

# Differential Equations in Banach Spaces

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# 1. Semigroup, Generator and their properties

## Definition

We say that  $S : [0, +\infty) \rightarrow L(X)$  is **strongly continuous semigroup** ( **$C_0$ -semigroup**) if

- (i)  $S(0) = I$
- (ii)  $S(t + s) = S(t)S(s)$  for all  $t, s \geq 0$
- (iii)  $t \mapsto S(t)x$  is continuous for each  $x \in X$ .

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## Definition

We say that an operator  $A : D(A) \subset X \rightarrow X$  is a generator of the  $C_0$ -semigroup  $S$  if

- (i)  $D(A) = \{x \in X : \lim_{h \rightarrow 0^+} \frac{1}{h}(S(h)x - x) \text{ exists}\}$
- (ii)  $Ax = \lim_{h \rightarrow 0^+} \frac{1}{h}(S(h)x - x)$  for all  $x \in D(A)$ .

## Proposition 1

Let  $S$  be a  $C_0$ -semigroup. Then there exists  $C > 0$ ,  $\omega \in \mathbb{R}$  such that

$$\|S(t)\| \leq Ce^{\omega t}$$

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## Proposition 2

Let  $S$  be a  $C_0$ -semigroup and  $A$  its generator. Then

- (i)  $x \in D(A)$  implies  $S(t)x \in D(A)$  for all  $t \geq 0$ ,
- (ii) if  $x \in D(A)$  then  $S(t)Ax = AS(t)x = \frac{d}{dt}S(t)x$ ,
- (iii) if  $x \in X$  then  $\int_0^t S(s)x \in D(A)$  and  $A \int_0^t S(s)x = S(t)x - x$ ,

## Proposition 3

Let  $S$  be a  $C_0$ -semigroup and  $A$  its generator. Then

- (i)  $D(A)$  is dense in  $X$ ,
- (ii)  $A$  is closed and linear,
- (iii)  $A$  determines  $S$  uniquely.

## Definition

**Resolvent set** of a linear operator  $A : D(A) \subset X \rightarrow X$  is  $\rho(A) = \{\lambda \in \mathbb{C} : (\lambda I - A) : D(A) \rightarrow X \text{ is a bijection}\}$ .

**Resolvent operator** of  $A$  is  $R(\lambda, A) : X \rightarrow D(A)$  such that  $(\lambda I - A)R(\lambda, A) = I_X$ .

## Proposition 4

Let  $S$  be a  $C_0$ -semigroup satisfying  $\|S(t)\| \leq Ce^{\omega t}$  for all  $t \geq 0$  and let  $A$  be its generator. Then the following hold for all  $\lambda \in \mathbb{C}$ ,  $\Re \lambda > \omega$

- (i)  $\lambda \in \rho(A)$ ,
- (ii)  $R(\lambda, A) = \int_0^{+\infty} e^{-\lambda s} S(s) ds$  and
- (iii)  $\|R(\lambda, A)\| \leq \frac{C}{\Re \lambda - \omega}$ .

# 2. Semigroup and differential equation

# Abstract Cauchy Problem

$$x' = Ax, \quad x(0) = x_0 \quad (\text{ACP})$$

## Definition

A function  $x : \mathbb{R}_+ \rightarrow X$  is **a classical solution** of (ACP) if  $x \in C(\mathbb{R}_+, D(A)) \cup C^1(\mathbb{R}_+, X)$ ,  $x(0) = x_0$  and for every  $t \geq 0$  satisfies  $x'(t) = Ax(t)$ .

# Abstract Cauchy Problem

$$x' = Ax, \quad x(0) = x_0 \quad (\text{ACP})$$

## Definition

A function  $x : \mathbb{R}_+ \rightarrow X$  is a **classical solution** of (ACP) if  $x \in C(\mathbb{R}_+, D(A)) \cup C^1(\mathbb{R}_+, X)$ ,  $x(0) = x_0$  and for every  $t \geq 0$  satisfies  $x'(t) = Ax(t)$ . Function  $x$  is called a **mild solution** of (ACP) if  $x \in C(\mathbb{R}_+, X)$  and for every  $t \geq 0$  satisfies

$$x(t) = x_0 + A \int_0^t x(s) ds.$$

Note:  $\mathbb{R}_+ = [0, +\infty)$  !!!

## Definition

We say that (ACP) is **well-posed** if

- (i) for every  $x \in D(A)$  there exists a unique classical solution and
- (ii) if  $x_{0n} \rightarrow x_0$  then  $x_n \rightrightarrows x$  on  $[0, T]$  for every  $T > 0$ .

