Every evolution equation with a strict Lyapunov function is a gradient system

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Abstract

It was shown in Bárta et al (2012) that every ordinary differential equation with a strict Lyapunov function is a gradient system for an appropriate Riemannian metric. We extend this result to evolution equations in Hilbert spaces including most of PDEs. We further study extensions of the gradient structure to stationary points.

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1 Introduction

It was shown in Bárta et al (2012) that every ordinary differential equation (even on a Riemannian manifold) with a strict Lyapunov function is a gradient system for an appropriate Riemannian metric. In particular, if an ODE

$$\dot{u} + F(u) = 0 \tag{ODE}$$

is given and $\mathbb{E}$ is a strict Lyapunov function for (ODE), i.e. $\langle \nabla E(u), F(u)\rangle > 0$ whenever $F(u) \neq 0$, then we can find a Riemannian metric r such that the gradient of $\mathbb{E}$ with respect to r is equal to F, i.e.

$$\nabla_r \mathbb{E}(u) = F(u),$$

where $\nabla_r \mathbb{E}(u)$ is defined by $\langle \nabla_r \mathbb{E}(u), v\rangle = \mathbb{E}'(u)v$ for every $v \in \mathbb{R}^n$.

The main result in Bárta et al (2012) remains valid if F is a contiuous operator on a Hilbert space H (with the same proof and the same setting). In the present paper we extend the result to unbounded operators, i.e. F continuous from a Hilbert space $W \hookrightarrow H$ to H. The proof is still the same but the settings change.

Unlike Bárta et al (2012), we prove existence of the gradient structure on the whole domain including stationary points, if they are non-degenerate. For ODE's this was done in Bílý (2014). The problem was then further studied in Brooks and Maas (2024) and it was shown e.g. how to construct a smooth gradient metric. However, some questions concerning stationary points from Bárta et al (2012) have remained unanswered. We give (negative) answers to some of them here.

The idea of finding an appropriate metric on a Hilbert space to interpret a given PDE as a gradient system occurs, e.g. in Jordan et al (1998) (Fokker-Planck equation), Heida (2015) (Allen–Cahn equation, Cahn–Hilliard equation), Erbar (2024) (Boltzmann equation), Erbar and Maas (2014) (discrete porous medium equation), Erbar et al (2022) (equations on graphs, McKean–Vlasov equation).

The paper is organized as follows. Section 2 contains definitions, settings, and the main result. Section 3 deals with existence of gradient metric at non-stationary points while Section 4 focuses on a neighborhood of stationary points. Section 5 contains some examples and counterexamples.

2 Settings and the main result

Let $W \hookrightarrow H$ be two infinite-dimensional Hilbert spaces and let the embedding be dense. Let M be an open subset of W. Consider an evolution equation

$$\dot{u} + F(u) = 0, \tag{1}$$

where $F : M \to H$ is continuous. By N we denote the set of stationary points $N = \{u \in M : \ F(u) = 0\}$. Moreover, we assume that $\mathbb{E} \in C^1(M)$ is a Lyapunov function to (1) according to the following definition.

Definition 1 (strict Lyapunov function)**.** *We say that $\mathbb{E} \in C^1(M)$ is a strict Lyapunov function to* (1) *if for every $w \in M$*

(i) $\mathbb{E}'(w)$ extends to a bounded linear functional on H and
(ii) $\mathbb{E}'(w)F(w) > 0$ provided $w \in M \setminus N$.

Here, $\mathbb{E}'(w)F(w)$ is the extended linear functional $\mathbb{E}'(w) : H \to \mathbb{R}$ applied to $F(w) \in H$. Condition (i) in fact says that there exists a gradient of $\mathbb{E}$ on M (by the Riesz representation theorem); see the definition below. An inner product on H is a continuous bilinear form from $H \times H$ to $\mathbb{R}$ that is symmetric and positive definite.

Definition 2 (gradient)**.** *Let $w \in M$ and let g be an inner product on H. Let $u \in H$ satisfy $\langle u, v\rangle_g = \mathbb{E}'(w)v$ for all $v \in H$. Then we say that u is a gradient of $\mathbb{E}$ in w with respect to g, i.e. $u = \nabla_g \mathbb{E}(w)$.*

If a gradient exists, then obviously it is unique. We denote by $\mathrm{Inner}(H)$ the space of all inner products on H with the topology of strong convergence, i.e. $g_n \to g$ means $\langle u, v\rangle_{g_n} \to \langle u, v\rangle_g$ for all $u, v \in H$.

Definition 3 (metric, gradient metric, gradient system)**.** *Any continuous mapping $r : M \to \mathrm{Inner}(H)$ is called* a metric *on M. If $\nabla_{r(w)}\mathbb{E}(w) = F(w)$ for all $w \in M$, then r is called* a gradient metric *on M. If there exists a gradient metric r to* (1)*, then* (1) *is called a* gradient system *(with respect to the metric r).*

Observe that continuity in this definition means: if $\|w_n - w\|_W \to 0$, then $\langle u, v\rangle_{r(w_n)} \to \langle u, v\rangle_{r(w)}$ for all $u, v \in H$. By Observation 22, this is equivalent to $\langle u_n, v_n\rangle_{r(w_n)} \to \langle u, v\rangle_{r(w)}$ for all $u_n \to u$, $v_n \to v$ in H. Let us also note, that the norm associated with an inner product is not necessarily equivalent to the original norm of H. In particular, $r(w_1)$, $r(w_2)$ are not necessarily equivalent.

Let us define uniform continuity of a metric, this notion is needed below.

Definition 4. *We say that a metric r is norm-continuous at u, if*

$$\sup_{\|x\|,\|y\|\le 1} \left| \langle x, y\rangle_{r(u_n)} - \langle x, y\rangle_{r(u)} \right| \to 0$$

whenever $u_n \to u$ in W.

