Bifurcations

The following equation is given:

$$x' = f(x, \mu)\,, \tag{1}$$

where $\mu \in \mathbb{R}$ is a parameter, $x \in \mathbb{R}^n$ is an unknown function. We suppose that the right hand side f is smooth.

Definition 1 (Informal)**.** We say that the equation (1) has a bifurcation in the point (x_0, μ_0), if the behaviour of the solution on a neighbourhood of the point x_0 *radically* changes when $\mu = \mu_0$.

By a radical change in behaviour we typically understand either the genesis or termination of an equilibrium or the change of its stability.

Basic 1D bifurcations

In the following we will mention basic types of one-dimensional bifurcations, i.e. the case $x \in \mathbb{R}$. We represent the bifurcations on a so called bifurcation diagram: we draw equilibria into the plane (μ, x), that is solutions of the equation $f(x, \mu) = 0$ (in blue in the graf). It is customary to draw the stable equilibria by a continuous line and unstable equilibria by a dashed line. The direction of the solution while μ is fixed – which corresponds to the sign of $f(x, \mu)$ – is indicated by arrows.

Example 1. A „saddle-node“ bifurcation arise when we study the equation $x' = \mu - x^2$. For $\mu < 0$ we have no equilibria; for $\mu = 0$ a single equilibrium arise in the origin, which for $\mu > 0$ breaks up into two: a stable one and an unstable one (Figure 1).

Remark. The term „saddle-node“ is a bit confusing in the one-dimensional case; the analogous case in $\mathbb{R}^2$ corresponds to the situation, when a new equilibrium arise, which consequently breaks into the (stable) node and the (unstable) saddle. See Exercise 5 below.

Example 2. A „transcritical“ bifurcation arise when we solve the equation $x' = \mu x - x^2$. For $\mu < 0$ we see a stable and unstable equilibrium, which approach with increasing μ. When $\mu = 0$ their paths cross and their types of stability switch (Figure 2).

Example 3. A „pitchfork“ bifurcation is a combination of the preceding two. An example is the equation $x' = \mu x - x^3$. For $\mu < 0$ we have a stable equilibrium, from which at $\mu = 0$ two new equilibria arise, whereas the original equilibrium loses its stability (Figure 3).

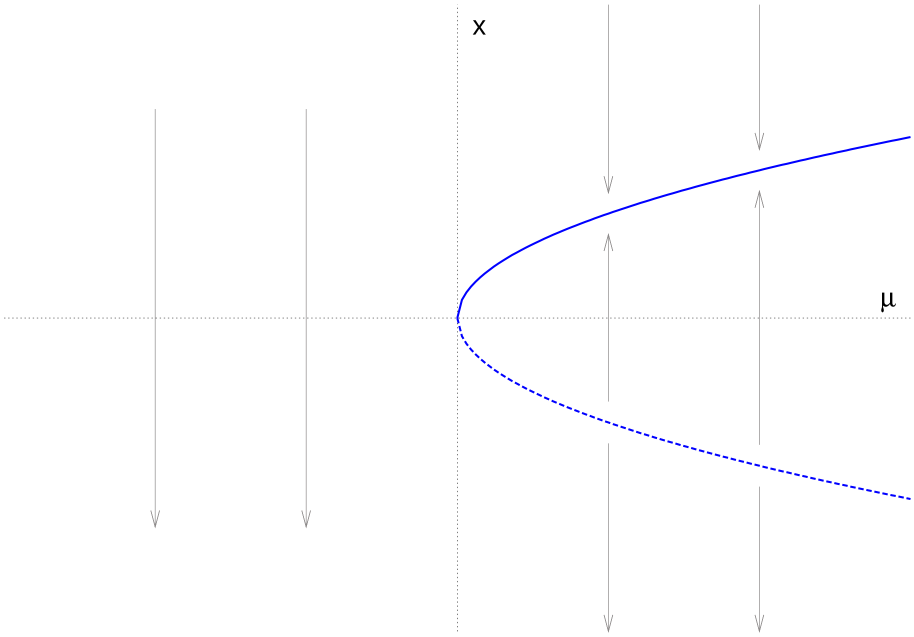


Figure 1: A „saddle-node “ bifurcation in 1d

Remark. In every mentioned case there are bifurcations in a point, i.e. for $(x_0, \mu_0) = (0, 0)$ the following equations hold:

$$\begin{aligned} f(x, \mu) &= 0 \\ \partial_x f(x, \mu) &= 0 \end{aligned}$$

It is possible to show that (for a one-dimensional equation) they are *necessary* conditions for a bifurcation to occur. Another important property: in no neighbourhood of the bifurcation is it possible to express the set $\{f(x, \mu) = 0\}$ as a graf of a function of the variable μ.

Example 4 (No bifurcation)**.** The equation $x' = x - \sin \mu$ has a single equilibrium $x = \sin \mu$. While μ changes, this point moves, however it stays unstable. Therefore there is no bifurcation in this case (Figure 4).

