MAXIMAL REGULARITY OF STOKES PROBLEM WITH DYNAMIC BOUNDARY CONDITION — HILBERT SETTING

T. BÁRTA, P. DAVIS, P. KAPLICKÝ

ABSTRACT. For the evolutionary Stokes problem with dynamic boundary conditions we show the maximal regularity of weak solutions in time. Due to the characteriation of R-sectorial operators on Hilbert spaces, the proof reduces to finding the correct functional analytic setting and proving that the corresponding operator is sectorial, i.e. generates an analytic semigroup.

1. INTRODUCTION

There are materials like polymer melts that may slip over solid surfaces. Such boundary behavior is described by slip velocity models, see [Hatzikiriakos, 2012, Section 6] for an overview. Moreover, it has been observed that the slip is often not constant but changes over time as it depends on the current state of the fluid. Such fluids need to be represented using dynamic slip models. They were first proposed in [Pearson and Petrie, 1968] in a general form

$$u_\tau + \lambda_\tau \partial_t u_\tau = \varphi(\sigma_w),$$

where u_τ is the slip velocity, t stands for the time, λ_τ is the slip relaxation time, σ_w stands for the wall shear stress and φ should be determined based on the rheological properties of the fluid under consideration.

The mathematical studies of problems with dynamic boundary conditions in the context of fluid mechanics started by the thesis of Maringová, [Maringová, 2019]. She studied existence of solutions to systems of (Navier)-Stokes type under various constitutive relations for the extra stress tensor and the modified dynamic boundary condition $s(u_\tau) + \partial_t u_\tau = -\sigma_w$ with a given - perhaps nonlinear - function s. These results were later published in [Abbatiello et al., 2021].

We are interested in the optimal regularity of problems with dynamic boundary conditions in the scale of Lebesgue spaces. In this paper we focus on the linear Stokes problem. First of all we think that the result is interesting in itself. Second, it could be used to study the regularity of more complicated systems. Moreover, the linear theory can be considered as a tool for the reconstruction of pressure, see [Sohr and von Wahl, 1986].

Key words and phrases. Stokes problem, dynamic boundary conditions, maximal regularity, analytic semigroup.

We study the problem

$$\partial_t u - \Delta u + \nabla p = f \qquad \text{in } I \times \Omega, \tag{1}$$

$$\operatorname{div} u = 0 \qquad \text{in } I \times \Omega, \tag{2}$$

$$\beta \partial_t u + (2Du \cdot \nu)_\tau + \alpha u_\tau = \beta g \qquad \text{in } I \times \partial\Omega, \tag{3}$$

$$u_\nu = 0 \qquad \text{in } I \times \partial\Omega, \tag{4}$$

$$u = u_0 \qquad \text{in } \{0\} \times \Omega \tag{5}$$

$$u = v_0 \qquad \text{in } \{0\} \times \partial\Omega \tag{6}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$ with $C^{2,1}$ boundary and a time interval $I = (0, T)$, $T > 0$. The constants $\alpha \in \mathbb{R}$, $\beta > 0$, the functions $f : I \times \Omega \to \mathbb{R}^d$, $g : I \times \partial\Omega \to \mathbb{R}^d$, $u_0 : \Omega \to \mathbb{R}^d$ and $v_0 : \partial\Omega \to \mathbb{R}^d$ are given. Subscripts $(\cdot)_\tau$ and $(\cdot)_\nu$ denote the tangential and the normal part of the vectors. We look for unknown functions $u : I \times \Omega \to \mathbb{R}^d$ and $p : I \times \Omega \to \mathbb{R}$. Let us mention that we permit $\alpha < 0$, however, only $\alpha \geq 0$ seem to be physically relevant.
We adopt the notion of the weak solution (with a small modification) from [Maringová, 2019, Section 5]. We work in Banach spaces

$$\mathcal{V} = \{(u, u_b) \in H^1_\sigma(\Omega) \times L^2_\nu(\partial\Omega) : u_b = \gamma(u)\}, \quad \mathcal{H} = L^2_\sigma(\Omega) \times L^2_\nu(\partial\Omega)$$

with norms

$$\|(u, u_b)\|^2_{\mathcal{V}} = 2\|Du\|^2_{L^2(\Omega)} + \|u_b\|^2_{L^2(\partial\Omega)}, \quad \|(u, u_b)\|^2_{\mathcal{H}} = \|u\|^2_{L^2(\Omega)} + \beta\|u_b\|^2_{L^2(\partial\Omega)}.$$

Definitions of all mentioned function spaces can be found in Subsection 2.1.
The duality pairing between $\mathcal{V}$ and its dual space $\mathcal{V}^*$ is denoted $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ and it is defined as an extension of the scalar product in $\mathcal{H}$, see [Maringová, 2019, Section 3.1].
When dealing with a function from $\mathcal{V}$ or $\mathcal{V} \cap \mathcal{H}$ we write only the first component of the vector. The trace of the function is automatically considered as the second component.

Definition 1. *Let $0 < T \leq +\infty$, $\alpha \in \mathbb{R}$, $\beta > 0$, $\Omega \subset \mathbb{R}^3$, $\Omega \in C^{0,1}$, $f \in L^1_{loc}([0, T), H^1_\sigma(\Omega)^*)$, $g \in L^1_{loc}([0, T), L^2_\nu(\partial\Omega))$, $u_0 \in L^2_\sigma(\Omega)$ and $v_0 \in L^2_\nu(\partial\Omega)$. We say that u is a weak solution to the problem (1)-(6) if $u \in L^2_{loc}([0, T), \mathcal{V}) \cap C_{loc}([0, T), \mathcal{H}) \cap L^\infty_{loc}([0, T), \mathcal{H})$, $\partial_t u \in L^1_{loc}(0, T, \mathcal{V}^*)$ and the balance of linear momentum is satisfied in the weak sense, i.e.,*

$$\langle \partial_t u, \varphi \rangle_{\mathcal{V}} + 2 \int_\Omega Du : D\varphi + \alpha \int_{\partial\Omega} u\varphi = \langle (f, g), \varphi \rangle_{\mathcal{V}} \tag{7}$$

almost everywhere on $(0, T)$ and for all $\varphi \in \mathcal{V}$. The initial condition is attained in the strong sense; $u(0) = (u_0, v_0)$ in $\mathcal{H}$.

Note that β is hidden in (7) in the definition of $\langle \cdot, \cdot \rangle_{\mathcal{V}}$. We are interested in the maximal regularity of weak solutions with respect to the problem data, i.e. the right hand side functions f and g, and the initial values u_0 and v_0. In order to state the precise conditions for the initial values we need to introduce spaces

$$\mathcal{X}_0 = L^2_\sigma(\Omega) \times H^{1/2}_\nu(\partial\Omega), \quad \mathcal{X}_1 = \{(u, u_b) \in H^2_\sigma(\Omega) \times H^{1/2}_\nu(\partial\Omega) : \gamma(u) = u_b\},$$

$$\|(f, g)\|_{\mathcal{X}_0} = \|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}, \quad \|(u, v)\|_{\mathcal{X}_1} = \|u\|_{H^2(\Omega)} + \|v\|_{H^{1/2}(\partial\Omega)}$$

$$\mathcal{X}_{1-\frac{1}{q},q} = (\mathcal{X}_0, \mathcal{X}_1)_{1-\frac{1}{q},q}.$$

Before we formulate the main theorem, we need some preparation for nonaxisymmetric domains, see also Lemma 4 below.

Lemma 2. *Let Ω be nonaxisymmetric. There is $\alpha_0 < 0$ such that*

$$2\alpha_0\|u\|^2_{L^2(\partial\Omega)} + 4\|Du\|^2_{L^2(\Omega)} \geq 0$$

for all $u \in H^1(\Omega)$ with $u \cdot \nu = 0$ on $\partial\Omega$.