Here  $x_n$  and  $x$  denote the solutions corresponding to  $x_{0n}$  and  $x_0$ , respectively.

## Theorem 5

*Let  $A$  be closed and densely defined linear operator. Then the following is equivalent*

- (i)  $A$  generates a  $C_0$ -semigroup,*
- (ii) (ACP) is well-posed.*

## Theorem 5

*Let  $A$  be closed and densely defined linear operator. Then the following is equivalent*

- (i)  *$A$  generates a  $C_0$ -semigroup,*
- (ii) *(ACP) is well-posed.*

## Lemma 6

*Let  $(B, D(B))$  generate a  $C_0$ -semigroup  $S$  on  $X$  and  $D \subset D(B)$  s.t.  $D$  is dense in  $X$  and invariant w.r.t.  $S(t)$  for all  $t \geq 0$ . Then  $D$  is dense in  $D(B)$  in the graph norm*

$$\|x\|_B := \|x\| + \|Bx\|$$

## Theorem 7

*Let  $A$  generate a  $C_0$ -semigroup  $S$ . Then  $x(t) := S(t)x_0$  is a mild solution to (ACP) for each  $x_0 \in X$ .*

# 3. Hille–Yosida and Lumer–Phillips Theorems

# Hille–Yosida Theorem (Contractive Case)

## Theorem 8 (Hille–Yosida, contractive case)

Let  $A : D(A) \subset X \rightarrow X$  be a linear operator. Then  $A$  generates a **contraction  $C_0$ -semigroup**  $S$ , i.e.  $\|S(t)\| \leq 1$  for all  $t \geq 0$ , on  $X$  if and only if

- (i)  $A$  is closed and densely defined,
- (ii)  $(0, +\infty) \subset \rho(A)$ , and
- (iii)  $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$  for all  $\lambda > 0$ .

# Corollary: Exponentially Bounded Semigroup, Constant 1

## Corollary 9

Let  $A : D(A) \subset X \rightarrow X$  be a linear operator. Then  $A$  generates a  $C_0$ -semigroup  $S$  satisfying  $\|S(t)\| \leq e^{\omega t}$  for all  $t \geq 0$  if and only if

- (i)  $A$  is closed and densely defined,
- (ii)  $(\omega, +\infty) \subset \rho(A)$ , and
- (iii)  $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda - \omega}$  for all  $\lambda > \omega$ .

# General Hille–Yosida Theorem

## Theorem 10 (Hille–Yosida, general case)

Let  $A : D(A) \subset X \rightarrow X$  be linear. Then  $A$  generates a  $C_0$ -semigroup  $S$  satisfying  $\|S(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$  if and only if

- (i)  $A$  is closed and densely defined,
- (ii)  $(\omega, +\infty) \subset \rho(A)$ , and
- (iii) for all  $n \in \mathbb{N}$  and  $\lambda > \omega$ ,

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}.$$

# Dissipative Operators

## Definition

A linear operator  $A : D(A) \subset X \rightarrow X$  is **dissipative** if it is closed, densely defined, and

$$\|(\lambda - A)x\| \geq \lambda\|x\|$$

for every  $\lambda > 0$  and  $x \in D(A)$ .

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for every  $\lambda > 0$  and  $x \in D(A)$ .

## Proposition 11

*If  $A$  is dissipative, then*

- (i)  $\lambda I - A$  is injective for all  $\lambda > 0$  and  $\|(\lambda I - A)z\| \leq \frac{1}{\lambda}\|z\|$  for all  $z \in \text{Rg}(\lambda I - A)$
- (ii) If  $\lambda I - A$  is surjective for one  $\lambda > 0$ , then it is surjective for all  $\lambda > 0$  and  $(0, +\infty) \subset \rho(A)$ .

# Lumer–Phillips Theorem

## Theorem 12 (Lumer–Phillips)

*A dissipative operator  $A : D(A) \subset X \rightarrow X$  generates a contraction  $C_0$ -semigroup on  $X$  if and only if  $\lambda I - A$  is surjective for some/all  $\lambda > 0$ .*

## Proposition 13

Let  $A : D(A) \subset X \rightarrow X$  be a densely defined closed linear operator. Assume that for all  $x \in D(A)$  there exists  $x^* \in X^*$  such that  $\langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2$  and

$$\Re \langle Ax, x^* \rangle \leq 0.$$

Then  $A$  is dissipative.