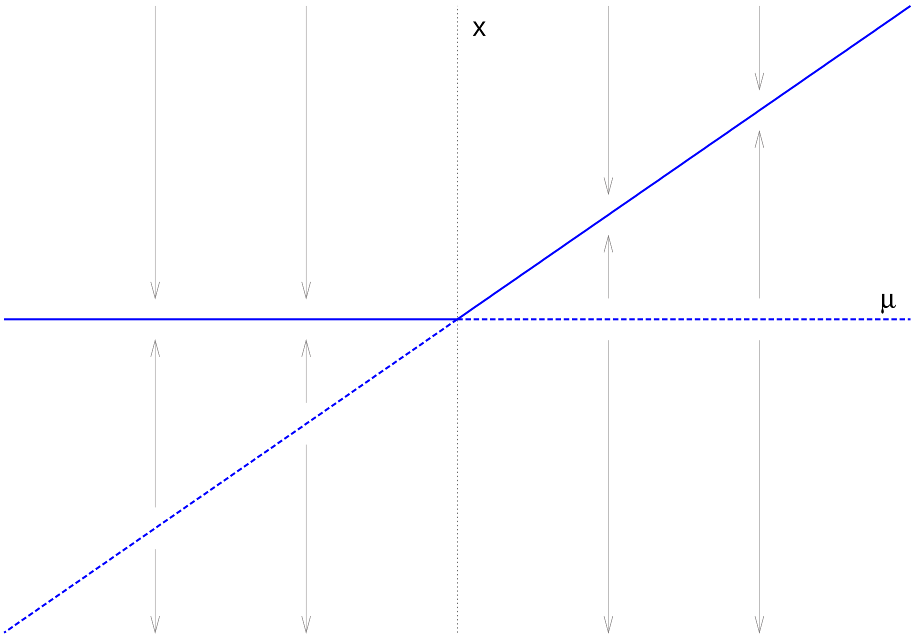


Figure 2: A „transcritical“ bifurcation

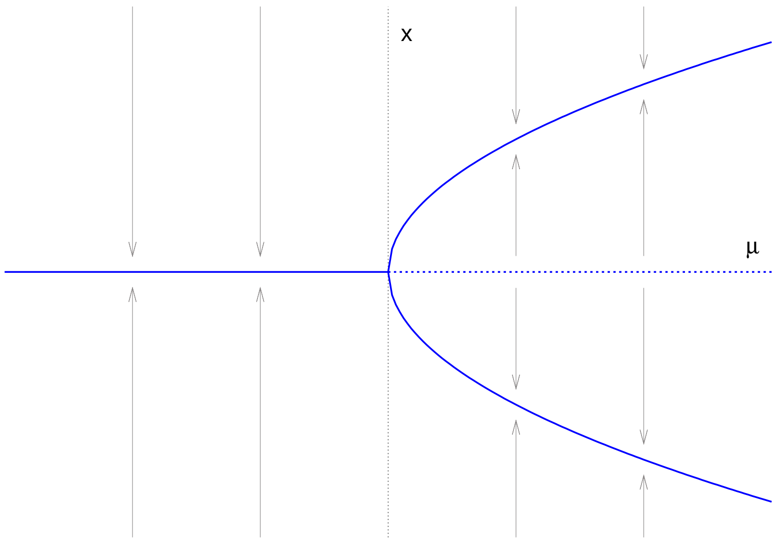


Figure 3: A „pitchfork“ bifurcation

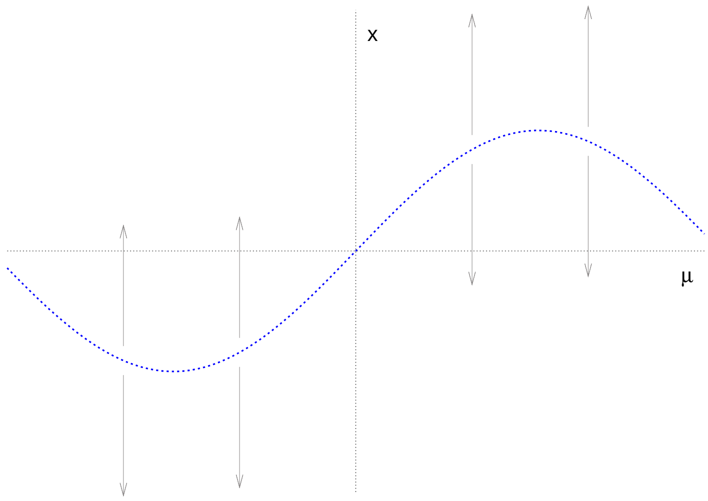


Figure 4: No bifurcation

Solve the following exercises on bifurcations.

1. Sketch bifurcation diagrams, determine the type of bifurcation:

(a) $x' = \mu - x^2$

(b) $x' = \mu + x^2 - x^3$

(c) $x' = \mu^2 + x^2 - 1$

(d) $x' = 1 - \mu x$

(e) $x' = \mu - \sin x$

(f) $x' = \sin \mu - x$

(g) $x' = \mu^2 - x^2$

(h) $x' = \mu x$

2. Let the polynomial $p(x)$ have a simple root at x_0, i.e. $p(x_0) = 0$, $p'(x_0) \neq 0$. Then the polynomial $\tilde{p}(x)$ with slightly changed coeficients once again has a simple root and it is located near x_0.

Hints and solutions.

