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Data Envelopment Analysis within Evaluation of the Efficiency of Firm Productivity

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ROBUST 2018
Rybník, January 22, 2018

- *productivity growth*
  - output variations that are not explained by input variations
  - traditional approach: neglect inefficiencies in input/output usage
  - (total) productivity growth = shift in technologies
- Farrell, 1957
  - idea to measure productivity efficiency using all inputs (not only a selected one)
  - technical efficiency = multiplicative inverse of the Malmquist (1953) and Shephard (1957) input *distance function*
- Charnes, Cooper, Rhodes (1978)
  - successful attempt to compute productivity efficiency using linear optimization model
  - nonparametric approach
  - *data envelopment analysis* (DEA): the efficiency frontier made up as the boundary of a convex hull of the data points
  - different extensions to the model adopted

- DMU_k ... k-th decision making unit ($k = 1, \dots, K$)
- $X := (x_{ik}) \in \mathbb{R}^{m \times K}$ ... input matrix
  - $x_{\cdot k} := (x_{1k}, \dots, x_{mk})$ ... input vector of DMU_k
  - $x_{i\cdot} := (x_{i1}, \dots, x_{iK})$ ... values for i-th input ($i = 1, \dots, m$)
- $Y := (y_{jk}) \in \mathbb{R}^{n \times K}$ ... output matrix
  - $y_{\cdot k} := (y_{1k}, \dots, y_{nk})$ ... output vector of DMU_k
  - $y_{j\cdot} := (y_{j1}, \dots, y_{jK})$ ... values for j-th output ($j = 1, \dots, n$)
- PPS ... production possibility set – combination of allowed inputs and outputs
- DMU_0 with $(x_{\cdot 0}, y_{\cdot 0})$ ... DMU to be analyzed

Definition 1

DMU_1 dominates DMU_2 wrt. PPS if $x \leq x_{\cdot 0}$ and $y \geq y_{\cdot 0}$ with at least one (one-dimensional, input or output) inequality strict

Definition 2

DMU_0 is efficient wrt. PPS if $\nexists (x, y) \in \text{PPS}$ dominating $(x_{\cdot 0}, y_{\cdot 0})$.

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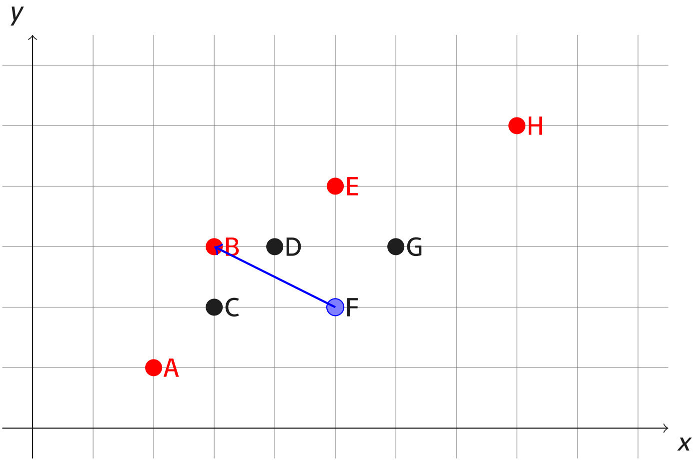
Discrete PPS (Bowlin, Brennan et al, 1984): $\text{PPS}_I = \{(x_k, y_k)\}_{k=1}^{K}$

Dominance wrt. PPS_I: additive model with integer constraints

$$
\begin{aligned}
&\max\left(\sum_j s_j^+ + \sum_i s_i^-\right) \text{ subject to} \\
&\qquad \sum_k x_{ik}\lambda_k + s_i^- = x_{i0} \qquad \forall i \quad \text{(inputs)} \\
&\qquad \sum_k y_{ik}\lambda_k - s_i^+ = y_{j0} \qquad \forall j \quad \text{(outputs)} \\
&\sum_k \lambda_k = 1,\ \lambda_k \in \{0,1\}^K,\ s_i^-, s_j^+ \geq 0
\end{aligned}
\tag{1}
$$

- s^- ... slack for $X\lambda \leq x_0$ // s^+ ... slack (surplus) for $Y\lambda \geq y_0$
- DMU_0 is efficient wrt. PPS_I if no slack is greater than 0 (i. e., both inequalities are active) in optimal solution