Our main theorem follows.

Theorem 3. *Let one of the following conditions be met:*

(a) $T \in (0, +\infty)$,
(b) $T = +\infty$, $\alpha > 0$,
(c) $T = +\infty$, Ω *nonaxisymmetric*, $\alpha \in (\alpha_0, 0]$.

For every $q \in (1, +\infty)$ there exists $C > 0$ such that for every $\mathcal{F} = (f, g) \in L^q(I, \mathcal{X}_0)$ and $(u_0, v_0) \in \mathcal{X}_{1-1/q,q}$ the weak solution u of (1)–(6) is unique and satisfies $u \in L^q(I, H^2(\Omega))$, $\partial_t u \in L^q(I, L^2(\Omega))$. Moreover, there exists a function $p \in L^q(I, H^1(\Omega)$ such that (1)-(6) hold pointwisely almost everywhere and

$$\|\partial_t u(t)\|_{L^q(I,L^2(\Omega))} + \|u(t)\|_{L^q(I,H^2(\Omega))} + \|p(t)\|_{L^q(I,H^1(\Omega))} \leq C(\|\mathcal{F}\|_{L^q(I,\mathcal{X}_0)} + \|(u_0, v_0)\|_{\mathcal{X}_{1-1/q,q}}). \tag{8}$$

Our approach to the problem is as follows. We rewrite the problem (1)–(4) as an abstract Cauchy problem

$$\partial_t \mathcal{U} = \mathcal{A}\mathcal{U} + \mathcal{F}(t), \tag{9}$$

on a Hilbert space $\mathcal{X}_0$. Since the problem combines evolutionary equations in the interior of Ω and on its boundary, the space $\mathcal{X}_0$ must be a product of spaces in the interior and on the boundary of Ω, compare [Escher, 1992, Denk et al., 2008]. We show below that $\mathcal{A}$ is the generator of an analytic semigroup $\mathcal{T}$. Then the Variation-Of-Constants-Formula

$$\mathcal{U}(t) = \mathcal{T}(t)\mathcal{U}_0 + \int_0^t \mathcal{T}(t-s)\mathcal{F}(s)ds \tag{10}$$

defines a mild solution to (9). This mild solution actually has better properties if $\mathcal{U}_0$ and $\mathcal{F}$ are sufficiently good. Namely, since $\mathcal{X}_0$ is a Hilbert space we have maximal L^p regularity, i.e. for every $\mathcal{F} \in L^p(I, \mathcal{X}_0)$ the solution $\mathcal{U}$ given by (10) with $\mathcal{U}_0 = 0$ satisfies $\mathcal{A}\mathcal{U}, \dot{\mathcal{U}} \in L^p(I, \mathcal{X}_0)$ and

$$\|\dot{\mathcal{U}}\|_{L^p(I,\mathcal{X}_0)} + \|\mathcal{A}\mathcal{U}\|_{L^p(I,\mathcal{X}_0)} \leq C\|\mathcal{F}\|_{L^p(I,\mathcal{X}_0)}$$

with a constant $C > 0$ independent of $\mathcal{F}$, compare [**?**] or [Kunstmann and Weis, 2004, Corollary 1.7]. Since the mild solution is very regular we show that it is actually a weak solution from Definition 1. Uniqueness of the weak solution then concludes the argumentation.

Apart from articles [Maringová, 2019] and [Abbatiello et al., 2021] we are aware only of the article [**?**], that appeared recently. In this article its authors study, if the problem (1)–(6) generates an analytic semigroup in spaces $L^p_\sigma(\Omega) \times L^p_\nu(\partial\Omega)$ with $p > 1$. The result is rather involved but does not cover our result since we work in $\mathcal{X}_0 = L^2_\sigma(\Omega) \times H^{1/2}_\nu(\partial\Omega)$. A variant of dynamic boundary conditions appeared also in a different context. In [Ventcel′, 1959] they appeared as general boundary conditions that turn a given elliptic differential operator to

the generator of a semigroup of positive contraction operators. There are many works on dynamic boundary conditions (or Wentzell[1] boundary conditions) in the context of parabolic and hyperbolic equations without the incompressibility constraint and without the pressure. Our main example are the results in [Denk et al., 2008] where the maximal L^p regularity is proved for a very general class of parabolic systems equipped with a general dynamic boundary condition. The presented article can be considered as the first step to a parallel theory for the Stokes problem.
In the following section we give the basic notation and define the operator $\mathcal{A}$. Elliptic theory is studied in Section 3. The proof of Theorem 3 is given in Section 4.

2. Notation and functional analytic setting

2.1. **Notation and function spaces.** If $z : \mathbb{R}^d \to \mathbb{R}^d$ then $(\nabla z)_{ij} = \partial_j z_i$ and $(Dz)_{ij} = \frac{1}{2}(\partial_j z_i + \partial_i z_j)$ for $i, j \in \{1, \ldots, d\}$. If A, B are matrices, then AB denotes the matrix product, e.g. $([\nabla z]z)_i = \partial_k z_i z_k$ for $i \in \{1, \ldots, d\}$, while $A : B = a_{ij} b_{ij}$. We use the summation convention over repeated indices. For two vectors $a, b \in \mathbb{R}^d$, $a \cdot b$ denotes the scalar product in $\mathbb{R}^d$.
We recall that $\Omega \subset \mathbb{R}^d$ is a bounded domain with $C^{2,1}$ boundary, $I = (0, T)$ for some $T > 0$. If w is a function defined on Ω and with trace on $\partial\Omega$ we denote w_ν its normal part and w_τ its tangential part on $\partial\Omega$. By $\nu(x)$ we denote the unit outer normal vector to $\partial\Omega$ at point $x \in \partial\Omega$. Equalities of functions are understood almost everywhere with respect to the corresponding Hausdorff measure.
We denote standard Sobolev and Sobolev-Slobodeckii spaces over Ω with integrability 2 and differentiability $s > 0$ by H^s. Further,

$$\mathcal{D}_\sigma = \{u \in C_0^\infty(\Omega); \operatorname{div} u = 0\}, \quad L^2_\sigma(\Omega) = \text{closure of } \mathcal{D}_\sigma \text{ in } L^2(\Omega),$$
$$H^1_\sigma(\Omega) = H^1 \cap L^2_\sigma(\Omega), \quad H^2_\sigma(\Omega) = H^2 \cap L^2_\sigma(\Omega).$$

The trace operator is denoted by γ. We remark that if $w \in H^1_\sigma(\Omega)$ then $\operatorname{div} w = 0$ in Ω in the weak sense and $\gamma(w)_\nu = 0$ on $\partial\Omega$. Consequently, if we define

$$L^2_\nu(\partial\Omega) = \{w \in L^2(\partial\Omega) : w_\nu = 0 \quad \text{a.e. on } \partial\Omega\},$$
$$H^{\frac{1}{2}}_\nu(\partial\Omega) = \{w \in H^{\frac{1}{2}}(\partial\Omega) : w_\nu = 0 \text{ on } \partial\Omega\},$$
$$H^{\frac{3}{2}}_\nu(\partial\Omega) = \{w \in H^{\frac{3}{2}}(\partial\Omega); w_\nu = 0 \text{ on } \partial\Omega\},$$

then $H^{1/2}_\nu(\partial\Omega) = \gamma(H^1_\sigma(\Omega))$ and $H^{3/2}_\nu(\partial\Omega) = \gamma(H^2_\sigma(\Omega))$.
The Helmholtz-Weyl decomposition yields $L^2(\Omega) = G_2(\Omega) \otimes L^2_\sigma(\Omega)$ where

$$G_2(\Omega) = \{w \in L^2(\Omega); w = \nabla p, p \in H^1\},$$

see e.g. [Galdi, 2011, Theorem III.1.1]. The continuous Leray projection of $L^2(\Omega)$ to $L^2_\sigma(\Omega)$ is denoted $P : L^2(\Omega) \to L^2_\sigma(\Omega)$.