1) (a) a saddle-node at $(\mu, x) = (0, 0)$

(b) a saddle-node at $(0, 0)$ and $(-4/27, 2/3)$

(c) a saddle-node at $(-1, 0)$ and $(1, 0)$

(d) no bifurcation

(e) a saddle-node at $(-1, (2k-1)\pi/2)$, $(1, (2k+1)\pi/2)$

(f) no bifurcation

(g) a transcritical bifurcation at $(0, 0)$

(h) a bifurcation at $(0, 0)$

2) Use the implicit function theorem on

$$F(a_0, \ldots, a_n, x) = a_0 x^n + a_1 x^{n-1} + \ldots a_0$$

Bifurcations in the plane.

Example 5 (A saddle-node bifurcation – the symbiosis formation)**.** We are interested in the system

$$a' = a(K-a) + \frac{ap}{p+1} \tag{2}$$

$$p' = -\frac{p}{2} + \frac{ap}{p+1} \tag{3}$$

describing the symbiosis in between a plant p and insects (pollinators) a. In blue we have higlighted the term describing the mutual positive bond. In the absence of insects ($a = 0$) the plants exponentially die whereas for $p = 0$ (no plants), a has the limit K ($=$ the inherent capacity of nature).

We are studying the behaviour of the solution in the first quadrant (a, $p \geq 0$) depending on the parameter K. The system always has equilibria $(0, 0)$ and $(K, 0)$; strictly inside of the first quadrant we have

$$a' = 0 \iff p = \frac{a-K}{K+1-a} \tag{4}$$

$$p' = 0 \iff p = 2a - 1 \tag{5}$$

At the same time, if these equations are satisfie, then there exist equilibria and it leads to the following quadratic equation

$$2a^2 - 2a(K+1) + 1 = 0 \tag{6}$$

with the discriminant $D = 4(K+1)^2 - 8$, which is equal to 0 if and only if K reaches the critical value

$$K_0 = \sqrt{2} - 1\,.$$

We distinguish the following three cases (see Figures 5, 6, 7).

(i) $K < K_0$, see Figure 5, the equation (6) has no solution, i.e. the curves (4), (5) (in red) do not cross. By examining the direction of the course of the solutions (grey arrows) it is obvious, that all the solutions have for $t \to \infty$ the limit in $(K, 0)$ (blue circle).

(ii) For $K = K_0$, see Figure 6, the curves (4), (5) touch, which is how an equilibrium emerge

$$(x_0, y_0) = (1/\sqrt{2}, \sqrt{2} - 1)\,.$$

The linearization matrix at this point has one zero and one negative eigenvalue. The stable manifold (corresponding to the negative eigenvalue) is approximately indicated by arrows; the solutions above them (i.e. in the B-sector) have the

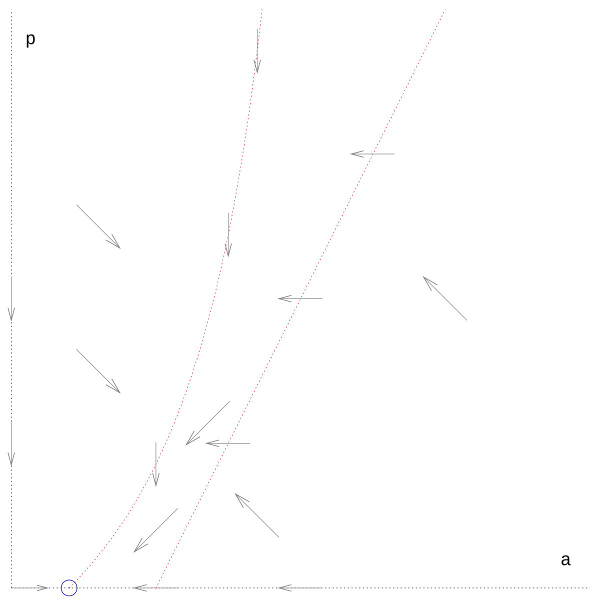


Figure 5: The subcritical case $K < K_0$

limit (x_0, y_0), whereas solutions in the A-sector head towards $(K, 0)$, and thus (x_0, y_0) is not stable.

(iii) Another increase in K, see Figure 7, leads to the split of (x_0, y_0) to a stable node and an unstable saddle. Specifically for $K = K_0 + 0.05$ we have

$$\begin{aligned} (x_1, y_1) &\doteq (0.542, 0.085) \qquad &\text{(saddle)} \\ (x_2, y_2) &\doteq (0.922, 0.844) \qquad &\text{(node)} \end{aligned}$$

The solutions in the sector C head towards the point (x_2, y_2).

Remark. Notice that in the bifurcation case ($K = K_0$) the equilibrium (x_0, y_0) is not hyperbolic. It is possible to show that this is a necessary condition for a bifurcation, because hyperbolic equilibria are "robust" (the behaviour of the solutions in their neighbourhood does not change if the system is smoothly perturbated).