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Continuous (convex) PPS (Banker, Cooper, Charnes, 1984):

$$\mathrm{PPS}_C = \{(x,y) \mid x = X\lambda, y = Y\lambda, \sum \lambda_k = 1, \lambda \geq 0\}$$

Dominance wrt. PPS_C: BCC-I model

$$\min \theta + \epsilon\left(\sum_j s_j^+ + \sum_i s_i^-\right) \text{ subject to}$$
$$\sum_k x_{ik}\lambda_k + s_i^- = \theta x_{i0} \qquad \forall i \quad \text{(inputs)}$$
$$\sum_k y_{ik}\lambda_k - s_i^+ = y_{j0} \qquad \forall j \quad \text{(outputs)} \tag{2}$$
$$\sum_k \lambda_k = 1,\ \lambda_k \geq 0, s_i^-, s_j^+ \geq 0, \theta \text{ unconstrained}$$

ϵ ... non-Archimedean infinitesimal

Dual problem:

$$\begin{aligned} \max v^T y_{.0} + q \text{ subject to} & \\ -u^T x_{.k} + v^T y_{.k} + q \leq 0 & \quad \forall k \quad \text{(DMUs)} \\ u^T x_{.0} = 1 & \quad \text{(dual for } \theta\text{)} \\ u \geq \epsilon \mathbf{1},\ v \geq \epsilon \mathbf{1},\ q \text{ unconstrained} & \end{aligned} \tag{3}$$

q (dual for $\sum_k \lambda_k = 1$) ... variable returns to scale (VRS) factor

BCC-I DEA problem of fractional programming:

$$\begin{aligned} \max \frac{v^T y_{.0} + q}{u^T x_{.0}} & \text{ subject to} \\ \frac{v^T y_{.k} + q}{u^T x_{.k}} \leq 1 & \quad \forall k \quad \text{(DMUs)} \\ u^T x_{.0} = 1, u/u^T x_{.0} \geq \epsilon \mathbf{1},\ v/u^T x_{.0} \geq \epsilon \mathbf{1},\ q \text{ unconstrained} & \end{aligned} \tag{4}$$

Dual problem:

$$\begin{aligned} \max v^T y_{\cdot 0} + q \text{ subject to} & \\ -u^T x_{\cdot k} + v^T y_{\cdot k} + q \leq 0 & \quad \forall k \quad \text{(DMUs)} \\ u^T x_{\cdot 0} = 1 & \quad \text{(dual for } \theta) \\ u \geq \epsilon \mathbf{1},\ v \geq \epsilon \mathbf{1},\ q \text{ unconstrained} & \end{aligned} \tag{3}$$

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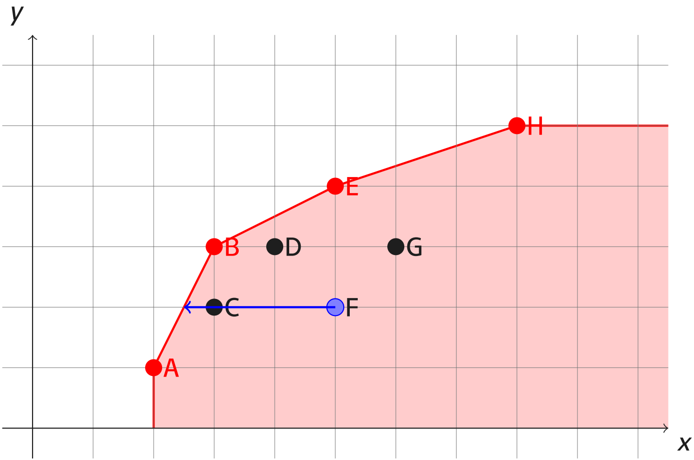
Definition 3 (DEA Efficiency)