Let r be any metric on M and $\tilde{\mathbb{E}} \in C^1(M)$. Let us define $F = \nabla_r \tilde{\mathbb{E}}$. Then obviously $F : M \to H$ is continuous and $\tilde{\mathbb{E}}$ is a strict Lyapunov function for equation (1) (see Observation 22). The main result states in some sense the opposite implication. Before we formulate it, let us write down one more definition

Definition 5 (non-degenerate stationary point)**.** *Let us consider equation* (1) *with a strict Lyapunov function $\mathbb{E}$. We say that a stationary point $w \in N$ is non-degenerate, if $\mathbb{E}'(w) = 0$, $F'(w)$ and $\mathbb{E}''(w)$ exist, $F'(w)$ has a bounded inverse $F'(w)^{-1} : H \to W$, and the mapping $\Phi_w : (X, Y) \mapsto (\mathbb{E}''(w)F'(w)^{-1}X)Y$ belongs to* Inner H *and the associated norm is equivalent to the norm of H.*

Theorem 6. *Let $M \subset W$, $F : M \to H$ continuous, $\mathbb{E} : M \to \mathbb{R}$ be a strict Lyapunov function for* (1)*, and let each $w \in N$ be non-degenerate. Then there exists a gradient metric g on M.*

Proof. Stationary points of F are isolated, since otherwise they would have an accumulation point $\bar{w}$, $F(\bar{w}) = 0$ and necessarily either $F'(\bar{w}) = 0$ or $F'(\bar{w})$ does not exist, which is a contradiction in both cases. So, we can cover M by open sets each of them containing at most one stationary point, and consequently exactly one. Let us denote G_w the open set containing the stationary point w. Then consider a partition of unity (ρ_w) subordinate to this cover. On each G_w we define a gradient metric g_w by Theorem 12. Then $g = \sum_w \rho_w g_w$ is obviously a gradient metric on M. □

Remark 7. *1. A gradient metric on $M \setminus N$ exists even without the non-degeneracy condition. The gradient metric is not unique. Each metric on $M \setminus N$ (and each Lyapunov function) yields a gradient metric on $M \setminus N$, these gradient metrics are different (in general), see Theorem 9 for details.*
2. Problem of extending the gradient metric to stationary points is more delicate. For fixed $\mathbb{E}$ the gradient metric at stationary points is unique, see Proposition 14. But it may vary with $\mathbb{E}$. Non-degeneracy of a stationary points is not a necessary condition for existence of a gradient metric on M. At degenerate stationary points, existence of a gradient metric depends on the choice of a suitable Lyapunov function in some cases, in other cases gradient metric does not exist for any Lyapunov function. For more details see Section 4 and examples and counterexamples in Section 5 where this topic is discussed.

Remark 8. *Througout the paper we assume H, W to be infinite-dimensional. However, the results remain true in finite-dimensional case, there is only one restriction in Proposition 13 where we need* $\dim W \geq 3$ *in the backward implication.*

3 Gradient metric at non-stationary points

Theorem 9. *Let $M \subset W$, $F : M \to H$ continuous, and let $\mathbb{E} : M \to \mathbb{R}$ be a strict Lyapunov function for (1). Then there exists a gradient metric g on the open set*

$$\tilde{M} := M \setminus N.$$

Proof. By assumption (ii) in Definition 1, $0 \neq F(w) \notin \ker \mathbb{E}'(w)$ and $\ker \mathbb{E}'(w) \neq H$ for every $w \in \tilde{M}$. As a consequence, for every $w \in \tilde{M}$ we have

$$H = \ker \mathbb{E}'(w) \oplus \langle F(w) \rangle. \tag{2}$$

For every $u \in H$ and $w \in \tilde{M}$ let us define

$$u_{w0} := u - \frac{\langle \mathbb{E}'(w), u \rangle}{\langle \mathbb{E}'(w), F(w) \rangle} F(w) \text{ and } u_{w1} := \frac{\langle \mathbb{E}'(w), u \rangle}{\langle \mathbb{E}'(w), F(w) \rangle} F(w). \tag{3}$$

Then $u_{w0} \in \ker \mathbb{E}'(w)$, $u_{w1} \in \langle F(w) \rangle$ and the mappings $w \mapsto u_{w0}$, $w \mapsto u_{w1}$ are continuous from W to H. Now we choose an arbitrary metric r on H. Starting from this metric, we define a new metric on $\tilde{M}$ by setting

$$\begin{aligned} \langle u, v \rangle_{g(w)} &:= \langle u_{w0}, v_{w0} \rangle_{r(w)} + \frac{1}{\langle \mathbb{E}'(w), F(w) \rangle} \langle \mathbb{E}'(w), u \rangle \langle \mathbb{E}'(w), v \rangle \\ &= \langle u_{w0}, v_{w0} \rangle_{r(w)} + \frac{1}{\langle \mathbb{E}'(w), F(w) \rangle} \langle \mathbb{E}'(w), u_{w1} \rangle \langle \mathbb{E}'(w), v_{w1} \rangle . \end{aligned} \tag{4}$$

Precisely at this point we use the assumption that $\mathbb{E}$ is a strict Lyapunov function, that is, $\langle \mathbb{E}', F \rangle > 0$ on $\tilde{M}$, because this assumption implies that g really is a metric (in particular: positive definite). Continuity of g follows from continuity of the mappings $w \mapsto u_{w0}$, $w \mapsto u_{w1}$, Observation 21 and continuity of r, $\mathbb{E}'$ and F.
By definition of the metric g and by definition of the gradient $\nabla_g \mathbb{E}$, we have for every $v \in H$, $w \in \tilde{M}$

$$\langle F(w), v \rangle_{g(w)} = 0 + \langle \mathbb{E}'(w), v \rangle = \left\langle \nabla_{g(w)} \mathbb{E}(w), v \right\rangle_{g(w)},$$

so g is a gradient metric on $\tilde{M}$. □

Remark 10. *1. One can see from the proof that the metric g is not unique, different r's in general yield different g's. On the other hand, every gradient metric comes from some r via the construction described above. In fact, if r is a gradient metric, then g defined in the proof of Theorem 9 is equal to r.*
2. If $\|\cdot\|_{r(w)}$ is equivalent to $\|\cdot\|_H$ then also $\|\cdot\|_{g(w)}$ is equivalent to these norms (and, in particular, it is complete). On the other hand, if $\|\cdot\|_{r(w)}$ is not equivalent to $\|\cdot\|_H$, then $\|\cdot\|_{g(w)}$ can, but does not have to, be equivalent to $\|\cdot\|_{r(w)}$. See Proposition 11 for more details.