Example 6 (An easy saddle-node bifurcation)**.** Let us consider the system

$$\begin{aligned} x' &= (1 + \varepsilon)x - y + x^2 \\ y' &= (1 + \varepsilon)y - x + y^2 \end{aligned}$$

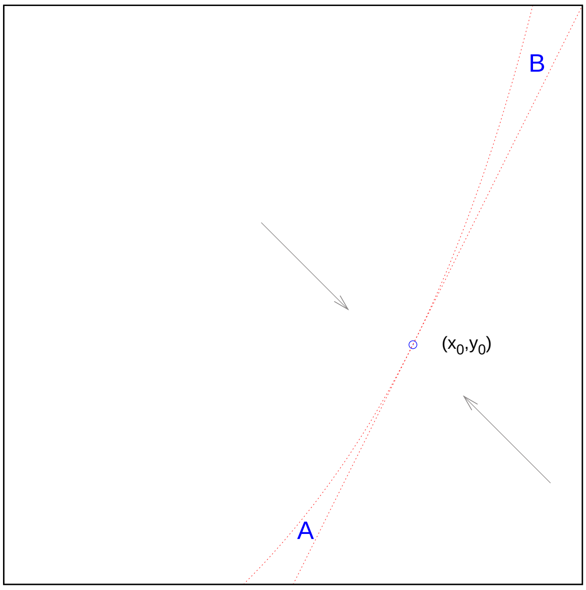


Figure 6: The critical case $K = K_0$

For $\varepsilon = 0$ there is a (unique) equilibrium that is ustable and non-hyperbolic (the spectrum of the linearization is $\{0, 2\}$). Figure 8 – isolines $\{y' = 0\}$ in red, $\{x' = 0\}$ blue; the solutions in black.

For small $\varepsilon > 0$ the origin becomes an unstable node (the spectrum is $\{\varepsilon, 2 + \varepsilon\}$). Another equilibrium emerges $(-\varepsilon, -\varepsilon)$: a saddle with the spectrum $\{-\varepsilon, 2 - \varepsilon\}$. See Figure 9; the orbit connecting equilibria is in bright blue.

Example 7 (A rupture of a homoclinic orbit)**.** Consider the system

$$\begin{aligned} x' &= y \\ y' &= \varepsilon y + x - x^2 \end{aligned}$$

For $\varepsilon = 0$ we have two equilibria : the origin (a saddle with the spectrum $\{\pm 1\}$) and $(1, 0)$, whose stability can not be determined by linearization (spectrum $\{\pm i\}$). The function $V(x, y) = y^2 - x^2 + 2x^3/3$ is a first integral, i.e. for every solution it holds $V(x, y) = C$ for a suitable C, see Figure 10. For $C = 0$ we get the origin and the corresponding stable (in green) and unstable (in red) manifold, merging in the right half-plane in one homoclinic orbit (in dark blue).

For $C \in (-1/3, 0)$, $V(x, y) = C$ are simple closed curves, correspoding

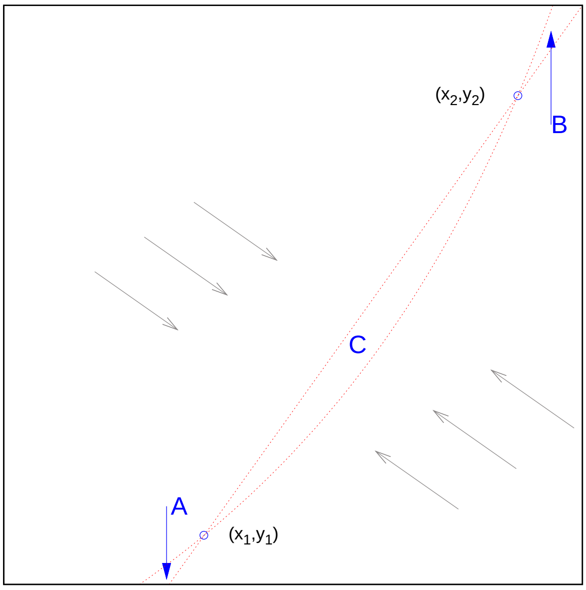


Figure 7: Supercritical case $K > K_0$

to the periodic solutions around the point $(1, 0)$. From this we can easily conclude that this point is stable, but not asymptotically stable.

For $\varepsilon > 0$ and $\varepsilon < 0$ respectively we have that $V(x, y)$ is increasing and decreasing respectively along the solution. The point $(1, 0)$ is unstable and asymptotically stable respectively.

The homoclinic orbit gets broken („ruptured“). For $\varepsilon > 0$ the stable part becomes a heteroclinic orbit connecting the origin with the (now unstable) point $(1, 0)$, see Figure 11(a) in green. The unstable part (in red) goes to infinity.

For $\varepsilon < 0$ the now unstable manifold (see Figure 11(b), in red) connects the origin with the (stable) equilibrium $(1, 0)$; the stable part of the manifold (in red) comes from infinity.

Remark. For $\varepsilon = 0$ in the point $(1, 0)$ we get a so called *Hopf bifurcation*, which we will study in detail below.