[1]Note that Wentzell and Ventcel' are different spelling of the same name. The first form is used in MathSciNet, the second one in literature.

2.2. **Definition of the operator $\mathcal{A}$.** The operator $\mathcal{A}$ is considered on the space $\mathcal{X}_0 = L^2_\sigma(\Omega) \times H^{1/2}_\nu(\partial\Omega)$. The domain of $\mathcal{A}$ is defined as $D(\mathcal{A}) = \mathcal{X}_1$. Finally, we set

$$\mathcal{A}\begin{pmatrix} u \\ u_b \end{pmatrix} = \begin{pmatrix} P\Delta u \\ -\beta^{-1}[(2Du\cdot\nu)_\tau + \alpha u_b] \end{pmatrix} \quad \text{for } \begin{pmatrix} u \\ u_b \end{pmatrix} \in D(\mathcal{A}) \tag{11}$$

3. Regularity theory for the elliptic problem

Before we show that $(\mathcal{A}, D(\mathcal{A}))$ is the generator of an analytic semigroup in $\mathcal{X}_0$, we need some preliminary results on existence and regularity of solutions to the following system.

$$\lambda u - \Delta u + \nabla\pi = f \qquad \text{in } \Omega, \tag{12}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega, \tag{13}$$

$$\lambda u + \beta^{-1}[(2Du\cdot\nu)_\tau + \alpha u_\tau] = h \qquad \text{in } \partial\Omega, \tag{14}$$

$$u_\nu = 0 \qquad \text{in } \partial\Omega \tag{15}$$

Since the operator $\mathcal{A}$ is defined on a product space, we keep this fact also in this part of the presentation. Actually, it is not necessary because the second component of the space is just trace of the first one.
In this part we work in the space

$$Z = \{(u, u_b) \in H^1_\sigma(\Omega) \times L^2_\nu(\partial\Omega) : u_b = \gamma(u)\}$$

with norm

$$\|(u, u_b)\|_Z^2 = 2\|Du\|^2_{L^2(\Omega)} + \|u_b\|^2_{L^2(\partial\Omega)}.$$

The norm in Z is equivalent to the norm in $H^1(\Omega)$ by Korn's and Poincaré's inequalities originating from [Hlaváček and Nečas, 1970], see also [Acevedo Tapia et al., 2021, Proposition 3.13]. We present it here for readers convenience. The essential part of the lemma below is taken from [Acevedo Tapia et al., 2021, Proposition 3.13]. The last equivalence of norms is the standard Korn's inequality.

Lemma 4. *Let Ω be a bounded Lipschitz domain. Then, for all $u \in H^1(\Omega)$ with $u\cdot\nu = 0$ on $\partial\Omega$, we have*

$$\|u\|_{H^1(\Omega)} \sim \|Du\|_{L^2(\Omega)}$$

if Ω is nonaxisymmetric, and

$$\|u\|_{H^1(\Omega)} \sim \|Du\|_{L^2(\Omega)} + \|u_\tau\|_{L^2(\partial\Omega)} \sim \|Du\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$$

if Ω is arbitrary. Here, "$\sim$" denotes the equivalence of two norms.

The first part of this lemma proves Lemma 2. We continue with the definition of a weak solution to (12)-(15).

Definition 5. *Let $(f, h) \in L^2(\Omega) \times L^2(\partial\Omega)$ (complex valued) and let $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{C}$. We say that $(u, u_b) \in Z$ is a weak solution to (12)-(15) if*

$$\lambda\int_\Omega u\bar{\varphi} + \int_\Omega 2Du : \nabla\bar{\varphi} + (\beta\lambda + \alpha)\int_{\partial\Omega} u_b\bar{\varphi}_b = \int_\Omega f\bar{\varphi} + \int_{\partial\Omega}\beta h\bar{\varphi}_b \tag{16}$$

holds for every $(\varphi, \varphi_b) \in Z$.

In the next proposition we prove the existence and uniqueness of weak solutions in the set Z.

Proposition 6. *Let* $(f,h) \in L^2(\Omega) \times L^2(\partial\Omega)$*,* $\alpha \in \mathbb{R}$*,* $\beta > 0$*,* $\lambda \in \mathbb{C}$ *and let one of the following conditions be met:*

(a) $\beta \operatorname{Re}\lambda \geq \max(1, -4\alpha)$,
(b) $\alpha > 0$, $\operatorname{Re}\lambda \geq 0$
(c) $\alpha \in (\alpha_0, 0]$, Ω *nonaxisymmetric,* $\operatorname{Re}\lambda \geq 0$.

Then there exists a unique weak (complex-valued) solution $(u, u_b) \in Z$ *of* (12)-(15)*. Moreover,* $u_b \in H^{1/2}_\nu(\partial\Omega)$ *and there exists* $C > 0$ *independent of* λ *such that*

$$\|u\|_{H^1(\Omega)} + \|u_b\|_{H^{1/2}(\partial\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|h\|_{L^2(\partial\Omega)}).$$

Proof. Let us define a sesquilinear form

$$B(\mathcal{U}, \mathcal{V}) = \lambda \int_\Omega u\bar{v} + \int_\Omega 2Du : \nabla\bar{v} + (\beta\lambda + \alpha)\int_{\partial\Omega} u_b\bar{v}_b$$

on $Z \times Z$ where $\mathcal{U} = (u, u_b)^T$, $\mathcal{V} = (v, v_b)^T$. We define $a := \operatorname{Re}\lambda$. By Lemmas 2 and 4 any of conditions (1)-(3) implies the existence of $C > 0$ independent of $\mathcal{U}$ and λ such that

$$\begin{aligned} |B(\mathcal{U},\mathcal{U})| &= \left|\lambda\|u\|^2_{L^2(\Omega)} + 2\|Du\|^2_{L^2(\Omega)} + (\beta\lambda + \alpha)\|u_b\|^2_{L^2(\partial\Omega)}\right| \\ &\geq \left|a\|u\|^2_{L^2(\Omega)} + 2\|Du\|^2_{L^2(\Omega)} + (\beta a + \alpha)\|u_b\|^2_{L^2(\partial\Omega)}\right| \\ &\geq C\|\mathcal{U}\|^2_Z \end{aligned}$$

Moreover, the form B is bounded from above on Z.
By the Lax-Milgram theorem, see e.g. [Petryshyn, 1965], for $\mathcal{F} \in Z^*$ defined by $\mathcal{F}(\Phi) = \int_\Omega f\bar{\varphi} + \int_{\partial\Omega} \beta h\bar{\varphi}_b$ for $\Phi \in Z$ there exists a unique $\mathcal{U} = (u, u_b) \in Z$ such that $B(\Phi, \mathcal{U}) = \mathcal{F}(\Phi)$ for every $\Phi = (\varphi, \varphi_b) \in Z$, i.e. (16) holds. By the trace theorem, $u_b = \gamma(u) \in H^{1/2}_\nu(\partial\Omega)$. The final estimate follows from $B(\mathcal{U},\mathcal{U}) = \mathcal{F}(\mathcal{U})$ and properties of B and $\mathcal{F}$. $\square$

Remark 7. *Since the parametres* $\alpha \in \mathbb{R}$ *and* $\beta > 0$ *are fixed, we do not track the dependence of the constant* C *on these parameters in Proposition 6 and also in all further estimates.*

In the next proposition we state the first spectral estimate.