# Hilbert Space Case

## Proposition 14

Let  $H$  be a Hilbert space and  $A : D(A) \subset H \rightarrow H$  a densely defined closed linear operator such that

$$\Re \langle Ax, x \rangle \leq 0 \quad \forall x \in D(A). \quad (1)$$

Then  $A$  is dissipative.

# Hilbert Space Case

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Let  $H$  be a Hilbert space and  $A : D(A) \subset H \rightarrow H$  a densely defined closed linear operator such that

$$\Re \langle Ax, x \rangle \leq 0 \quad \forall x \in D(A). \quad (1)$$

Then  $A$  is dissipative.

## Corollary 15

Let  $H$  be a Hilbert space and  $A : D(A) \subset H \rightarrow H$  a densely defined closed linear operator satisfying (1). If  $\lambda I - A$  is surjective for some  $\lambda > 0$ , then  $A$  generates a contraction  $C_0$ -semigroup.

# 4. Self-adjoint semigroups on Hilbert Spaces

## Definition

We say that  $(A, D(A))$ , an operator on a Hilbert space  $X$  is **self-adjoint** if the following hold:

- (i)  $\langle Ax, y \rangle = \langle x, Ay \rangle$  for all  $x, y \in D(A)$  .
- (ii) For every  $y \in X$  the following holds: if there exists  $z \in X$  such that  $\langle Ax, y \rangle = \langle x, z \rangle$  for all  $x \in D(A)$ , then  $y \in D(A)$  and  $z = Ay$ .

If  $A$  satisfies (i), it is called **symmetric**.

## Proposition 16

1. *For the spectrum of a symmetric operator one of the following holds:  $\sigma(A) = \mathbb{C}$  or  $\sigma(A) = \{\lambda : \operatorname{Im} \lambda \geq 0\}$  or  $\sigma(A) = \{\lambda : \operatorname{Im} \lambda \leq 0\}$  or  $\sigma(A) \subset \mathbb{R}$ .*
2. *A symmetric operator is self-adjoint if and only if  $\sigma(A) \subset \mathbb{R}$  if and only if  $(\pm i - A)$  are surjective.*

Proof: Z. Schnaubelt, Lecture notes on spectral theory, Thm 4.7

## Theorem 17 (Spectral Theorem)

*Let  $(A, D(A))$  be a self-adjoint operator on a separable (complex) Hilbert space. Then there exist a measure space  $(Y, \Sigma, \mu)$  with a finite measure, a unitary operator  $U : X \rightarrow L^2(\Sigma, \mu)$ , and a measurable function  $m : Y \rightarrow \mathbb{R}$  such that  $UAU^{-1} = A_m$  where  $A_m$  is the multiplication operator on  $L^2(\Sigma, \mu)$  corresponding to  $m$ .*

Proof: Davies, Thm 2.5.2 or Reed and Simon, Thm VIII.4

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## Proposition 18

*If  $(A, D(A))$  has a compact resolvent, then*

- 1.  $R(\lambda, A)$  is compact for all  $\lambda \in \rho(A)$ ,*
- 2.  $\sigma(A)$  is countable. If it is infinite, then it consists of a sequence of eigenvalues  $\lambda_n$  with  $\lambda_n \rightarrow \infty$ .*

## Theorem 19 (Spectral Theorem II)

Let  $(A, D(A))$  be a self-adjoint operator with compact resolvent on a separable Hilbert space. Then there exists a sequence  $(\alpha_n) \in \ell^2(\mathbb{R})$  and a unitary operator  $U : X \rightarrow \ell^2$  such that

$$UAU^{-1} = (\alpha_n x_n), \quad D(UAU^{-1}) = \{(x_n) \in \ell^2 : (\alpha_n x_n) \in \ell^2\}.$$

In particular, there exists an orthonormal basis  $(\varphi_n)$  in  $X$  consisting of eigenvectors of  $(A, D(A))$  such that

$$Ax = \sum_{k=1}^{\infty} \lambda_k \langle x, \varphi_k \rangle \varphi_k, \quad D(A) = \{x \in X : (\lambda_n \langle x, \varphi_n \rangle) \in \ell^2\}.$$

Proof: Davies, Thm 4.2.2 + Corollary

## Corollary 20

*Let  $(A, D(A))$  be a self-adjoint dissipative operator on a Hilbert space  $X$ . Then  $(A, D(A))$  generates a contraction  $C_0$ -semigroup  $T$ . Moreover,  $T(t)$  are self-adjoint and if  $A$  has compact resolvent then  $T(t)$  is compact for all  $t > 0$ .*

# 5. Regularity of semigroups

# Initial remarks

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- If a solution is differentiable for  $t = t_0$ , then it is differentiable for all  $t > t_0$  (since  $T(t_0)x_0 \in D(A)$  implies  $T(t)x_0 \in D(A)$  for all  $t > 0$ )

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- If  $x_0 \in D(A^n)$ , then the solution is  $n$ -times differentiable for all  $t \geq 0$ .