Proposition 11. *Let $w \in H$. Let us denote $\|\cdot\|$ the standard norm in the Hilbert space H, let r be an inner product on H and let g be defined by (4), (3) (with $g = g(w)$, $r = r(w)$). We denote by $\sim$ equivalence of two norms and by $\precnsim$ the fact that the norm on the left is bounded above by the norm on the right but not equivalent. Then the following holds.*

1. *If $\|\cdot\|_r \sim \|\cdot\|$, then $\|\cdot\|_r \sim \|\cdot\|_g$.*
2. *If $\|\cdot\|_r \precnsim \|\cdot\|$ and $\ker \mathbb{E}'(w)$ is closed in H w.r.t. $\|\cdot\|_r$, then $\|\cdot\|_r \sim \|\cdot\|_g$.*
3. *If $\|\cdot\|_r \precnsim \|\cdot\|$ and $\ker \mathbb{E}'(w)$ is not closed in H w.r.t. $\|\cdot\|_r$, then $\|\cdot\|_r \precnsim \|\cdot\|_g \precnsim \|\cdot\|$ and $\ker \mathbb{E}'(w)$ is closed w.r.t. g.*

Proof. 1. and 2. It is well known that if $H = G \oplus \mathbb{R}$ and G is a closed subspace of codimension 1 of a normed linear space H with the norm inherited from H, then it has a topological complement, and therefore $\|\cdot\|_H^2 \sim \|\cdot\|_G^2 + \|\cdot\|_{\mathbb{R}}^2$.
3. Since w is fixed, let us write $\mathbb{E}'$, F instead of $\mathbb{E}'(w)$, $F(w)$. Since H is a Hilbert space, the projections to $\operatorname{Ker} \mathbb{E}'$ and $\langle F\rangle$ are continuous, so for $z = u + dF$, $d \in \mathbb{R}$ we have $\|z\|_g^2 = \|u\|_r^2 + d^2\langle \mathbb{E}', F\rangle \le c\|u\|^2 + \|dF\|_g^2 \le c\|z\|^2$. Further, $\|z\|_r^2 \le (\|u\|_r + \|dF\|_r)^2 \le (\|u\|_g + c\|dF\|_g)^2 \le 2(\|u\|_g^2 + \|dF\|_g^2) = 2\|z\|_g$. So, $\|\cdot\|_r \prec \|\cdot\|_g \prec \|\cdot\|$. Obviously, since $\|\cdot\|_g = \|\cdot\|_r \precnsim \|\cdot\|$ on $\ker \mathbb{E}'$ we have $\|\cdot\|_g \precnsim \|\cdot\|$. Finally, since $\ker \mathbb{E}'$ is not closed w.r.t. r, we have $\overline{\ker \mathbb{E}'}^{\|\cdot\|_r} = H$. So, there exists $u_n \in \ker \mathbb{E}'$ with $u_n \to F$ in r. However, $\|u_n - F\|_g^2 = \|u_n\|_r^2 + \langle \mathbb{E}', F\rangle \to \|F\|_r^2 + \langle \mathbb{E}', F\rangle \ne 0$. Hence, $\|\cdot\|_r \precnsim \|\cdot\|_g$. Finally, if $z \ne \ker \mathbb{E}'$, then $z = u + dF$ for some $u \in \ker \mathbb{E}'$, $d \in \mathbb{R}$. Then for all $v \in \ker \mathbb{E}'$ we have $\|z - v\|_g^2 = \|u - v\|_r^2 + d^2\langle \mathbb{E}', F\rangle \ge d^2\langle \mathbb{E}', F\rangle$, so z is not in the g-closure of $\ker \mathbb{E}'$. □

4 Extension to stationary points

In this section we show that gradient metric can be extended to stationary points if they are non-degenerate.

Theorem 12. *Let $N = \{w\}$ and let w be non-degenerate. Then there exists a gradient metric on M.*

Proof. Let us consider a constant metric r defined by $\langle x, y\rangle_{r(u)} = \Phi_w(x, y)$. Then apply Theorem 9 to define g on $M \setminus \{w\}$ and the following Proposition to extend g to w. Obviously, the assumptions of the following Proposition are met since r is norm-continuous and satisfies $\langle x, y\rangle_{r(w)} = \Phi_w(x, y)$. □

Proposition 13. *Let $N = \{w\}$ and let w be non-degenerate. Let r be a metric on M and let g be the gradient metric on $M \setminus \{w\}$ defined in the proof of Theorem 9. Moreover, assume that r is norm-continuous at w. Then g has a contiunuous extension to w, if and only if $\langle x, y\rangle_{r(w)} = \Phi_w(x, y)$. In this case, $\langle x, y\rangle_{g(w)} = \Phi_w(x, y)$*