Proposition 8. *Under the assumptions of Proposition 6 there exists* $C > 0$ *independent of* λ *such that any weak solution* (u, u_b) *to* (12)-(15) *satisfies*

$$|\lambda|\left(\|u\|_{L^2(\Omega)} + \|u_b\|_{L^2(\partial\Omega)}\right) \leq C(\|f\|_{L^2(\Omega)} + \|h\|_{L^2(\partial\Omega)}). \tag{17}$$

Proof. Let us insert $\varphi = \lambda u$ into (16). We obtain

$$|\lambda|^2\|u\|^2_{L^2(\Omega)} + 2\bar{\lambda}\|Du\|^2_{L^2(\Omega)} + (\beta|\lambda|^2 + \alpha\bar{\lambda})\|u\|^2_{L^2(\partial\Omega)} = \int_\Omega f\overline{\lambda u} + \int_{\partial\Omega} \beta h\overline{\lambda u_b}.$$

We consider only the real part of this equality. Since the real part of the right hand side is estimated by its modulus, we get

$$|\lambda|^2\|u\|^2_{L^2(\Omega)} + (\beta|\lambda|^2 + \alpha\operatorname{Re}\lambda)\|u\|^2_{L^2(\partial\Omega)} + 2\operatorname{Re}\lambda\|Du\|^2_{L^2(\Omega)} \leq \int_\Omega |f\overline{\lambda u}| + \beta\int_{\partial\Omega} |h\overline{\lambda u_b}|.$$

Applying the Young inequality to the right-hand side we obtain

$$|\lambda|^2\|u\|^2_{L^2(\Omega)} + (\beta|\lambda|^2 + 2\alpha\operatorname{Re}\lambda)\|u\|^2_{L^2(\partial\Omega)} + 4\operatorname{Re}\lambda\|Du\|^2_{L^2(\Omega)} \leq \|f\|^2_{L^2(\Omega)} + \beta\|h\|^2_{L^2(\partial\Omega)}.$$

In the case of conditions (2), (3) we have

$$2\alpha\|u\|^2_{L^2(\partial\Omega)} + 4\|Du\|^2_{L^2(\Omega)} \geq 0,$$

if condition (1) applies we have $\operatorname{Re}\lambda > 1/\beta > 0$ and $\beta|\lambda|^2 + 2\alpha \operatorname{Re}\lambda \geq \beta|\lambda|^2/2$. The statement (17) follows. □

By a straightforward modification of [Galdi, 2011, Theorem III.5.3] one can associate a pressure $\pi \in L^2(\Omega)$ to any weak solution defined in Definition 5 such that

$$(18) \qquad \int_\Omega f\bar{\varphi} + \int_{\partial\Omega} \beta h\bar{\varphi}_b = \lambda \int_\Omega u\bar{\varphi} + \int_\Omega 2Du : \nabla\bar{\varphi} - \int_\Omega \pi \operatorname{div}\bar{\varphi} + (\beta\lambda + \alpha)\int_{\partial\Omega} u_b\bar{\varphi}_b$$

for any $\varphi \in H^1$, $\varphi_b = \gamma(\varphi)$ with $(\varphi_b)_\nu = 0$ on $\partial\Omega$. The pressure is defined uniquely up to an additive constant. Let us further require the constant to be chosen in such a way that the pressure has zero mean over Ω. Then the mapping $(f,h) \in L^2_\sigma(\Omega) \times L^2_\nu(\partial\Omega) \mapsto \pi \in L^2(\Omega)$ is linear and bounded.
Before we state our result on regularity of weak solutions we need to prove a lemma on existence of a special function satisfying boundary conditions.

Lemma 9. *There exists $C > 0$ such that for every $h \in H^{1/2}_\nu(\partial\Omega)$ there exists $w \in H^2(\Omega)$ with properties 1) $\operatorname{div}(w) = 0$ in Ω, 2) $w = 0$ on $\partial\Omega$, 3) $(2Dw \cdot \nu)_\tau = h$ on $\partial\Omega$ and 4) $\|w\|_{H^2(\Omega)} \leq C\|h\|_{H^{1/2}(\partial\Omega)}$.*

Remark. *Regularity of $w \in H^2(\Omega)$ together with 1) and 2) imply $w \in H^2_\sigma(\Omega)$.*

Proof of Lemma 8. Step 1: We find a function z in $H^2(\Omega)$ satisfying conditions 2)-4) and additionally satisfying 5) $\operatorname{div} z = 0$ on $\partial\Omega$. By the inverse trace theorem, see e.g. [Nečas, 2012, Theorem 2.5.8], there exists $z \in H^2(\Omega)$ such that $z = 0$ and $\partial_\nu z = h$ on $\partial\Omega$ and $\|z\|_{H^2(\Omega)} \leq C\|h\|_{H^{1/2}(\partial\Omega)}$ holds. This function z obviously satisfies 2) and 4). Since $z \equiv 0$ on $\partial\Omega$ we have $\partial_\xi z = (\nabla z)\xi = 0$ for any tangent vector ξ to $\partial\Omega$. Therefore,

$$[(\nabla z)^T\nu] \cdot \xi = \xi^T(\nabla z)^T\nu = [(\nabla z)\xi]^T\nu = 0\nu = 0$$

and $(2[Dz]\nu)_\tau = ([\nabla z]\nu)_\tau + ([\nabla z]^T\nu)_\tau = ([\nabla z]\nu)_\tau = h_\tau = h$ and 3) is satisfied. Further, Héron's formula (see [Héron, 1981, Lemme 3.3] or [Amrouche and Girault, 1994, Lemma 3.5]) yields

$$(19) \qquad \operatorname{div} z = \operatorname{div}_{\partial\Omega}(z_\tau) + \partial_\nu z \cdot \nu - 2Kz\cdot\nu \quad \text{on } \partial\Omega.$$

In the formula, K denotes the mean curvature of $\partial\Omega$ and $\operatorname{div}_{\partial\Omega}$ denotes the surface divergence. All three terms on the right-hand side of (19) are zero since $z \equiv 0$ on $\partial\Omega$ and $\partial_\nu z\cdot\nu = h\cdot\nu = 0$. So, 5) holds.
Step 2: It remains to correct the solenoidality of z without destroying the conditions $2) - 4)$. To do this we apply [Bogovskiĭ, 1980, Theorem 2] to the problem $\operatorname{div}\zeta = \operatorname{div} z$ in Ω. Since $\operatorname{div} z \in H^1_0(\Omega)$ and $\int_\Omega \operatorname{div} z = \int_{\partial\Omega} z = 0$ there exists a solution $\zeta \in H^2_0(\Omega)$ of this problem such that $\|\zeta\|_{H^2} \leq C\|\operatorname{div} z\|_{H^1} \leq C\|z\|_{H^2} \leq C\|h\|_{H^{1/2}}$.
Finally, it remains to define $w = z - \zeta$. This function satisfies all conditions 1)-4). □

Theorem 10. *Under the assumptions of Proposition 6 the unique weak solution (u, u_b) of the problem (12)-(15) and the associated pressure π satisfy $(u,u_b) \in D(\mathcal{A})$, $\pi \in H^1(\Omega)$ for every $f \in L^2(\Omega)$, $h \in H^{1/2}_\nu(\partial\Omega)$. Moreover, there exists $C > 0$ independent of λ such that*

$$(20) \qquad \|u_b\|_{H^{\frac{3}{2}}(\partial\Omega)} + \|u\|_{H^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|h\|_{H^{1/2}(\partial\Omega)}).$$

Proof. According to the definition of $D(\mathcal{A})$ it suffices to show $u \in H^2(\Omega)$, $\pi \in H^1(\Omega)$ together with the estimate (20). Note that the estimate of the boundary value u_b follows from the estimate of u in $H^2(\Omega)$ by the embedding theorem.
If $\lambda \in \mathbb{R}$ we rewrite the system as follows with $\eta = 3|\alpha| + 1$

$$-\Delta u + \nabla\pi = f - \lambda u, \qquad \operatorname{div} u = 0, \qquad \text{in } \Omega,$$
$$(2Du \cdot \nu)_\tau + (\beta\lambda + \eta + \alpha)u_\tau = \beta h + \eta u_\tau, \qquad u_\nu = 0 \qquad \text{in } \partial\Omega.$$