Example 8 (A rupture of a heteroclinic orbit)**.** Consider the system

$$x' = \lambda + 2xy$$
$$y' = 1 + x^2 - y^2$$

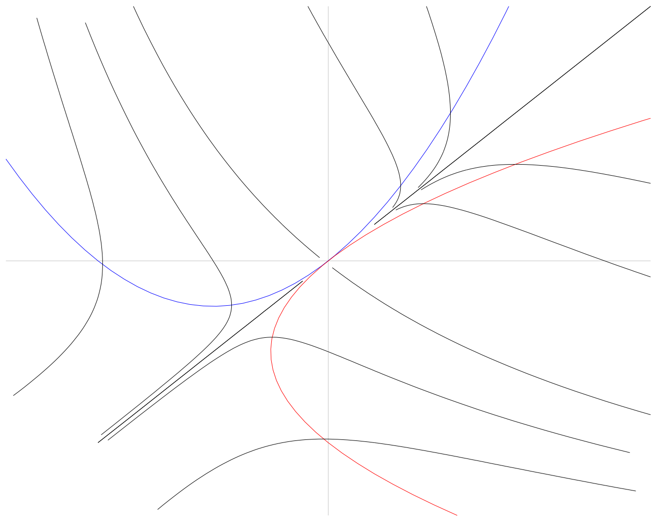


Figure 8: Exercise 6 for $\varepsilon = 0$

For $\lambda = 0$ we see the solution on the Figure 12(a). The set $x' = 0$ and $y' = 0$ is highlighted in blue and in red respectively. There exist two hyperbolic equilibria $(0, \pm 1)$. They are saddles connected by a heteroclinic orbit (in green).

For $\lambda < 0$ both saddle points slightly move, the type of stability will remain unchanged. The heteroclinic orbit however becomes a globally defined solution (see Figure 12(b), in green).

Remark. Homoclinic orbit = an orbit connecting a stable and an unstable manifold of *the same* equilibrium ($o\mu o\varsigma$, means „the same" in Greek). Heteroclinic orbit = an orbit connecting a stable and an unstable manifold of *two different* equilibria ($\varepsilon\tau\varepsilon\varrho o\varsigma$, means „different" in Greek).

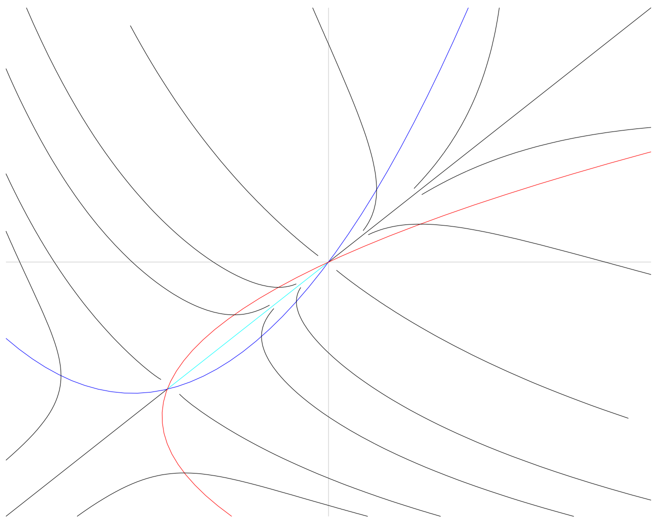


Figure 9: Exercise 6 for $\varepsilon > 0$

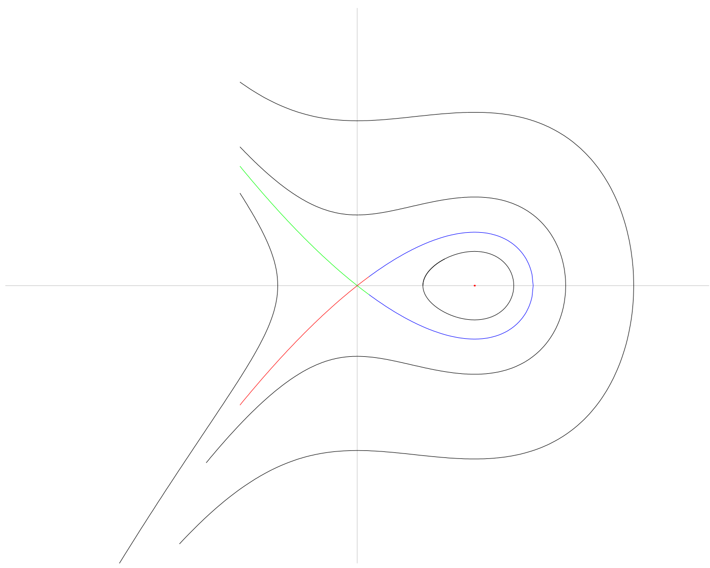
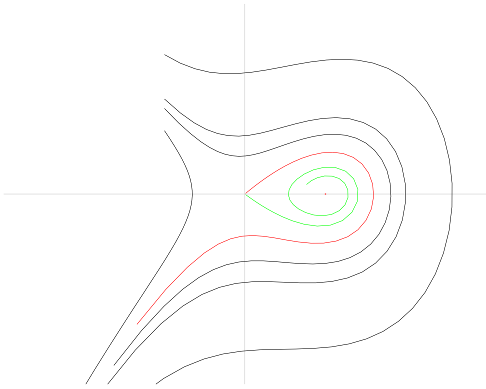
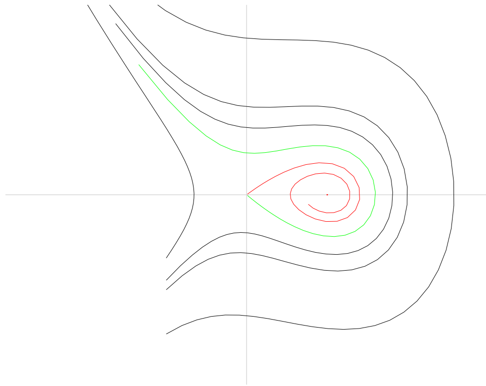


