the arity of c and $c_m = c(\underbrace{1, \ldots, 1}_{m}, 0, \ldots, 0)$
- But there exist $i, j \in \{\lfloor \frac{2}{5}n \rfloor + 1, \ldots, n\}$, $i \neq j$, such that $i + j = n$. Since c is polymorphism, $g(c_i) \neq g(c_j)$
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Thanks for your attention!