Since any of assumptions (1)-(3) of Proposition 6 implies $(\beta\lambda + \eta + \alpha) \geq 1$, we have $\|f - \lambda u\|_{L^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|h\|_{L^2(\partial\Omega)})$ by Proposition 7 and $\|\beta h + \eta u_\tau\|_{H^{1/2}(\partial\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|h\|_{L^2(\partial\Omega)})$ by Proposition 6. Therefore, we can apply [Acevedo Tapia et al., 2021, Theorem 4.5] to get the estimate (20).
If $\lambda \in \mathbb{C}$ we still have a weak solution u of (12)-(15) by Proposition 6. We would like to apply a complex valued analogue of [Acevedo Tapia et al., 2021, Theorem 4.5] to

$$\begin{aligned} &-\Delta v + \nabla\sigma = \tilde{f}, \qquad \operatorname{div} v = 0, \qquad \text{in } \Omega, \\ &(2Dv \cdot \nu)_\tau + \tilde{\alpha} v_\tau = \tilde{h}, \qquad v_\nu = 0 \qquad \text{in } \partial\Omega, \end{aligned} \tag{21}$$

where $\tilde{f} = f - \lambda u \in L^2(\Omega)$, $\tilde{\alpha} = \beta\lambda + \eta + \alpha$, $\operatorname{Re}\tilde{\alpha} \geq 1$ and $\tilde{h} = \beta h + \eta u_\tau \in H^{1/2}(\partial\Omega)$ with norms independent of λ. The proof presented in [Acevedo Tapia et al., 2021] works also in the complex valued situation with minor changes.
As in that article, we can again assume without loss of generality that $\tilde{h} = 0$. In fact, if $\tilde{h} \neq 0$ we consider a solenoidal function $w \in H^2_\sigma(\Omega)$ satisfying the equation $(21)_2$ on the boundary. Such a function exists due to Lemma 8 and satisfies $\|w\|_{H^2(\Omega)} \leq C\|\tilde{h}\|_{H^{1/2}(\partial\Omega)}$. Then it suffices to study the solution to (21) with the right hand side $\tilde{f} + \Delta w \in L^2(\Omega)$ and $\tilde{h} = 0$.
To show the regularity of a weak solution to (21) with $\tilde{h} = 0$ and of the associated pressure we apply the method of differences as in [Acevedo Tapia et al., 2021]. It can be followed almost line by line. The only difference is in obtaining regularity at the boundary in the tangent direction since our parameter λ is complex. We test the weak formulation of the equation (21) by the complex conjugate $\overline{D_k^{-h}(\zeta^2 D_k^h v)}$. The estimates from above of the terms on the right hand side are done as in [Acevedo Tapia et al., 2021]. The terms that should provide information, i.e.

$$I_1 = 2\int_\Omega \zeta^2 |D_k^h Dv|^2 \quad \text{and} \quad I_2 = \int_{\partial\Omega} \zeta^2 |D_k^h v_\tau|^2$$

can be treated as follows. On the left-hand side we get the term $I_1 + \tilde{\alpha} I_2$ which satisfies

$$|I_1 + \tilde{\alpha} I_2| \geq \operatorname{Re}(I_1 + \tilde{\alpha} I_2) = I_1 + \operatorname{Re}\tilde{\alpha} I_2 \geq I_1 + I_2.$$

Hence, one can continue as in [Acevedo Tapia et al., 2021, proof of Theorem 4.5] to conclude that solutions of (21) satisfy

$$\|v\|_{H^2(\Omega)} + \|\sigma\|_{H^1(\Omega)} \leq C(\|\tilde{f}\|_{L^2(\Omega)} + \|\tilde{h}\|_{H^{1/2}(\partial\Omega)})$$

which then implies (20). ◻

Remark 11. *It follows from Theorem 9 and the discussion below* (18) *that the mapping that associates the pressure with zero mean to the problem data,* $(f, h) \in L^2_\sigma(\Omega) \times H^{1/2}_\nu(\partial\Omega) \mapsto \pi \in H^1(\Omega)$*, is linear and bounded from* $L^2_\sigma(\Omega) \times H^{1/2}_\nu(\partial\Omega)$ *to* $H^1(\Omega)$*.*

Remark 12. *It seems to us that in* [Acevedo Tapia et al., 2021] *the result corresponding to the previous Theorem is announced for bounded domains* Ω *with* $C^{1,1}$ *boundary. As the main reference for the technique that allows to get the result in the neighborhood of the nonflat boundary is presented* [Beirão Da Veiga, 2004]. *We are not able to reconstruct the proof for* $C^{1,1}$ *domains and we want to remark that also in* [Solonnikov and Ščadilov, 1973, Beirão Da Veiga, 2004] *it is assumed that the boundary of* Ω *is* $C^{2,1}$ *or* C^3.

Proposition 13. *Under the assumptions of Proposition 6 let* $\mathcal{F} = (f, h) \in L^2_\sigma(\Omega) \times H^{1/2}_\nu(\partial\Omega)$. *The weak solution* $\mathcal{U} := (u, u_b) \in Z$ *of* (12)-(15) *belongs to* $D(\mathcal{A})$ *and satisfies* $\lambda\mathcal{U} - \mathcal{A}\mathcal{U} = \mathcal{F}$.

Proof. Since $(u, u_b) \in D(\mathcal{A})$ and the associated pressure $\pi \in H^1$ by Theorem 9, we have $u \in H^2$ and by (18)

(22)
$$\begin{aligned}\int_\Omega f\bar{\varphi} + \int_{\partial\Omega} \beta h\bar{\varphi}_b &= \lambda \int_\Omega u\bar{\varphi} + 2\int_\Omega Du : \nabla\bar{\varphi} - \int_\Omega \pi \operatorname{div}\bar{\varphi} + (\beta\lambda + \alpha)\int_{\partial\Omega} u_b\bar{\varphi}_b \\ &= \lambda \int_\Omega u\bar{\varphi} + 2\int_{\partial\Omega} [Du]\nu\bar{\varphi}_b - \int_\Omega \Delta u\bar{\varphi} + \int_\Omega \nabla\pi\bar{\varphi} + (\beta\lambda + \alpha)\int_{\partial\Omega} u_b\bar{\varphi}_b\end{aligned}$$

for any $\varphi \in H^1$, $\varphi_b = \gamma(\varphi)$ with $(\varphi_b)_\nu = 0$ on $\partial\Omega$. It follows $\lambda u - \Delta u + \nabla\pi = f$ a. e. in Ω which gives $\lambda u - P(\Delta u) = f$. Inserting the pointwise equality $\lambda u - \Delta u + \nabla\pi = f$ into (22) for a general $\varphi \in H^1$, $\varphi_b = \gamma(\varphi)$ with $(\varphi_b)_\nu = 0$ on $\partial\Omega$ we obtain

$$\int_{\partial\Omega} \beta h\bar{\varphi}_b = 2\int_{\partial\Omega} Du \cdot \nu\bar{\varphi}_b + (\beta\lambda + \alpha)\int_{\partial\Omega} u_b\bar{\varphi}_b.$$

Due to the regularity of (u, u_b) we can set $\varphi = \beta h_\tau - 2([Du]\nu)_\tau - (\beta\lambda + \alpha)(u_b)_\tau$ to get $\beta h = 2([Du]\nu)_\tau + (\beta\lambda + \alpha)u_b$ on $\partial\Omega$. ☐

4. Proof of the main theorem

4.1. **Uniqueness of the weak solutions.** Our Definition 1 of the weak solution differs from the one in [Maringová, 2019, Definition 5.1] in the assumption on regularity of the right-hand side function (f, g) and of the solution $\partial_t u$. That is why we present here a simple proof of the uniqueness of the weak solutions.