Figure 10: Exercise 7 for $\varepsilon = 0$

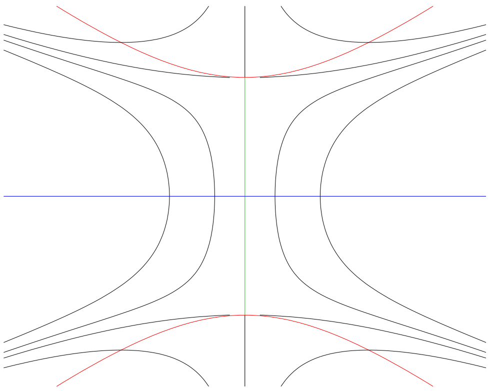


(a) $\varepsilon > 0$

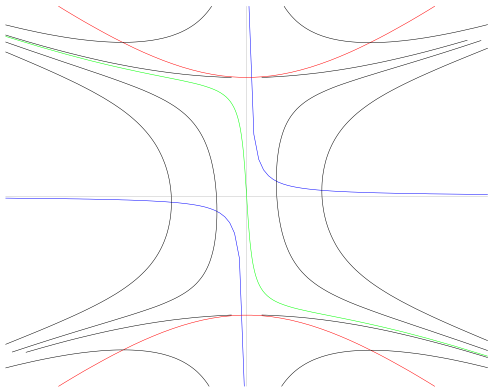


(b) $\varepsilon < 0$

Figure 11: Exercise 7 for $\varepsilon \neq 0$



(a) $\lambda = 0$



(b) $\lambda < 0$

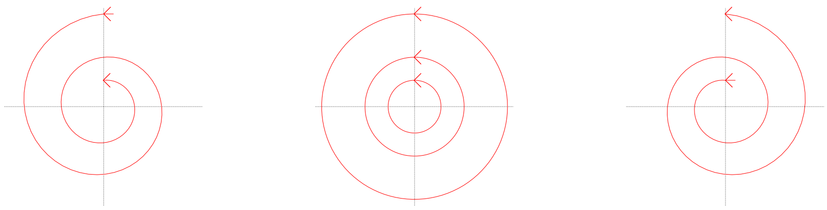
Figure 12: Exercise 8

Hopf bifurcation.

Motivational exercise. Consider the linear system

$$x' = \mu x - y$$
$$y' = x + \mu y\,.$$

The spectrum of the matrix is $\{\mu \pm i\}$; the origin loses its stability when $\mu = 0$. The course of the solutions can be seen on the Figure 13.



(a) $\mu < 0$ (b) $\mu = 0$ (c) $\mu > 0$

Figure 13: The loss of stability of a linear system

The solutions approach on a spiral, whose velocity of rolling/unrolling changes with the parameter μ. The fact that for a suitable critical value of μ the spiral degenerate into a closed curve, i.e. a periodic solution emerges, appears to be „robust“, i.e. it should occur even in a presence of terms of higher orders. This is precisely formulated in the Hopf bifurcation theorem.

Theorem 1 (Hopf)**.** *For (x, y, μ) in a neighbourhood of $(0, 0, 0)$ we consider the system*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A_\mu \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y,\mu) \\ g(x,y,\mu) \end{pmatrix}, \tag{7}$$

where the matrix A_μ depends on the parameter μ. Suppose that $\sigma(A_\mu) = \{\alpha(\mu) \pm i\omega(\mu)\}$, where $\alpha(\cdot)$, $\omega(\cdot)$ are C^2 functions and

$$\alpha(0) = 0\,, \qquad \alpha'(0) \neq 0\,, \qquad \omega(0) \neq 0\,. \tag{8}$$

The functions f, g are peturbations of higher order, i.e.

$$f = 0\,,\ g = 0\,, \qquad \nabla_{x,y} f = 0\,,\ \nabla_{x,y} g = 0\,, \qquad \text{at } (0, 0, \mu).$$

Then there exist δ, $\Delta > 0$ and a C^1 function $\varphi : (0, \delta) \to (-\Delta, \Delta)$ such that for each $a \in (0, \delta)$ the system (7) has a non-trivial periodic solution passing through $(x, y) = (a, 0)$ when we set $\mu = \varphi(a)$.

The key assumption (8) is guaranteeing that the spectrum will cross the imaginary axis in a non-zero velocity with a strictly non-zero imaginary part.

The main result of the theorem – the existence of non-trivial periodic solutions close to the origin – does not tell us much of importance from a practical standpoint. That is why the following specification (with an exeptionally complicated proof) is useful.