Proof. Let us assume (without loss of generality) that $w = 0$. Take $\rho > 0$ such that $B(0, \rho) \subset \tilde{M} \cup \{0\}$. Take an arbitrary $u \in W$, $u \ne 0$ and fix $\delta > 0$ such that $\delta u \in B(0, \rho)$. Then for every $x \in H$ and every $h \in (-\delta, \delta)$, $h \ne 0$ we have

$$\langle F(hu) - F(0), x\rangle_{g(hu)} = \langle F(hu), x\rangle_{g(hu)} = \langle \nabla_g \mathbb{E}(hu), x\rangle_{g(hu)} = \mathbb{E}'(hu)x = \mathbb{E}'(hu)x - \mathbb{E}'(0)x.$$

Dividing by h and taking limit for $h \to 0$ we obtain

$$\lim_{h\to 0} \left\langle \frac{1}{h}(F(hu) - F(0)), x \right\rangle_{g(hu)} = (\mathbb{E}''(0)u)x.$$

Let us first assume that g has a continuous extension to zero. Then $g(hu) \to g(0)$ and $\frac{1}{h}(F(hu) - F(0)) \to F'(0)u$ in H, so by Observation 21 we have

$$\langle F'(0)u, x\rangle_{g(0)} = (\mathbb{E}''(0)u)x. \tag{5}$$

Since $F'(0) : W \to H$ is a bijection we have $\langle v, x\rangle_{g(0)} = (\mathbb{E}''(0)F'(0)^{-1}v)x = \Phi_0(v, x)$ for all x, $v \in H$. Further, we follow Bílý (2014) to show $\langle x, y\rangle_{g(0)} = \langle x, y\rangle_{r(0)}$ for all x, $y \in H$. Let us fix x, $y \in H$. By definition of g we have for every $u \in W$ and for every small enough real $h \neq 0$

$$\begin{aligned}\langle x, y\rangle_{g(hu)} &= \left\langle x - \frac{\langle \mathbb{E}'(hu), x\rangle}{\langle \mathbb{E}'(hu), F(hu)\rangle} F(hu), y - \frac{\langle \mathbb{E}'(hu), y\rangle}{\langle \mathbb{E}'(hu), F(hu)\rangle} F(hu)\right\rangle_{r(hu)} \\ &\quad + \frac{1}{\langle \mathbb{E}'(hu), F(hu)\rangle} \langle \mathbb{E}'(hu), x\rangle \langle \mathbb{E}'(hu), y\rangle \\ &= \langle x, y\rangle_{r(hu)} - \frac{\left\langle \frac{\mathbb{E}'(hu)}{h}, x\right\rangle}{\left\langle \frac{\mathbb{E}'(hu)}{h}, \frac{F(hu)}{h}\right\rangle} \left\langle \frac{F(hu)}{h}, y\right\rangle_{r(u)} - \frac{\left\langle \frac{\mathbb{E}'(hu)}{h}, y\right\rangle}{\left\langle \frac{\mathbb{E}'(hu)}{h}, \frac{F(hu)}{h}\right\rangle} \left\langle \frac{F(hu)}{h}, y\right\rangle_{r(u)} \\ &\quad + \frac{\left\langle \frac{\mathbb{E}'(hu)}{h}, x\right\rangle}{\left\langle \frac{\mathbb{E}'(hu)}{h}, \frac{F(hu)}{h}\right\rangle} \frac{\left\langle \frac{\mathbb{E}'(hu)}{h}, y\right\rangle}{\left\langle \frac{\mathbb{E}'(hu)}{h}, \frac{F(hu)}{h}\right\rangle} \left\langle \frac{F(hu)}{h}, \frac{F(hu)}{h}\right\rangle_{r(u)} \\ &\quad + \frac{1}{\left\langle \frac{\mathbb{E}'(hu)}{h}, \frac{F(hu)}{h}\right\rangle} \left\langle \frac{\mathbb{E}'(hu)}{h}, x\right\rangle \left\langle \frac{\mathbb{E}'(hu)}{h}, y\right\rangle\end{aligned} \tag{6}$$

Taking limits we obtain

$$\begin{aligned}\langle x, y\rangle_{g(0)} &= \langle x, y\rangle_{r(0)} - \frac{\langle \mathbb{E}''(0)u, x\rangle \langle F'(0)u, y\rangle_{r(0)}}{\langle \mathbb{E}''(0)u, F'(0)u\rangle} - \frac{\langle \mathbb{E}''(0)u, y\rangle \langle F'(0)u, x\rangle_{r(0)}}{\langle \mathbb{E}''(0)u, F'(0)u\rangle} \\ &\quad + \frac{\langle \mathbb{E}''(0)u, x\rangle}{\langle \mathbb{E}''(0)u, F'(0)u\rangle} \frac{\langle \mathbb{E}''(0)u, y\rangle \langle F'(0)u, F'(0)u\rangle_{r(0)}}{\langle \mathbb{E}''(0)u, F'(0)u\rangle} + \frac{\langle \mathbb{E}''(0)u, x\rangle \langle \mathbb{E}''(0)u, y\rangle}{\langle \mathbb{E}''(0)u, F'(0)u\rangle}\end{aligned} \tag{7}$$

By (5) we have

$$\begin{aligned}\langle x, y\rangle_{g(0)} &= \langle x, y\rangle_{r(0)} - \frac{\langle F'(0)u, x\rangle_{g(0)} \langle F'(0)u, y\rangle_{r(0)}}{\|F'(0)u\|_{g(0)}} - \frac{\langle F'(0)u, y\rangle_{g(0)} \langle F'(0)u, x\rangle_{r(0)}}{\|F'(0)u\|_{g(0)}} \\ &\quad + \frac{\langle F'(0)u, x\rangle_{g(0)}}{\|F'(0)u\|_{g(0)}} \frac{\langle F'(0)u, y\rangle_{g(0)} \|F'(0)u\|_{r(0)}}{\|F'(0)u\|_{g(0)}} + \frac{\langle F'(0)u, x\rangle_{g(0)} \langle F'(0)u, y\rangle_{g(0)}}{\|F'(0)u\|_{g(0)}}.\end{aligned} \tag{8}$$