DMU_0 is BCC-I (fully) efficient wrt. PPS_C if

1. $\theta^* = 1$
2. $s^{+*} = s^{-*} = 0$

Remark

- weak DEA efficiency: $\theta^* = 1$ but some of s_i^{-*}, s_j^{+*} are not zero (efficient points which are not extreme points of PPS)
- two-stage solution procedure:
  1. solve the BCC-I problem with $\epsilon = 0$ to obtain θ^*
  2. solve the problem $\max \sum_j s_j^+ + \sum_i s_i^-$ subject to remaining constraints where $\epsilon = 0$ and $\theta = \theta^*$ to obtain maximal possible slacks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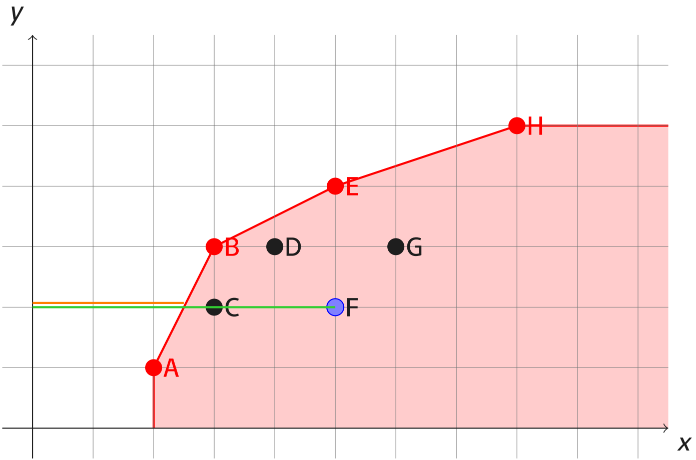

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Linear PPS (Charnes, Cooper, Rhodes (1978)):

$$\mathrm{PPS}_L = \left\{(x, y) \mid x = X\lambda, y = Y\lambda, \lambda \geq 0\right\}$$

Dominance wrt. PPS_L: CCR-I model

$$\begin{aligned} \min \theta + \epsilon\Big(\sum_j s_j^+ + \sum_i s_i^-\Big) \text{ subject to} \\ \sum_k x_{ik}\lambda_k + s_i^- &= \theta x_{i0} \quad \forall i \quad \text{(inputs)} \\ \sum_k y_{ik}\lambda_k - s_i^+ &= y_{j0} \quad \forall j \quad \text{(outputs)} \\ \lambda_k \geq 0, s_i^-, s_j^+ \geq 0, \theta \text{ unconstrained} \end{aligned} \tag{5}$$

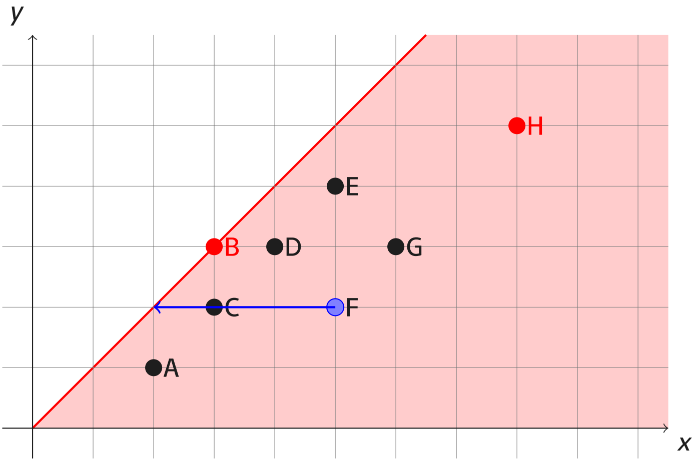
Dual problem:

$$\begin{aligned} \max v^T y_{\cdot 0} \text{ subject to} & \\ -u^T x_{\cdot k} + v^T y_{\cdot k} \leq 0 & \quad \forall k \quad \text{(DMUs)} \\ u^T x_{\cdot 0} = 1 & \quad \text{dual for } \theta) \\ u \geq \epsilon \mathbf{1}, \ v \geq \epsilon \mathbf{1} & \end{aligned} \tag{6}$$

$q = 0$ ... constant returns to scale (CRS)