Theorem 2. *Let the assupmtions of Theorem 1 hold, and let*

$$A_0 = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix} .$$

Then there exists a smooth change of coordinates such that $r = \sqrt{x^2 + y^2}$ *behaves qualitatively identically to the solution of the equation*

$$r' = d\mu r + ar^3 , \tag{9}$$

where $d = \alpha'(0)$ *and the constant* a *can be computed from the following relation*

$$\begin{aligned} 16a &= f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ &+ \frac{1}{\omega_0}\big[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\big] , \end{aligned}$$

where f_{xy} *etc. denote the corresponding partial derivative evaluated at* $(x, y, \mu) = (0, 0, 0)$*.*

Example 9. Let us consider the equation

$$\begin{aligned} x' &= -\mu x - y \\ y' &= x + y^3 \end{aligned}$$

This is a system in the form (7) with $f = 0$, $g = y^3$ and

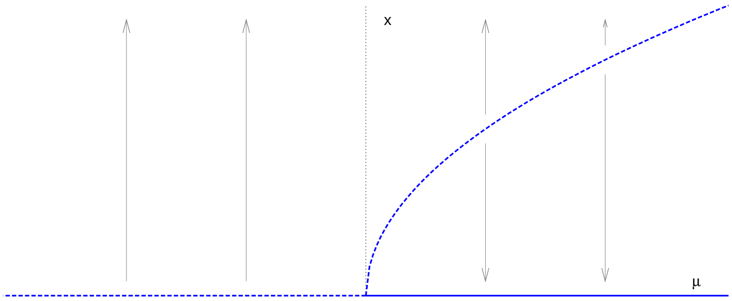
$$A_\mu = \begin{pmatrix} -\mu & -1 \\ 1 & 0 \end{pmatrix} .$$

The spectrum of A_μ is $\{-\mu/2 \pm i\sqrt{4 - \mu^2/4}\}$ (for small μ), the assumptions of both theorems hold.

We have $d = -1/2$, $\omega_0 = 1$ and $a = 3/8$; the equation (9) is in the form

$$r' = r\big(-\frac{\mu}{2} + \frac{3}{8}r^2\big) .$$

Let us sketch the course of the solutions into a bifurcation diagram in the plane (μ, r). For $r > 0$ the condition $r' = 0$ corresponds to a circle, i.e. a non-trivial periodic solution. Those solution emerge for $\mu > 0$ and they are unstable.



Remark. The theorem 2 specially enables us to determine the stability of the origin for $\mu = 0$, when (9) is reduced to $r' = ar^3$. Notice that for the given case, classical theorems about linearized stability can not be used since the spectrum of the matrix is purely imaginary. The stability is therefore decided by the terms of higher order and it is noteworthy, that the whole information is contained in the single constant a.

That can be used even in the simplified case where the bifurcation parameter is ignored (i.e. it is equal to 0).

Example 10. We have the system

$$x' = -y + x^2$$
$$y' = x + x^2$$

The linearization matrix in the origin has the required form

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

An easy computation shows that $a = -1/4$, and therefore the origin is stable. Attempt to deduce the stability of the origin through different methods – it is not simple!

Example 11. Consider the equation

$$x'' + (x^2 - \mu)x' + 2x + x^3 = 0\,,$$

which we transform by a usual substition $y = x'$ to the system

$$x' = y\,, \tag{10}$$
$$y' = -2x + \mu y - x^2 y - x^3\,. \tag{11}$$

The assumptions of Theorem 1 hold, however the assuptions of Theorem 2 do not because

$$A_\mu = \begin{pmatrix} 0 & 1 \\ -2 & \mu \end{pmatrix}$$

and thus $\sigma(A_\mu) = \{\mu/2 \pm i/2\sqrt{8-\mu^2}\}$, however A_μ does not have the required antisymetric form for $\mu = 0$. The problem is solved by the following

Auxiliary calculation. If A is a real 2×2 matrix with the spectrum $\{\alpha \pm i\omega\}$, it is necessarily similar to the matrix

$$B = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}.$$

We calculate the transition matrix using the following trick: we find a vector $U \in \mathbb{C}^2$ such that $AU = (\alpha - i\omega)U$. The decomposition to its real and imaginary part $U = u + iv$, where u, $v \in \mathbb{R}^2$, gives us

$$\begin{aligned} Au &= \alpha u + \omega v, \\ Av &= -\omega u + \alpha v, \end{aligned}$$

in other words, A with respect to the basis $\{u, v\}$ has the form B. For the transition matrix T with columns u, v (in this order) it therefore holds that

$$A = TBT^{-1}, \qquad B = T^{-1}AT.$$

If the original matrix has the form

$$X' = AX + F(X), \tag{12}$$

where $X \in \mathbb{R}^2$, the substitution $X = TY$ or $Y = T^{-1}X$ leads us to the equation

$$Y' = BY + T^{-1}F(TY). \tag{13}$$

Exercise continuation. In our specific case A_0 has the spectrum $\{\pm i\sqrt{2}\}$, i.e. we want to find $U = (U_1, U_2)$ satisfying the equation

$$\begin{pmatrix} -i\sqrt{2} & -1 \\ 2 & -i\sqrt{2} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A possible solution is $U = (i\sqrt{2}, 2)$; the corresponding transition matrices are

$$T = \begin{pmatrix} 0 & \sqrt{2} \\ 2 & 0 \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} 0 & 1/2 \\ 1/\sqrt{2} & 0 \end{pmatrix}.$$

We now understand (10-11) as a system (12), i.e. $X = (x, y)$, $A = A_\mu$ and

$$F(X) = \begin{pmatrix} 0 \\ -x^2y - x^3 \end{pmatrix}.$$

By substituting $X = TY$, where $Y = (u, v)$, we get – see (13) – for $\mu = 0$:

$$\begin{aligned} u' &= -\sqrt{2}v - 2uv^2 - \sqrt{2}v^3\,, \\ v' &= \sqrt{2}u\,, \end{aligned}$$

which is the required form. We can now apply Theorem 2, by which $r = \sqrt{u^2 + v^2}$ behaves identically as the solution of the equation

$$r' = d\mu r + ar^3\,.$$

Here $d = 1/2$, $a = -1/4$, from wich we get that the origin is stable for $\mu \leq 0$, whereas for $\mu > 0$ the origin is unstable and stable periodical orbits with the radius approximately $r = \sqrt{2\mu}$ emerge.