Now, let us choose u such that $\langle F'(0)u, x\rangle_{g(0)} = 0$ and $\langle F'(0)u, y\rangle_{g(0)} = 0$. This yields $\langle x, y\rangle_{g(0)} = \langle x, y\rangle_{r(0)}$.
To show the second implication let us fix x, $y \in H$ and prove $\lim_{u\to 0} \langle x, y\rangle_{g(u)} = \langle x, y\rangle_{r(0)}$. We again follow the proof for finite-dimensional setting in Bílý (2014). By definition of $g(u)$, $u \in M \setminus \{0\}$ we need to estimate

$$\left| \left\langle x - \frac{\langle \mathbb{E}'(u), x\rangle}{\langle \mathbb{E}'(u), F(u)\rangle} F(u), y - \frac{\langle \mathbb{E}'(u), y\rangle}{\langle \mathbb{E}'(u), F(u)\rangle} F(u)\right\rangle_{r(u)} + \frac{\langle \mathbb{E}'(u), x\rangle \langle \mathbb{E}'(u), y\rangle}{\langle \mathbb{E}'(u), F(u)\rangle} - \langle x, y\rangle_{r(0)} \right|$$

Since $\langle x, y\rangle_{r(u)} \to \langle x, y\rangle_{r(0)}$ it remains to show that the following expressions tend to zero

$$\begin{aligned}&\left| \frac{\langle \mathbb{E}'(u), x\rangle}{\langle \mathbb{E}'(u), F(u)\rangle} \left(\langle \mathbb{E}'(u), y\rangle - \langle F(u), y\rangle_{r(u)} \right) \right|, \\ &\left| \frac{\langle \mathbb{E}'(u), y\rangle}{\langle \mathbb{E}'(u), F(u)\rangle} \left(\langle \mathbb{E}'(u), x\rangle - \langle F(u), x\rangle_{r(u)} \right) \right|, \\ &\left| \frac{\langle \mathbb{E}'(u), x\rangle \langle \mathbb{E}'(u), y\rangle}{\langle \mathbb{E}'(u), F(u)\rangle^2} \left(\|F(u)\|^2_{r(u)} - \langle \mathbb{E}'(u), F(u)\rangle \right) \right|.\end{aligned} \tag{9}$$

We have $\mathbb{E}'(u) = \mathbb{E}'(0)+\mathbb{E}''(0)u+o(\|u\|_W) = \|u\|_W(\mathbb{E}''(0)\tilde{u}+o(1))$ and $F(u) = F(0)+F'(0)u+o(\|u\|_W) = \|u\|_W(F'(0)\tilde{u} + o(1))$ where $\tilde{u} = \frac{u}{\|u\|_W}$. Therefore,

$$\|u\|_W \frac{\langle \mathbb{E}'(u), x\rangle}{\langle \mathbb{E}'(u), F(u)\rangle} = \frac{\langle \mathbb{E}''(0)\tilde{u}, x\rangle + o(1)}{\langle \mathbb{E}''(0)\tilde{u}, F'(0)\tilde{u}\rangle + o(1)}$$

and the right-hand side is bounded by a constant independent of $\tilde{u}$ since

$$\langle \mathbb{E}''(0)\tilde{u}, F'(0)\tilde{u}\rangle = \left\langle \mathbb{E}''(0)F'(0)^{-1}F'(0)\tilde{u}, F'(0)\tilde{u}\right\rangle = \Phi_0(F'(0)\tilde{u}, F'(0)\tilde{u}) \geq c\|F'(0)\tilde{u}\| \geq c' > 0$$

since $F'(0)$ has bounded inverse and $\|\tilde{u}\|_W = 1$. Similarly, the term

$$\|u\|_W^2 \frac{\langle \mathbb{E}'(u), x\rangle \langle \mathbb{E}'(u), y\rangle}{\langle \mathbb{E}'(u), F(u)\rangle^2} \tag{10}$$

is bounded. Further, we have

$$\begin{aligned}\frac{1}{\|u\|_W} \langle F(u), y\rangle_{r(u)} &= \langle F'(0)\tilde{u}, y\rangle_{r(u)} + o(1) \\ &= \langle F'(0)\tilde{u}, y\rangle_{r(u)} - \langle F'(0)\tilde{u}, y\rangle_{r(0)} + \langle F'(0)\tilde{x}, y\rangle_{r(0)} + o(1) \\ &= \langle F'(0)\tilde{u}, y\rangle_{r(u)} - \langle F'(0)\tilde{u}, y\rangle_{r(0)} + \langle \mathbb{E}''(0)\tilde{u}, y\rangle + o(1).\end{aligned}$$

Hence,

$$\frac{1}{\|u\|_W} \left|\langle \mathbb{E}'(u), y\rangle - \langle F(u), y\rangle_{r(u)}\right| = \left|\langle F'(0)\tilde{u}, y\rangle_{r(u)} - \langle F'(0)\tilde{u}, y\rangle_{r(0)} + o(1)\right|$$

which tends to zero by norm-continuity of r. So, the first term in (9) converges to zero and the same is obviously true for the second term.
Convergence to zero of the third term follows from boundedness of (10), the following equalities

$$\begin{aligned}\frac{1}{\|u\|_W^2} \left(\|F(u)\|_{r(u)}^2 - \langle \mathbb{E}'(u), F(u)\rangle\right) &= \langle F'(0)\tilde{u}, F'(0)\tilde{u}\rangle_{r(u)} - \langle \mathbb{E}''(0)\tilde{u}, F'(0)\tilde{u}\rangle + o(1) \\ &= \langle F'(0)\tilde{u}, F'(0)\tilde{u}\rangle_{r(u)} - \langle F'(0)\tilde{u}, F'(0)\tilde{u}\rangle_{r(0)} + o(1)\end{aligned}$$

and norm-continuity of r. □

One can see from the proof that one implication of Proposition 13 holds under weaker assumptions on the stationary point w. Namely, norm-continuity of r is not needed and we also do not need that the norm associated with Φ_w is equivalent. The following proposition is an immediate consequence of the proof of Proposition 13