CCR-I DEA problem of fractional programming:

$$\begin{aligned} \max \frac{v^T y_{\cdot 0}}{u^T x_{\cdot 0}} \text{ subject to} & \\ \frac{v^T y_{\cdot k}}{u^T x_{\cdot k}} \geq 1 & \quad \forall k \quad \text{(DMUs)} \\ u^T x_{\cdot 0} = 1, u/u^T x_{\cdot 0} \geq \epsilon \mathbf{1}, \ v/u^T x_{\cdot 0} \geq \epsilon \mathbf{1} & \end{aligned} \tag{7}$$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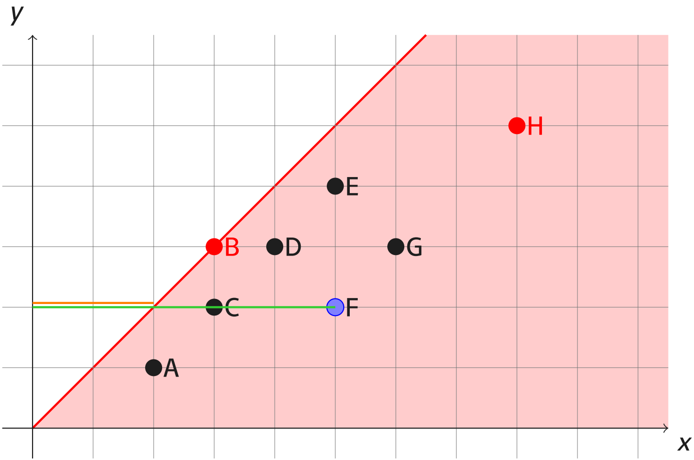

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Directional Distance Models (Chambers, Chung, Färe (1996, 1998)):

- dealing with negative data
- g^x, g^y ... vectors of improvement directions

Generic Directional Distance Model (wrt. PPS_C):

$$\begin{aligned} &\max \beta \text{ subject to} \\ &\quad \sum_k x_{ik}\lambda_k \le x_{i0} - \beta g_i^x \qquad \forall i \quad \text{(inputs)} \\ &\quad \sum_k y_{ik}\lambda_k \ge y_{j0} + \beta g_j^y \qquad \forall j \quad \text{(outputs)} \\ &\quad \sum_k \lambda_k = 1,\ \lambda_k \ge 0,\ \beta_0 \ge 0 \end{aligned} \tag{8}$$

- efficiency of DMU_0: $\beta^* = 0$
- special case: $g^x = x_0$, $g^y = 0$, $\theta = 1 - \beta$: BCC-I case

Range Directional Model

- range of possible improvements:

$$g_i^x = x_{i0} - \min_k x_{ik}$$
$$g_j^y = \max_k y_{jk} - y_{j0}$$

$I = (\min_k x_{\cdot k}, \max_k y_{\cdot k})$ ... ideal point

$$\max \beta \text{ subject to}$$
$$\sum_k x_{ik}\lambda_k \leq (1-\beta)x_{i0} + \beta \min_k x_{ik} \quad \forall i \quad \text{(inputs)}$$
$$\sum_k y_{ik}\lambda_k \geq (1-\beta)y_{j0} + \beta \min_k y_{jk} \quad \forall j \quad \text{(outputs)} \tag{9}$$
$$\sum_k \lambda_k = 1, \ \lambda_k \geq 0, \ \beta_0 \geq 0$$

Production possibility sets (available technology):

$$\text{PPS}_L := \{(x,y) \mid x = X\lambda, y = Y\lambda, \lambda \geq 0\}$$

$$\text{PPS}_C := \{(x,y) \mid x = X\lambda, y = Y\lambda, \sum \lambda_k = 1, \lambda \geq 0\}$$

$$\text{PPS}_C^s := \{(x,y) \mid x = X\lambda + s^+, y = Y\lambda - s^-, \sum \lambda_k = 1, \lambda \geq 0, s^+, s^- \geq 0\}$$

$$\text{PPS}_I := \{(x,y) \mid x = X\lambda, y = Y\lambda, \sum \lambda_k = 1, \lambda \in \{0,1\}^K\}$$

$$\text{PPS} := \{(x,y) \mid y \text{ can be produced from } x\} \qquad \text{(general PPS)}$$