# Initial remarks

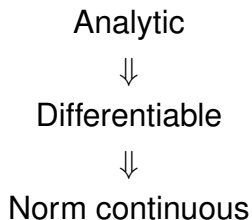
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- From PDE's: solution to transport equation preserves regularity of the initial value ... low regularity of the shift semigroup.

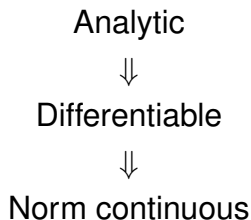
# Types of regularity

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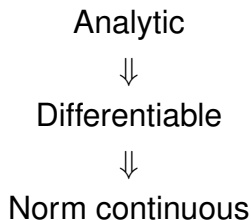
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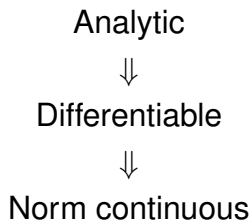


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**Differentiable** means that the mapping  $t \mapsto S(t)x$  is differentiable for all  $x \in X$  and all  $t > 0$  (strictly!).

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Let us concentrate on ANALYTIC SEMIGROUPS.

# Analytic semigroups

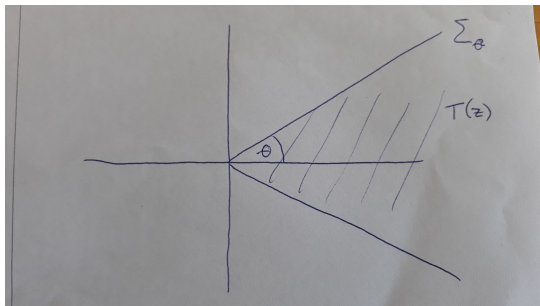
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# Analytic semigroups

## Definition

A  $C_0$ -semigroup  $T$  is called analytic, if there exists  $\theta \in (0, \pi/2]$  s.t.  $T$  has an analytic extension  $\tilde{T} : \Sigma_\theta \rightarrow (X)$  satisfying

1.  $\tilde{T}(z + w) = \tilde{T}(z)\tilde{T}(w)$  for all  $z, w \in \Sigma_\theta$
2.  $z \mapsto T(z)$  is analytic in  $\Sigma_\theta \setminus \{0\}$
3.  $\lim_{\Sigma_{\theta'} \ni z \rightarrow 0} T(z)x = x$  for all  $x \in X$ ,  $\theta' \in (0, \theta)$ .

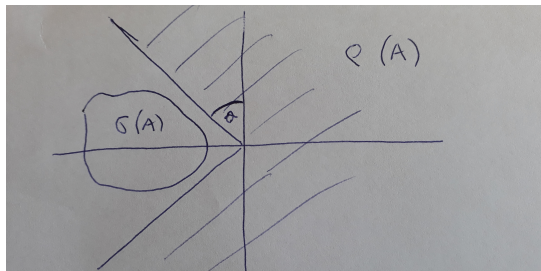
An analytic semigroup is called *bounded analytic semigroup* if it is bounded on each  $\Sigma_{\theta'}$ ,  $\theta' \in (0, \theta)$ .

# Sectorial operators

## Definition

An operator  $(A, D(A))$  is called *sectorial* if there exists  $\delta \in (0, \frac{\pi}{2}]$  such that

- $\Sigma_{\frac{\pi}{2}+\delta} \setminus \{0\} \subset \rho(A)$
- for every  $\varepsilon \in (0, \delta)$  there exists  $M_\varepsilon$ ,  $\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda|}$  for all  $\lambda \in \overline{\Sigma_{\frac{\pi}{2}+\delta-\varepsilon}} \setminus \{0\}$



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## Remark

The resolvent estimate holds with  $|\lambda|$  instead of  $\Re \lambda$ .