Proposition 14. *Let $N = \{w\}$ and let $\mathbb{E}'(w) = 0$, $F'(w)$ and $\mathbb{E}''(w)$ exist, and $F'(w)$ has a bounded inverse $F'(w)^{-1} : H \to W$. Let g be a gradient metric on M, then $\langle x, y\rangle_{g(w)} = \Phi_w(x, y)$. If, moreover, g comes from a metric r on $M \setminus \{w\}$ as in the proof of Theorem 9, then $\langle x, y\rangle_{g(w)} = \langle x, y\rangle_{r(w)} = \Phi_w(x, y)$.*

5 Examples

This section is devoted to examples and counterexamples demonstrating necessity and non-necessity of various assumptions of the results proved in previous sections and other related phenomena.
The following example shows that various r's and various Lyapunov functions yield various gradient metrics in non-stationary points.

Example 15 (Heat equation)**.** *Let us consider the heat equation*

$$u_t - \Delta u = 0 \tag{11}$$

with Dirichlet boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary. This fits in our settings with $H = L^2(\Omega)$, $M = W = H_0^1(\Omega) \cap H^2(\Omega)$ and $F(w) = -\Delta w$. It is known that (11) is a gradient system for $\mathbb{E}(w) = \frac{1}{2}\int_\Omega |\nabla w|^2$ and r being the scalar product on H. On the other hand, one can consider $r1$ defined as $\langle u, v\rangle_{r1(w)} = \int_\Omega (1 + \|x\|)u(x)v(x)\mathrm{d}x$ and obtain a gradient metric on $M \setminus \{0\}$ given by

$$\begin{aligned}\langle u, v\rangle_{g1(w)} = &\int (1+\|x\|)uv + \frac{1}{\|\Delta w\|^2}\Big(-\int \nabla w \nabla u \int (1+\|x\|)\nabla w \nabla v \\ &- \int \nabla w \nabla v \int (1+\|x\|)\nabla w \nabla u + \int \nabla w \nabla u \int \nabla w \nabla v\Big) \\ &+ \frac{1}{\|\Delta w\|^4}\int \nabla w \nabla u \int \nabla w \nabla v \int (1+\|x\|)\Delta w \Delta w\end{aligned}$$

or one can replace the term $\|x\|$ by $\|w\|$ to get a non-constant metric $r2$ and the corresponding gradient metric $g2$.
Further, one can consider $\mathbb{E}(w) = \frac{1}{2}\int_\Omega |w|^2$ which is another Lyapunov function for (11) and keep the original r. By Theorem 9, there exists a gradient metric g on $M \setminus \{0\}$ for this Lyapunov function and the gradient metric is given (for $w \not\equiv 0$) by the formula

$$\langle u, v\rangle_{g(w)} = a_w \int uv - \int uw \int v\Delta w - \int vw \int u \Delta w + \frac{b_w}{a_w}\int uw \int vw$$

where $a_w = \int w\Delta w$, $b_w = 1 + \int \Delta w \Delta w$ (all integrals over Ω). It is not immediately clear whether this g can be extended to the origin.
The following example shows that it can happen: 1st For various Lyapunov functions there exist gradient metrics extendable to stationary points. 2nd Not every gradient metric is extendable to stationary points, one needs to choose r appropriately. 3rd Some gradient metrics are not equivalent to the original norm on H. 4th For some $\mathbb{E}$ the unique gradient metric at stationary points is not equivalent to original norm on H. 5th For some $\mathbb{E}$ no gradient metric at stationary points exists (for any metric r).
Example 16 (1D heat equation as multiplication operator)**.** *Let us consider the Dirichlet Laplacian on $[0, 1]$ and represent it via Fourier series. So, $H = l^2$, $M = W = \{u \in l^2 : (n^2u(n)) \in l^2\}$, $F(u) = (n^2u(n))$.*
1. The most natural Lyapunov function for this problem is $\mathbb{E}(u) = \frac{1}{2}\sum n^2u(n)^2$, which corresponds to $\frac{1}{2}\int |\nabla u|^2$ and $F(u) = \nabla\mathbb{E}(u)$ w.r.t. $\langle x, y\rangle = \sum x(n)y(n)$.
2. Let us now consider $\mathbb{E}(u) = \frac{1}{2}\sum u(n)^2$, which corresponds to $\frac{1}{2}\int |u|^2$. Then

$$\mathbb{E}'(u)x = \sum u(n)x(n) = \langle \nabla_g \mathbb{E}(u), x\rangle_{g(u)}.$$