Lemma 14. *In the situation of Definition 1, let* u, v *be two weak solutions corresponding to the same data* f, g, u_0, v_0. *Then* $u = v$.

Proof. We define $w = u - v$. Then w is a weak solution corresponding to the trivial data. In particular, it solves (7) with zero right hand side. From this equation we read that actually $\partial_t w \in L^2_{loc}([0, T), \mathcal{V}^*)$ and consequently w is the unique weak solution on any $(0, T^*)$ with $T^* \in (0, T)$ in the spirit of [Maringová, 2019, Definition 5.1], for uniqueness see [Maringová, 2019, Theorem 5.1]. It follows that $w = 0$ and $u = v$. ☐

4.2. **The operator $\mathcal{A}$ generates an analytic semigroup.** We show that $(\mathcal{A}, D(\mathcal{A}))$ is densely defined, closed, its resolvent set contains a sector and resolvent estimates are satisfied there, see (23). We start with

Proposition 15. $D(\mathcal{A})$ *is dense in* $\mathcal{X}_0$ *and* $(\mathcal{A}, D(\mathcal{A}))$ *is a closed operator.*

Proof. We first prove density of $D(\mathcal{A})$ in $\mathcal{X}_0$. Let $(f,h) \in \mathcal{X}_0$ and $\varepsilon > 0$. Due to density of $H^{3/2}(\partial\Omega)$ in $H^{1/2}(\partial\Omega)$ there exists $\tilde{h}_1 \in H^{3/2}(\partial\Omega)$ s.t. $\|h - \tilde{h}_1\|_{L^2(\partial\Omega)} < \varepsilon$. Then we orthonormally project $\tilde{h}_1$ to the tangent bundle of $\partial\Omega$ and denote the resulting function h_1. Since Ω has $C^{2,1}$ boundary, the ortonormal projection does not spoil the regularity of h. Indeed, h_1 can be written as $h_1(x) = \tilde{h}_1(x) - \langle \tilde{h}_1(x), \nu(x)\rangle \nu(x)$. Consequently, $h_1 \in H^{3/2}_\nu(\partial\Omega)$. Moreover, since $h_\nu = 0$ we have $\|h - h_1\|_{L^2(\partial\Omega)} \leq \|h - \tilde{h}_1\|_{L^2(\partial\Omega)} < \varepsilon$. We now find $f_1 \in H^2_\sigma(\Omega)$ such that $\gamma(f_1) = h_1$. To do that, we first find $\tilde{f}_1 \in H^2(\Omega)$ such that $\gamma(\tilde{f}_1) = h_1$ and then correct its divergence to zero by [Amrouche and Girault, 1994, Corollary 3.8]. Finally, by definition of $L^2_\sigma(\Omega)$ there exists $f_2 \in \mathcal{D}_\sigma$ such that $\|(f - f_1) - f_2\|_{L^2_\sigma(\Omega)} < \varepsilon$. Then $(f_1 + f_2, h_1) \in D(\mathcal{A})$ is the desired approximation of $(f,h) \in \mathcal{X}_0$.
To show closedness of $\mathcal{A}$ let $\mathcal{U} = (u,b)^T, \mathcal{U}_n = (u_n, b_n)^T \in D(\mathcal{A})$ be such that $\mathcal{U}_n \to \mathcal{U}$ in $\mathcal{X}_0$ and $\mathcal{F}_n := \mathcal{A}\mathcal{U}_n \to \mathcal{F}$ in $\mathcal{X}_0$ as $n \to +\infty$. In particular,

$$\gamma(u_n) = b_n, \quad u_n \to u \quad \text{ in } L^2_\sigma(\Omega), \quad b_n \to b \quad \text{ in } H^{1/2}_\nu(\partial\Omega) \text{ as } n \to +\infty.$$

Since $\{\mathcal{U}_n\}$ is a bounded sequence in $\mathcal{X}_0 = L^2_\sigma(\Omega) \times H^{1/2}_\nu(\partial\Omega)$ we can apply Theorem 9 to the equation $\lambda\mathcal{U}_n - \mathcal{A}\mathcal{U}_n = \mathcal{F}_n + \lambda\mathcal{U}$ with $\lambda = \max(1, -4\alpha)/\beta$. We get that the sequence $\{\|u_n\|_{H^2}\}$ is bounded, so a subsequence $\{v_n\}$ of $\{u_n\}$ converges weakly in $H^2(\Omega)$ to some v. By the convergence $u_n \to u$ in $L^2_\sigma(\Omega)$ we have $v = u$ and necessarily $u \in H^2(\Omega)$. Due to the continuity of the trace mapping, the embeddings and the Leray projection P we also get $\gamma(v_n) \rightharpoonup \gamma(u)$ in $H^{3/2}(\partial\Omega)$, $\operatorname{div} v_n \rightharpoonup \operatorname{div} u$ in $H^1(\Omega)$, and $P\Delta v_n \rightharpoonup P\Delta u$ in $L^2_\sigma(\Omega)$. So $b = \gamma(u)$, $u \in H^2_\sigma(\Omega)$, $(u,b) \in D(\mathcal{A})$, and $\mathcal{A}\mathcal{U} = \mathcal{F}$. □

Definition 16. *For $\omega \in \mathbb{R}$, $\theta \in (0,\pi)$ we define*

$$S_{\theta,\omega} = \{\lambda \in \mathbb{C}; \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}.$$

Theorem 17. *Let $\alpha, \beta \in \mathbb{R}$. The operator $(\mathcal{A}, D(\mathcal{A}))$ is sectorial. More precisely, there is $\theta \in (\pi/2, \pi)$, $C > 0$ that for $\omega = \max(1, -4\alpha)/\beta$*

$$S_{\theta,\omega} \subset \rho(\mathcal{A}), \quad \forall \lambda \in S_{\theta,\omega} : \|(\lambda - \mathcal{A})^{-1}\| \leq \frac{C}{|\lambda - \omega|}. \tag{23}$$

If moreover 1) $\alpha > 0$ or 2) $\alpha > \alpha_0$ and Ω nonaxisymmetric, then there are $\theta \in (\pi/2, \pi)$, $\omega < 0$ and $C > 0$ such that (23) holds.

Proof. Take any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \max(1, -4\alpha)/\beta$.
By Proposition 12 there exists a solution $\mathcal{U} = (u, u_b) \in D(\mathcal{A})$ to $(\lambda - \mathcal{A})\mathcal{U} = \mathcal{F}$ for every $\mathcal{F} = (f,h) \in \mathcal{X}_0$, i.e. the operator $\lambda - \mathcal{A} : D(\mathcal{A}) \to \mathcal{X}_0$ is surjective. Since any solution to $(\lambda - \mathcal{A})\mathcal{U} = \mathcal{F}$ gives rise to a weak solution of (12)-(15) which is unique by Proposition 6, $\lambda - \mathcal{A}$ is also injective. The operator $\lambda - \mathcal{A}$ is closed by Proposition 14 and we obtain $\lambda \in \rho(\mathcal{A})$. In particular, $(\lambda - \mathcal{A})^{-1}$ is bounded.
Now we show the resolvent estimate. It is sufficient to show that there is $C > 0$ such that for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \max(1, -4\alpha)/\beta$ and any $\mathcal{F} \in \mathcal{X}_0$ the solution $\mathcal{U} \in D(\mathcal{A})$ of $(\lambda - \mathcal{A})\mathcal{U} = \mathcal{F}$ satisfies

$$|\lambda|\|\mathcal{U}\|_{\mathcal{X}_0} \leq C\|\mathcal{F}\|_{\mathcal{X}_0},$$

see [Lunardi, 1995, Proposition 2.1.11]. This can be reformulated for $\mathcal{U} = (u, u_b)$ and $\mathcal{F} = (f, h)$ as

$$|\lambda| \left(\|u\|_{L^2(\Omega)} + \|u_b\|_{H^{1/2}_\nu(\partial\Omega)} \right) \le C(\|f\|_{L^2(\Omega)} + \|h\|_{H^{1/2}(\partial\Omega)})$$