Desirable properties for PPS

1. convexity: if $(x_{\cdot k}, y_{\cdot k}) \in$ PPS and $\lambda \geq 0, \sum \lambda_k = 1$ then $(X\lambda, Y\lambda) \in$ PPS
2. free (strong) disposability of inputs and outputs:
    1. if $(x,y) \in$ PPS and $x^+ := x + s^+$ with $s^+ \geq 0$ then $(x^+, y) \in$ PPS
    2. if $(x,y) \in$ PPS and $y^- := y - s^-$ with $s^- \geq 0$ then $(x, y^-) \in$ PPS
3. minimum intersection: PPS is the intersection of all sets $\widehat{\text{PPS}}$ satisfying properties 1 and 2, subject to $(x,y) \in \widehat{\text{PPS}}$

Production possibility sets (available technology):

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$$\text{PPS} := \{(x,y) \mid y \text{ can be produced from } x\} \qquad \text{(general PPS)}$$

Additional desirable properties for PPS

4. no free lunch: if $(0, y) \in$ PPS then $y = 0$
5. no infinite outputs: $A(x) := \{(u, y) \mid u \leq x\}$ is bounded $\forall x$
6. closeness: PPS is closed (technical property)

Usual assumption (may be eliminated by some extensions)

7. no negative inputs and outputs

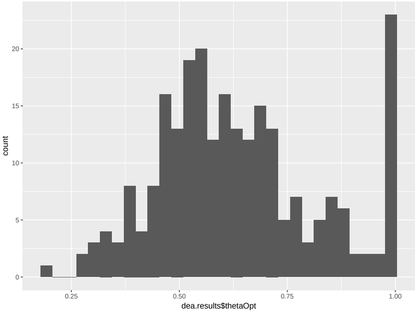
- Data: annual accounts of 380 Czech companies from the food industry (NACE C.10) [selected year: 2014]
- Implementation:
  - grouping the companies (according to the EC classification of economic activities)
  - choosing appropriate inputs and outputs to be analysed
  - choosing the model (returns to scale)
  - computer implementation
- Issues:
  - missing or implausible data
  - negative inputs/outputs

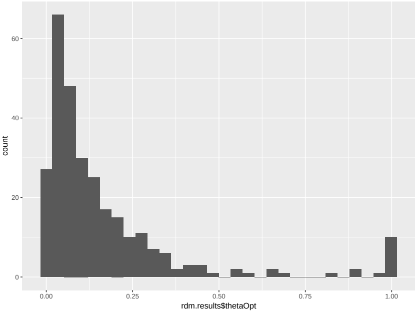
- Companies:
  - the whole group C.10 (Manufacture of food products)
- Inputs
  - `SPMAAEN`: material and energy consumption – 89 companies with no costs reported
  - `ON`: personnel costs
  - `STALAA`: fixed assets (buildings, equipments)
  - `POSN`: percentage of the personnel costs
- Outputs
  - `VYKONY`: business performance
  - `ROA`: return on assets (earning before interest and taxes per total assets) – 70 companies having negative ROA
- Models and feasible solutions:
  - input-oriented with variable returns to scale: 244 companies
  - range directional model: 291 companies

Considered alternatives (not in today's presentation):

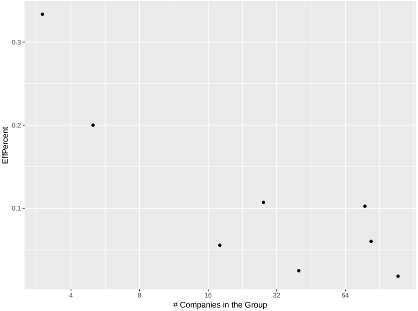
- Groups of companies:
  - manufacture and processing of meat (78) / fish (3) /fruit and vegetables (18) / oils and fats (5) / dairy produts (28) / grain mills products (15) / bakery and farinaceous products (109) / other food products (83) / prepared animal feeds (40)
- Inputs:
  - production consumption / depreciations / tangible and intangible fixed assets / cost of capital
- Outputs:
  - value of sales of goods and services / operating income / EBIT (earnings before interest and taxes) / value added
- Models: Cooper, Seiford, Tone (2007), Cooper, Seiford, Zhoe (2011):
  - input/output oriented CRS/VRS DEA models with discretionary/non-discretionary ou
  - alternative DEA models: additive (translation invariant), slack-based, Russel, free disposal hull, other directional distance