Solve the following exercises on the Hopf bifurcation.

3. Show that the system

$$x' = f(x, y)\,, \qquad y' = g(x, y)$$

using polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ transforms into

$$\begin{aligned} r' &= \cos\varphi\, f(r\cos\varphi, r\sin\varphi) + \sin\varphi\, g(r\cos\varphi, r\sin\varphi) \\ r\varphi' &= -\sin\varphi\, f(r\cos\varphi, r\sin\varphi) + \cos\varphi\, g(r\cos\varphi, r\sin\varphi) \end{aligned}$$

4. Using the polar coordinates, show that for μ close to 0 for the system

$$\begin{aligned} x' &= \mu x - y + f(x, y) \\ y' &= x + \mu y + g(x, y) \end{aligned}$$

non-trivial periodical solutions emerge in the neighbourhood of $(0, 0)$ Sketch a bifurcation diagram for (r, μ). — Compare this with the appropriate Hopf bifurcation theorems.

(a)

$$\begin{aligned} f(x, y) &= x\sqrt{x^2 + y^2} \\ g(x, y) &= y\sqrt{x^2 + y^2} \end{aligned}$$

(b)

$$\begin{aligned} f(x, y) &= -y\sqrt{x^2 + y^2} \\ g(x, y) &= x\sqrt{x^2 + y^2} \end{aligned}$$

(c)

$$\begin{aligned} f(x, y) &= -x(x^2 + y^2) \\ g(x, y) &= -y(x^2 + y^2) \end{aligned}$$

5. Show the existence of the Hopf bifurcation in $(x, y, \mu) = (0, 0, 0)$ for the following systems. Examine the stability of the origin and of the periodical solutions.

(a) $x' = -\mu x - y$, $y' = x + y^3$

(b) $x'' + (x')^3 - \mu x' + x = 0$

(c) $x' = \mu x + y + \mu x^2 - x^2 - xy^2$, $y' = -x + y^2$

(d) $x' = y - x^3$, $y' = -x + \mu y - x^2 y$

(e) $x' = \mu x + y - x^3 \cos x$, $y' = -x + \mu y$

(f) $x' = y - 2x^2$, $y' = -x + \mu y - x^2(1 + y)$

(g) $x' = -\mu x + y + x^2$, $y' = -x + x^2 - yx^2$

Hints and solutions.

3) Start with the relations

$$x' = r' \cos\varphi - r\varphi' \sin\varphi$$
$$y' = r' \sin\varphi + r\varphi' \cos\varphi$$

4) (a) $r' = \mu r + r^2$, $\varphi' = 1$; a circle $r = -\mu$ for $\mu < 0$

(b) $r' = \mu r$, $\varphi' = 1 + r$; a circle (with an arbitrary radius) for $\mu = 0$

(c) $r' = \mu r - r^3$, $\varphi' = 1$; a circle $r = \sqrt{\mu}$ for $\mu > 0$

5) (a) $d = -1/2$, $\omega_0 = -1$, $a = 3/8$; and thus $r' = r/2(-\mu + 3r^2/4)$; unstable origin for $\mu \leq 0$; a stable origin and unstable periodical solutions for $\mu > 0$.

(b) $d = 1/2$, $\omega_0 = -1$, $a = -3/8$; and thus $r' = -r/8(3r^2 - 4\mu)$; stable origin for $\mu \leq 0$; unstable origin and stable periodical solutions for $\mu > 0$.

(c) $d = 1/2$, $\omega_0 = -1$, $a = -1/8$; and thus $r' = -r/8(r^2 - 4\mu)$; stable origin for $\mu \leq 0$; unstable origin and stable periodical solutions for $\mu > 0$.

(d) $d = 1/2$, $\omega_0 = -1$, $a = -1/2$; and thus $r' = -r/2(r^2 - \mu)$; stable origin for $\mu \leq 0$; unstable origin and stable periodical solutions for$\mu > 0$.

(e) $d = 1/2$, $\omega_0 = -1$, $a = -3/8$; and thus $r' = -r/8(3r^2 - 4\mu)$; stable origin for $\mu \leq 0$; unstable origin and stable periodical solutions for $\mu > 0$.

(f) $d = 1/2$, $\omega_0 = -1$, $a = 3/8$; and thus $r' = r/2(\mu + 3r^2/4)$; unstable origin for $\mu \geq 0$; stable origin and unstable periodical solutions for $\mu < 0$.

(g) $d = -1/2$, $\omega_0 = 1$, $a = -3/8$; and thus $r' = -r/8(4\mu + 3r^2)$; stable origin for $\mu \geq 0$; unstable origin and stable periodical solutions for $\mu < 0$.