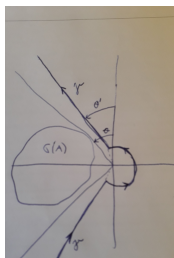
# Generator of an analytic semigroup

## Theorem 21

Let  $(A, D(A))$  be a  $\theta$ -sectorial operator. Then it generates a bounded analytic semigroup on  $\Sigma_\theta$  given by

$$T(z) = \frac{1}{2\pi i} \int_\gamma e^{\mu z} R(\mu, A) d\mu$$

where  $\gamma$  is as follows with  $\theta > \theta' > |\arg z|$ .



# Characterization of analytic semigroups

## Theorem 22

*Let  $(A, D(A))$  is the generator of a  $C_0$ -semigroup  $T$ . Then the following assertions are equivalent*

- *$A$  is  $\theta$ -sectorial*
- *$T$  is a bounded analytic semigroup on  $\Sigma_\theta$ .*
- *$T(t)X \subset D(A)$  and  $\{\|tAT(t)\| : t \in (0, 1]\}$  is bounded.*
- *$e^{\pm i\theta'} A$  generate bounded  $C_0$ -semigroups  $T(e^{\pm i\theta'} t)$  for all  $\theta' \in (0, \theta)$ .*

# Multiplicative sectorial operators

## Proposition 23

*A multiplicative operator  $A_m$  on  $L^2(\Omega)$  is  $\theta$ -sectorial if and only if the essential range of  $m$  is contained in*

$$\mathbb{C} \setminus \Sigma_{\frac{\pi}{2} + \theta} \cup \{0\}.$$

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## Corollary 24

*Every self-adjoint dissipative operator on a Hilbert space generates a bounded analytic semigroup on  $\Sigma_{\frac{\pi}{2}}$ .*

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*Every self-adjoint dissipative operator on a Hilbert space generates a bounded analytic semigroup on  $\Sigma_{\frac{\pi}{2}}$ .*

## Example

Dirichlet Laplacian on  $L^2(\Omega)$ ,  $\Omega$  bounded domain in  $\mathbb{R}^n$  is self-adjoint and dissipative. Therefore, the heat semigroup is analytic and solutions to the corresponding heat equation are analytic for  $t > 0$ .

# 6. Asymptotic behavior

# Spectral bound and growth bound

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Let  $(A, D(A))$  be the generator of a  $C_0$ -semigroup  $T$ .

- The **spectral bound** of  $A$  is

$$s(A) = \sup\{\Re\lambda : \lambda \in \sigma(A)\}.$$

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- The **spectral bound** of  $A$  is

$$s(A) = \sup\{\Re\lambda : \lambda \in \sigma(A)\}.$$

- The **growth bound** of  $T$  is

$$\omega(T) = \inf\{\omega \in \mathbb{R} : \exists M \geq 1 \ \|T(t)\| \leq Me^{\omega t} \ \forall t \geq 0\}.$$

# Does $s(A) = \omega(T)$ hold?

## Remark

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2. For a matrix  $A$  we have  $s(A) = \omega(e^{tA})$ . In fact,

$$\|e^{tA}\| \leq Me^{ct}(1+t)^{n-1} \leq \tilde{M}e^{(c+\varepsilon)t} \quad \forall t \geq 0$$

where  $c = \max\{\Re \lambda : \lambda \in \sigma(A)\}$  and  $n$  is the size of the biggest Jordan cell of  $A$ .

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1. By Proposition 4 (Chapter 1) we have  $s(A) \leq \omega(T)$ .
2. For a matrix  $A$  we have  $s(A) = \omega(e^{tA})$ . In fact,

$$\|e^{tA}\| \leq Me^{ct}(1+t)^{n-1} \leq \tilde{M}e^{(c+\varepsilon)t} \quad \forall t \geq 0$$

where  $c = \max\{\Re \lambda : \lambda \in \sigma(A)\}$  and  $n$  is the size of the biggest Jordan cell of  $A$ .

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**NO!**

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$$s(A) = \omega(T)$$

hold in general?

**NO!**

Two counterexamples can be found in [EN], Chapter IV.3.

# Does $s(A) = \omega(T)$ hold?

Split the equality  $s(A) = \omega(T)$  into two parts:

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$$\|T(t)\| \stackrel{(1)}{=} r(T(t)) = \sup_{\lambda \in \sigma(T(t))} |\lambda| = \sup_{e^{\lambda t} \in \sigma(T(t))} |e^{\lambda t}| \stackrel{(2)}{=} \sup_{\lambda \in \sigma(A)} |e^{\lambda t}|$$

## Proposition 25

Let  $T$  be a  $C_0$ -semigroup. Then  $\omega(T) = \frac{1}{t} \ln r(T(t))$  for each  $t \geq 0$

Proof: see [EN], Proposition IV.2.2

# Spectral inclusion

## Proposition 26

*Let  $T$  be a  $C_0$ -semigroup and  $A$  its generator. Then  $e^{t\sigma(A)} \subset \sigma(T(t))$ .*

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The other inclusion is more important.