In Proposition 7 we have already proved

$$|\lambda| \|u\|_{L^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|h\|_{H^{1/2}(\partial\Omega)}).$$

From the second component of the equation $(\lambda - \mathcal{A})\mathcal{U} = \mathcal{F}$, see (14), we have

$$\|\lambda u_b\|_{H^{1/2}(\partial\Omega)} \le C(\|u\|_{H^{1/2}(\partial\Omega)} + \|Du\|_{H^{1/2}(\partial\Omega)} + \|h\|_{H^{1/2}(\partial\Omega)}).$$

We estimate the terms containing u on the right-hand side by the trace theorem and by Theorem 9 as

$$\|u\|_{H^{1/2}(\partial\Omega)} + \|Du\|_{H^{1/2}(\partial\Omega)} \le C\|u\|_{H^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|h\|_{H^{1/2}(\partial\Omega)}).$$

This concludes the proof of the first statement.
If moreover 1) or 2) holds then the previous argument works for all $\lambda \in \{z \in \mathbb{C} : \operatorname{Re} z \ge 0\} \setminus \{0\}$ by Proposition 7 with conditions (2)-(3) from Proposition 6. It follows that there exists $\theta \in (\pi/2, \pi)$ such that $S_{\theta,0} \in \rho(\mathcal{A})$ and the resolvent estimate (23) holds with $\omega = 0$, see [Lunardi, 1995, Proposition 2.1.11]. Since in this case $0 \in \rho(\mathcal{A})$, it follows that we can actually consider some $\omega < 0$ (with possibly smaller θ) in the estimate (23). This concludes the proof. ☐

Corollary 18. *Operator $(\mathcal{A}, D(\mathcal{A}))$ generates an analytic semigroup $\{\mathcal{T}(t)\}_{t>0} \subset \mathcal{L}(\mathcal{X}_0)$. There is $\omega \in \mathbb{R}$ and $C > 0$ such that the semigroup satisfies for any $t > 0$*

$$\|\mathcal{T}(t)\|_{\mathcal{L}(\mathcal{X}_0)} \le Ce^{\omega t}. \tag{24}$$

If moreover 1) $\alpha > 0$ or 2) $\alpha > \alpha_0$ and Ω nonaxisymmetric, ω is less than 0.

Proof. The statement follows directly from [Lunardi, 1995, Proposition 2.1.1]. ☐

Now we are ready to prove the main result, Theorem 3.

4.3. **Proof of Theorem 3.** We start with the proof under additional assumptions (2) or (3). Let one of them hold, in particular $I = (0, +\infty)$. Then $\mathcal{A}$ is densely defined, closed and generates a bounded analytic semigroup $\mathcal{T}$ on the Hilbert space $\mathcal{X}_0$ by Proposition 14 and Corollary 17. We moreover have the estimate (24) with $\omega < 0$ at our disposal. From [Kunstmann and Weis, 2004, Theorem 1.1, Corollary 1.7 and (1.9)] the operator $\mathcal{A}$ has maximal L^q regularity, i.e., the mild solution $\mathcal{U}^0$ of (9) (see (10)) with $\mathcal{U}(0) = 0$ satisfies $\dot{\mathcal{U}}^0$, $\mathcal{A}\mathcal{U}^0 \in L^q(I, \mathcal{X}_0)$ and

$$\|\dot{\mathcal{U}}^0\|_{L^q(I,\mathcal{X}_0)} + \|\mathcal{A}\mathcal{U}^0\|_{L^q(I,\mathcal{X}_0)} \le C\|\mathcal{F}\|_{L^q(I,\mathcal{X}_0)}.$$

Let us denote $\mathcal{U}^1(t) := \mathcal{T}(t)\mathcal{U}_0$ for $t > 0$ the mild solution of (9) with $\mathcal{F} = 0$ and $\mathcal{U}(0) = \mathcal{U}_0$. Since $\mathcal{U}_0 \in \mathcal{X}_{1-1/q,q}$, the function $t \mapsto \mathcal{A}\mathcal{U}^1(t)$ (and therefore also $t \mapsto \dot{\mathcal{U}}^1(t)$) belongs to $L^q((0,1), \mathcal{X}_0)$ and the inequality

$$\|\dot{\mathcal{U}}^1\|_{L^q((0,1),\mathcal{X}_0)} + \|\mathcal{A}\mathcal{U}^1\|_{L^q((0,1),\mathcal{X}_0)} \le C\|\mathcal{U}_0\|_{\mathcal{X}_{1-1/q,q}}$$

holds (see [Lunardi, 1995, Proposition 2.2.2 and formula (2.2.3)]). From the properties of analytic semigroups, see [Lunardi, 1995, Proposition 2.1.1], we get for $t > 0$

$$\|\mathcal{A}\mathcal{U}^1(t)\|_{\mathcal{X}_0} = \frac{1}{t}\|t\mathcal{A}\mathcal{T}(t)\mathcal{U}_0\|_{\mathcal{X}_0} \leq \frac{Ce^{\omega t}}{t}\|\mathcal{U}_0\|_{\mathcal{X}_0} \leq \frac{Ce^{\omega t}}{t}\|\mathcal{U}_0\|_{\mathcal{X}_{1-1/q,q}}.$$

Since $\kappa(t) := e^{\omega t}/t$ satisfies $\kappa \in L^q(1, +\infty)$ it follows that

$$\|\dot{\mathcal{U}}^1\|_{L^q(I,\mathcal{X}_0)} + \|\mathcal{A}\mathcal{U}^1\|_{L^q(I,\mathcal{X}_0)} \leq C\|\mathcal{U}_0\|_{\mathcal{X}_{1-1/q,q}}.$$

Hence, the solution $\mathcal{U} = (u, u_b) = \mathcal{U}^0 + \mathcal{U}^1$ of (9) satisfies $\dot{\mathcal{U}}$, $\mathcal{A}\mathcal{U} \in L^q(I, \mathcal{X}_0)$, and

$$\|\dot{\mathcal{U}}\|_{L^q(I,\mathcal{X}_0)} + \|\mathcal{A}\mathcal{U}\|_{L^q(I,\mathcal{X}_0)} \leq C(\|\mathcal{F}\|_{L^q(I,\mathcal{X}_0)} + \|\mathcal{U}_0\|_{\mathcal{X}_{1-1/q,q}}).$$

For a.e. $t > 0$

$$-\mathcal{A}\mathcal{U}(t) = \mathcal{F}(t) - \dot{\mathcal{U}}(t) \quad \text{in } \mathcal{X}_0$$

and $\mathcal{U}(t)$ is also the unique weak solution to (12)-(15) with $\lambda = 0$ and the right hand side $(\tilde{f}, \tilde{h}) = (f(t) - \partial_t u(t), h(t) - \partial_t u_b) \in \mathcal{X}_0$. It follows from Theorem 9, with assumptions (2) or (3), that the functions $u(t)$, $u_b(t)$ and the associated pressure $\pi(t)$ satisfy the estimate

$$\begin{aligned} (25) \quad & \|u_b(t)\|_{H^{\frac{3}{2}}(\partial\Omega)} + \|u(t)\|_{H^2(\Omega)} + \|\pi(t)\|_{H^1(\Omega)} \\ & \leq C(\|f(t)\|_{L^2(\Omega)} + \|\partial_t u(t)\|_{L^2(\Omega)} + \|h(t)\|_{H^{1/2}(\partial\Omega)} + \|\partial_t u_b(t)\|_{H^{1/2}(\partial\Omega)}). \end{aligned}$$