To get a gradient system, we need the previous epression to be equal to $\langle F(u), x\rangle_{g(u)}$. We can define $\langle x, y\rangle_{g(u)} = \sum \frac{1}{n^2}x(n)y(n)$, then $\langle F(u), x\rangle_{g(u)} = \sum \frac{1}{n^2}n^2u(n)x(n) = \mathbb{E}'(u)x$. So, $\dot{u} = F(u)$ is a gradient system on W w.r.t. metric g, but the norm associated with g is not equivalent to the norm on H, it is not complete. Further, we have $(\mathbb{E}''(0)u)v = \sum u(n)v(n)$ and $F'(0)u = (n^2u(n))$, so $\Phi_0(x, y) = (\mathbb{E}''(0)F^{-1}(0)x)y = \sum \frac{1}{n^2}x(n)v(n)$. So, we can see that in this case a gradient metric exists on M even if the norm associated with Φ_0 is not equivalent.
3. Now, let us still consider $\mathbb{E}(u) = \frac{1}{2}\sum u(n)^2$ but take r as $\langle x, y\rangle_{r(w)} = \sum x(n)y(n)$ and the gradient metric g as in the proof of Theorem 9. Then the corresponding metric g cannot be extended to the origin, by Proposition 14.
4. Let us now consider $\mathbb{E}(u) = \frac{1}{4}u(1)^4 + \frac{1}{2}\sum_{n=2}^\infty n^2u(n)^2$. Obviously, this is a Lyapunov function, since $\mathbb{E}'(u)F(u) = u(1)^4 + \sum_{n=2}^\infty n^4u(n)^2 > 0$ if $u \not\equiv 0$. So, there exists a gradient metric g on $M \setminus \{0\}$. However, $\Phi_0(x, y) = (\mathbb{E}''(0)F'(0)^{-1}x)y = \sum_{n=2}^\infty \frac{1}{n^2}n^2x(n)y(n)$ which is positive semidefninite bilinear form. Therefore, there is no gradient metric on the whole M by Proposition 14, e.g. for any r, the g associated to r cannot be extended to the origin.
The following proposition provides examples where gradient metric cannot be extended to stationary points not only for any choice of r but even for any choice of a Lyapunov function. The example is finite-dimensional but can be easily extended to the infinite-dimensional case by adding further coordinates.

Typically, if there are orbits in the shape of spirals around the stationary point, then the extension of g is impossible. This is often the case of second order equations with damping.

Proposition 17. *Let $F \in C^1(\mathbb{R}^n)$ be such that $F(0) = 0$ and $F'(0)$ is a regular matrix with at least one eigenvalue having non-zero imaginary part. Then there does not exist a gradient metric on any neigborhood of 0 for any Lyapunov function $\mathbb{E} \in C^2$.*

Proof. Let us assume for contradiction that a Lyapunov function $\mathbb{E} \in C^2$ and a gradient metric g exist on a neighborhood of zero. Then, by Proposition 14 is the matrix $A = \mathbb{E}''(0)F'(0)^{-1}$ symmetric positive definite. In particular, it is invertible and $F'(0) = A^{-1}\mathbb{E}''(0)$. However, on the right-hand side is a product of two symmetric matrices where the first of them is positive definite. Such product has necessarily real spectrum which is a contradiction. □

Example 18. *Let us consider $F(x, y) = (-y, 2x - 2y)$. Then $F'(0) = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix}$ with eigenvalues $-1 \pm i$. Then $\mathbb{E}(x, y) = 2x^2 + y^2 - \varepsilon xy$ is a strict Lyapunov function if $\varepsilon > 0$ is small enough. So, there exist many Lyapunov functions and many gradient metrics on $\mathbb{R}^2 \setminus \{(0, 0)\}$. So, there are many ways how to write $(\dot{x}, \dot{y}) = F(x, y)$ as a gradient system on $\mathbb{R}^2 \setminus \{(0, 0)\}$ but it is not possible to write it as a gradient system on $\mathbb{R}^2$ (with C^2 Lyapunov function).*

The following example shows that non-degeneracy of a stationary point is not a necessary condition for existence of gradient metric. In particular, gradient metric may exist even if $F'(0)$ is not invertible. In fact, here F is a bounded operator and the example works even if restricted to a finite-dimensional space.

Example 19. *Let $H = l^2$, $M = W = H$, $F(u) = (u(1)^3, u(2), u(3), \dots)$. Then $F'(0)u = (0, u(2), u(3), \dots)$, so $F'(0)$ is not invertible. Let us further define $\mathbb{E}(u) = \frac{1}{4}u(1)^4 + \frac{1}{2}\sum_{n=2}^{\infty} u(n)^2$. Then $\mathbb{E}'(u)v = u(1)^3 v(1) + \sum_{n=2}^{\infty} u(n)v(n)$. So, $\nabla\mathbb{E}(u) = F(u)$ w.r.t. the standard inner product on l^2 for all $u \in l^2$. So, if we take r equal to the standard inner product, then $g = r$ on l^2, in particular g can be extended to the origin although $F'(0)$ is not invertible and definition of Φ_0 does not have sense.*

Example 20 (Damped wave equation)**.** *Let us consider the wave equation with linear damping*

$$u_{tt} + u_t - \Delta u = 0, \tag{12}$$

or more generally

$$u_{tt} + g(u_t)u_t + \nabla E(u) = 0 \tag{13}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. We can represent this problem as (1) as follows

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -v \\ g(v)v + \nabla E(u) \end{pmatrix} = 0 \tag{14}$$

where the second vector is $F(u, v)$. We take $H = H_0^1(\Omega) \times L^2(\Omega)$, $W = (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ and assume that $E \in C^1(H_0^1(\Omega))$ with $E'(w)$ extendable to a bounded functional on $L^2(\Omega)$ (then $\nabla E(w)$ is defined) for each $w \in H_0^1(\Omega) \cap H^2(\Omega)$. Then F is continuous from W to H, if $v \mapsto g(v)v$ is continuous from H_0^1 to L^2 and ∇E is continuous from $(H_0^1(\Omega) \cap H^2(\Omega))$ to L^2.

Here the first condition is satisfied if $g(v)(x) = \tilde{g}(v(x))$, $\tilde{g}$ continuous with $|\tilde{g}(v)| \le c|v|^{\frac{2}{n-2}}$. In fact, in this case the Nemytskii operator $v \mapsto vg(v)$ is continuous from L^q to L^2 for $q = \frac{2n}{n-2}$ (see e.g. (Appell and Zabrejko, 1990, Section 3.2)) and it holds that $H_0^1 \hookrightarrow L^q$.