# decomposition of spectrum

## Definition

Let  $(A, D(A))$  be a closed densely defined operator and  $\lambda \in \sigma(A)$ . We say that  $\lambda$  belongs to

- **the point spectrum**  $P\sigma(A)$  if  $\lambda - A$  is not injective,

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2. It can be shown that  $R_\sigma(A) = P_\sigma(A')$ , where  $A'$  is the adjoint of  $A$ .
3.  $\lambda \in A_\sigma(A)$  if and only if there exists *an approximate eigenvector*  $(x_n)_{n=1}^\infty \subset D(A)$  such that  $\|x_n\| = 1$  and  $\lim_{n \rightarrow \infty} (\lambda - A)x_n = 0$ .

# Spectral mapping theorem for point and residual spectrum

## Theorem 27

Let  $(A, D(A))$  be the generator of a  $C_0$ -semigroup  $T$ . Then  $P_\sigma(T(t)) \setminus \{0\} = e^{tP_\sigma(A)}$  and  $R_\sigma(T(t)) \setminus \{0\} = e^{tR_\sigma(A)}$ .

Proof can be found in [EN], Theorem IV.3.7.

# Spectral mapping theorem for point and residual spectrum

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Proof can be found in [EN], Theorem IV.3.7.

## Proposition 28

Let  $\mu \in \sigma(T(t))$  and  $\Lambda = \{\lambda \in \sigma(A) : e^{\lambda t} = \mu\}$ . Then

$$\text{Ker}(\mu - T(t)) = \overline{\text{lin}} \bigcup_{\lambda \in \Lambda} \text{Ker}(\lambda - A).$$

# Spectral mapping theorem for regular semigroups

## Theorem 29

*Let  $(A, D(A))$  be the generator of a  $C_0$ -semigroup  $T$  which is norm continuous for  $t > t_0$ . Then  $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$ .*

Proof can be found in [EN], Theorem IV.3.10

## Corollary 30

*Let  $(A, D(A))$  be the generator of an analytic semigroup. Then  $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$ .*

# 7. Regularity for non-homogeneous equations

# Motivation

Non-homogeneous Abstract Cauchy Problem

$$\dot{u} = Au + f(t), \quad u(0) = u_0. \quad (\text{nACP})$$

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Finite-dimensional case:  $u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s) ds$ .

## What is it useful for?

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- It allows for solving non-linear equations.
- It allows for solving perturbed equations  $\dot{u} = (A + B)u$ .

# Definitions of Solutions (nACP)

## Definition

- **Classical Solution** is a function  $u \in C([0, T], X)$ , s.t.  
 $\dot{u} \in L^1_{loc}([0, T], X)$ ,  $u(t) = u_0 + \int_0^t \dot{u}(s) ds$  for all  
 $t \in [0, T)$ , (nACP) holds for all  $t \in (0, T)$ .

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ATTENTION: Lower regularity of classical solution.

# Relationship Between Solutions

## Proposition 31

*Every classical solution is a strong solution; every strong solution is a mild solution.*

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## Proposition 32

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## Proposition 33

*Let  $f \in L^1_{loc}([0, T], X)$ . Then there exists a mild solution of (nACP).*

# Regularity Conditions

## Theorem 34

*Let  $u$  be a mild solution. Then  $u$  is strong if and only if  $u(t) \in D(A)$  a.e. and  $Au \in L^1_{loc}$ .*

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## Proposition 35

*Let  $X$  be a reflexive Banach space or a Hilbert space, and let  $f \in \text{Lip}([0, T], X)$ . Then*

$$v(t) = \int_0^t e^{(t-s)A} f(s) ds$$

*is a strong solution of (nACP) for  $u_0 = 0$ .*

# Corollaries for Strong Solutions

## Corollary 36

*Let  $X$  be a reflexive Banach or Hilbert space,  $f \in Lip_{loc}([0, T], X)$ , and either  $u_0 \in D(A)$  or  $e^{tA}$  is a differentiable semigroup. Then there exists a strong solution (given by the variation of constants formula).*

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