Since $(\tilde{f}, \tilde{h}) \in L^q(I, L^2(\Omega)) \times L^q(I, H^{1/2}(\partial\Omega))$ measurability of the mapping $t > 0 \mapsto \pi(t) \in H^1(\Omega)$ follows from Remark 10. Integrating (25) and using regularity of $\dot{\mathcal{U}}$ we obtain (8). It remains to show that $\mathcal{U}$ is actually the unique weak solution of (1)-(6). The function $\mathcal{U}$ clearly satisfies $\mathcal{U} \in C([0, +\infty), \mathcal{H}) \cap L^\infty_{loc}([0, +\infty), \mathcal{H}), \dot{\mathcal{U}} \in L^1_{loc}([0, T), \mathcal{V}^*)$ and the equation (7) holds almost everywhere in $(0, +\infty)$. As $\mathcal{U} \in L^\infty_{loc}([0, +\infty), \mathcal{H})$ we have $u_b \in L^2_{loc}([0, \infty), L^2(\Omega))$. The initial values are atained by [Lunardi, 1995, Proposition 2.1.1 and Proposition 2.1.4 (i)] and Proposition 14. To show that u is a weak solution of (1)-(6) it remains to prove $u \in L^2_{loc}([0, +\infty), H^1(\Omega))$. We know $\mathcal{U} \in L^q(I, D(\mathcal{A}))$, $\mathcal{U} \in L^\infty(I, \mathcal{X}_0)$, consequently $u \in L^q(I, H^2(\Omega))$, $u \in L^\infty(I, L^2(\Omega))$ and the interpolation gives

$$\|u(t)\|^2_{H^1(\Omega)} \leq C\|u(t)\|_{H^2(\Omega)}\|u(t)\|_{L^2(\Omega)}.$$

It is enough to integrate this inequality over the time interval to get $u \in L^2_{loc}([0, +\infty), H^1(\Omega))$. This concludes the proof in the case of assumptions (2) or (3).
If the assumption (1) holds we can proceed similarly. Recall $I = (0, T)$ with $T \in (0, +\infty)$. First we note that from analyticity of $\mathcal{A}$ we get that the mild solution $\mathcal{U}$ defined in (10) satisfies $U \in L^\infty(I, \mathcal{X}_0)$. Then we rewrite the equation (9) as

$$\partial_t\mathcal{U} = \tilde{\mathcal{A}}\mathcal{U} + \tilde{\mathcal{F}}(t),$$

with $\tilde{\mathcal{A}} := \mathcal{A} - \lambda_0$, $\tilde{\mathcal{F}}(t) := \mathcal{F}(t) + \lambda_0\mathcal{U}$ and $\lambda_0 = \max(1, -4\alpha)/\beta$. The function $\tilde{\mathcal{F}}$ and the semigroup $\tilde{\mathcal{T}}$ generated by the operator $\tilde{\mathcal{A}}$ can be estimated

$$\|\tilde{\mathcal{F}}\|_{L^q(I,\mathcal{X}_0)} \leq C(\|\mathcal{F}\|_{L^q(I,\mathcal{X}_0)} + \|\mathcal{U}_0\|_{\mathcal{X}_0}) \quad \text{and} \quad \exists\omega < 0, \forall t > 0 : \|\tilde{\mathcal{T}}(t)\|_{\mathcal{L}(\mathcal{X}_0)} \leq Ce^{\omega t}.$$

The rest of the proof can be done as in the cases (2) or (3). □

Acknowledgements

This work is a part of the Czech Science Foundation project 20-11027X.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest declaration

The authors declare that there is no conflict of interest.

References

[Abbatiello et al., 2021] Abbatiello, A., Bulíček, M., and Maringová, E. (2021). On the dynamic slip boundary condition for Navier-Stokes-like problems. *Math. Models Methods Appl. Sci.*, 31(11):2165–2212.

[Acevedo Tapia et al., 2021] Acevedo Tapia, P., Amrouche, C., Conca, C., and Ghosh, A. (2021). Stokes and Navier-Stokes equations with Navier boundary conditions. *J. Differential Equations*, 285:258–320.

[Amrouche and Girault, 1994] Amrouche, C. and Girault, V. (1994). Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension. *Czechoslovak Math. J.*, 44(119)(1):109–140.

[Beirão Da Veiga, 2004] Beirão Da Veiga, H. (2004). Regularity for Stokes and generalized Stokes systems under nonhomogeneous slip-type boundary conditions. *Adv. Differential Equations*, 9(9-10):1079–1114.

[Bogovskiĭ, 1980] Bogovskiĭ, M. E. (1980). Solutions of some problems of vector analysis, associated with the operators div and grad. In *Theory of cubature formulas and the application of functional analysis to problems of mathematical physics*, volume 1980 of *Trudy Sem. S. L. Soboleva, No. 1*, pages 5–40, 149. Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk.

[Denk et al., 2008] Denk, R., Prüss, J., and Zacher, R. (2008). Maximal L_p-regularity of parabolic problems with boundary dynamics of relaxation type. *J. Funct. Anal.*, 255(11):3149–3187.

[Escher, 1992] Escher, J. (1992). Nonlinear elliptic systems with dynamic boundary conditions. *Math. Z.*, 210(3):413–439.

[Galdi, 2011] Galdi, G. P. (2011). *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition. Steady-state problems.

[Hatzikiriakos, 2012] Hatzikiriakos, S. G. (2012). Wall slip of molten polymers. *PROGRESS IN POLYMER SCIENCE*, 37(4):624–643.

[Héron, 1981] Héron, B. (1981). Quelques propriétés des applications de trace dans des espaces de champs de vecteurs à divergence nulle. *Comm. Partial Differential Equations*, 6(12):1301–1334.

[Hlaváček and Nečas, 1970] Hlaváček, I. and Nečas, J. (1970). On inequalities of Korn's type. II. Applications to linear elasticity. *Arch. Rational Mech. Anal.*, 36:312–334.

[Kunstmann and Weis, 2004] Kunstmann, P. C. and Weis, L. (2004). Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 65–311. Springer, Berlin.

[Lunardi, 1995] Lunardi, A. (1995). *Analytic semigroups and optimal regularity in parabolic problems*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel. [2013 reprint of the 1995 original] [MR1329547].

[Maringová, 2019] Maringová, E. (2019). Mathematical analysis of models arising in continuum mechanics with implicitly given rheology and boundary conditions. Doctoral Thesis, Charles University, Prague.

[Nečas, 2012] Nečas, J. (2012). *Direct methods in the theory of elliptic equations*. Springer Monographs in Mathematics. Springer, Heidelberg. Translated from the 1967 French original by Gerard Tronel and Alois Kufner, Editorial coordination and preface by Šárka Nečasová and a contribution by Christian G. Simader.

[Pearson and Petrie, 1968] Pearson, J. R. A. and Petrie, C. J. S. (1968). On the melt-flow instability of extruded polymers. In *Polymer systems: deformation and flow*.

[Petryshyn, 1965] Petryshyn, W. V. (1965). Constructional proof of Lax-Milgram lemma and its application to non-$K-$p.d. abstract and differential operator equations. *J. Soc. Indust. Appl. Math. Ser. B Numer. Anal.*, 2:404–420.

[Sohr and von Wahl, 1986] Sohr, H. and von Wahl, W. (1986). On the regularity of the pressure of weak solutions of Navier-Stokes equations. *Arch. Math. (Basel)*, 46(5):428–439.

[Solonnikov and Ščadilov, 1973] Solonnikov, V. A. and Ščadilov, V. E. (1973). A certain boundary value problem for the stationary system of Navier-Stokes equations. *Trudy Mat. Inst. Steklov.*, 125:196–210, 235. Boundary value problems of mathematical physics, 8.

[Ventcel′, 1959] Ventcel′, A. D. (1959). On boundary conditions for multi-dimensional diffusion processes. *Theor. Probability Appl.*, 4:164–177.