It is easy to show that

$$\mathbb{E}_1(u, v) = \frac{1}{2}\|v\|^2 + E(u)$$

is a Lyapunov function for (14) which is not strict. One can show that (under additional assumptions)

$$\mathbb{E}(u, v) = \mathbb{E}_1(u, v) + \varepsilon B(\mathbb{E}_1(u, v)) \langle \nabla E(u), v \rangle_{-1}$$

is a strict Lyapunov function for an appropriate function B and $\varepsilon > 0$ small enough. In particular, if $g(v) = |v|^\alpha$ and E satisfies the Lojasiewicz gradient inequality

$$\|E'(u)\| \ge c\|E(u) - E(0)\|^{1-\theta}$$

and $\|E'(u)\| \le C\|E(u)-E(0)\|^{1/2}$, then $B(s) = s^\beta$ with $\beta = \alpha(1-\theta)$. This (slightly more general) result is shown in Hassen and Haraux (2011), for a further generalization you can see Bárta (2016). In fact, denoting

$$H(t) = \mathbb{E}(u(t), v(t))$$

for a solution $(u(\cdot), v(\cdot))$ of (14), we have

$$\langle \mathbb{E}'(u,v), F(u,v)\rangle = -\mathbb{E}'(u,v)\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = -H'(t)$$

and it is shown in (Hassen and Haraux, 2011, proof of Theorem 2.2), resp. (Bárta, 2016, proof of Theorem 2.1) that $-H'(t) > 0$, whenever $(\nabla E(u), v) \neq (0,0)$, i.e. $F(u,v) \neq 0$.
Therefore, equations (12) and (13) (under the assumptions from Hassen and Haraux (2011) or Bárta (2016)) are gradient systems on the set where $F \neq 0$ by Theorem 9.
There is no gradient metric on the whole domain for the Lyapunov function $\mathbb{E}$ defined above, since $\mathbb{E}''(F')^{-1}$ is not symmetric in stationary points. In fact, stationary points are $(u,0)$ where $\nabla E(u) = 0$. Then

$$F'(u,0) = \begin{pmatrix} 0 & -1\\ g'(0) & E''(u)\end{pmatrix} \text{ and } F'(u,0)^{-1} = \frac{1}{g'(0)}\begin{pmatrix} E''(u) & 1\\ -g'(0) & 0\end{pmatrix}.$$

Further,

$$\mathbb{E}''(u,0) = \begin{pmatrix} E''(u) & 0\\ 0 & 1\end{pmatrix} \text{ if } \varepsilon = 0,$$

so

$$\mathbb{E}''(u,0)F'(u,0)^{-1} = \begin{pmatrix} E''(u)g'(0)E''(u) & E''(u)\\ -1 & 0\end{pmatrix} + o(1) \text{ as } \varepsilon \to 0.$$

Hence, the bilinear form is not symmetric.

6 Appendix

Observation 21. *(i) (uniform boundedness principle for bilinear forms) Let A be a set of bilinear forms on H. If for every u, $v \in H$ the set $\{a(u,v) : a \in A\}$ is bounded, then $\{\|a\| : a \in A\}$ is bounded.*
(ii) (continuity of a Riemannian metric) Let r be a Riemannian metric $V \to \mathrm{Inner}(H)$. Let $w_n \to w$ in V, $v_n \to v$ in H and $u_n \to u$ in H. Then $\langle u_n, v_n\rangle_{r(w_n)} \to \langle u, v\rangle_{r(w)}$.

Proof. (i) Since $a(u,\cdot)$ are linear mappings, we have for each fixed u boundedness of $\{\|a(u,\cdot)\| : a \in A\}$. Since the mappings $L_a : u \mapsto a(u,\cdot)$ are linear and $\{L_a(u) : a \in A\}$ is bounded for every $u \in V$, boundedness of $\{\|L_a\| : a \in A\}$ follows and this is exactly what we need.
(ii) Let us estimate

$$\begin{aligned} |\langle u_n, v_n\rangle_{r(w_n)} - \langle u, v\rangle_{r(w)}| &\le |\langle u_n, v_n\rangle_{r(w_n)} - \langle u_n, v\rangle_{r(w_n)}| \\ &+ |\langle u_n, v\rangle_{r(w_n)} - \langle u, v\rangle_{r(w_n)}| \\ &+ |\langle u, v\rangle_{r(w_n)} - \langle u, v\rangle_{r(w)}|. \end{aligned}$$

Here the first term on the right-hand side is estimated by

$$\sup \|u_n\|_H \sup \|r(w_n)\|_{H\times H\to\mathbb{R}} \|v_n - v\|_H,$$

where the second supremum is finite due to boundedness of $\{\langle u, v\rangle_{r(w_n)} : n \in \mathbb{N}\}$ for every fixed u, $v \in H$ (follows from strong continuity of r) and (i). The other two terms on the right-hand side can be estimated analogously, so convergence to zero follows. ☐

Observation 22. *If $\mathbb{E} \in C^1(M)$, $F : M \to H$ and $F(w) = \nabla_{r(w)}\mathbb{E}(w)$ for every $w \in M$, then $\mathbb{E}$ is a strict Lyapunov function for (1).*

Proof. Since $\nabla_{r(w)}\mathbb{E}(w)$ exists, $\mathbb{E}'(w)$ can be extended to a bounded linear functional on H as follows $\mathbb{E}'(w)h = \left\langle \nabla_{r(w)}\mathbb{E}(w), h \right\rangle_{r(w)}$. Further,

$$\mathbb{E}'(w)F(w) = \left\langle \nabla_{r(w)}\mathbb{E}(w), F(w) \right\rangle_{r(w)} = \left\langle \nabla_{r(w)}\mathbb{E}(w), \nabla_{r(w)}\mathbb{E}(w) \right\rangle_{r(w)} > 0,$$

whenever $0 \neq \nabla_{r(w)}\mathbb{E}(w) = F(w)$. □

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