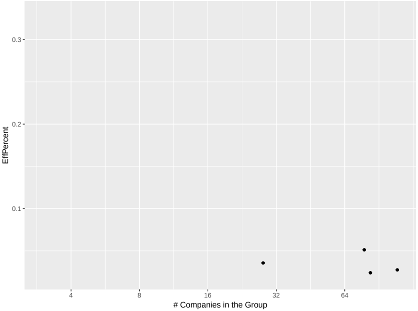
- BCC: 22 efficient companies (additional 3 with efficiency $> 95\,\%$)
- RDM: 10 efficient companies



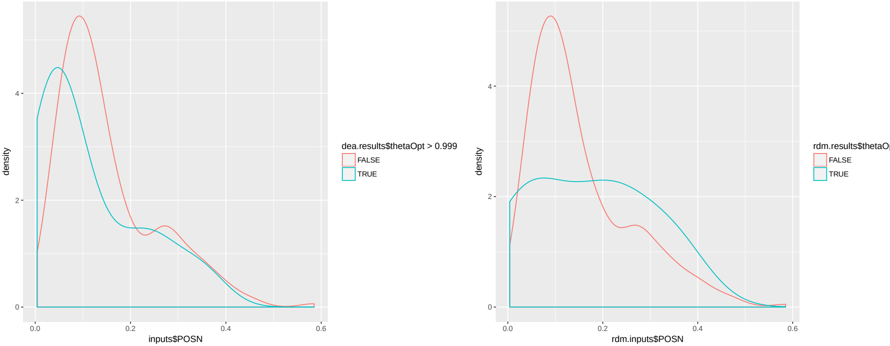


- BCC: from the groups, manufacture and processing of meat (8 of 78), manufacture of other food products (5 of 83)

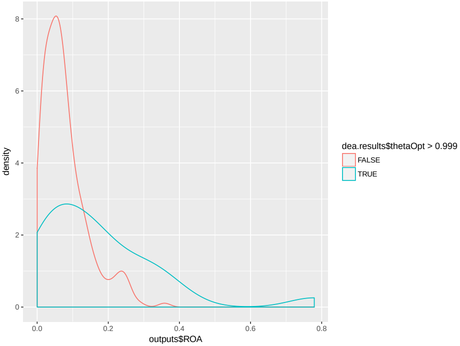


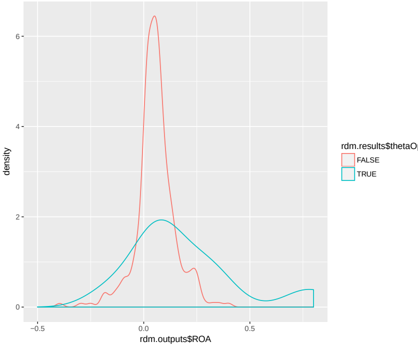


- distribution of a selected input for efficient and inefficient companies

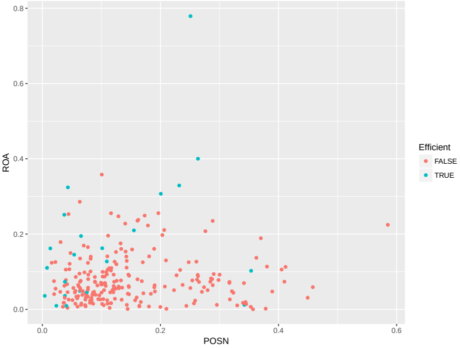


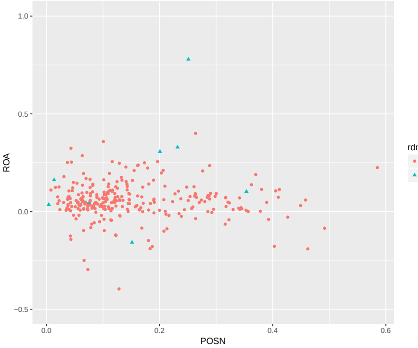
- distribution of a selected output for efficient and inefficient companies





- distribution of a selected input and a selected output for efficient and inefficient companies





- Caves, Christensen, Diewert (1982)
  - input-based Malmquist productivity index defined as the ratio of two input distance functions (optimal values of DEA problems)
- Färe, Grosskopf, Lindgren, Roos (1992)
  - introducing dynamics: the Malmquist index defined as the geometric mean of two indexes in Caves et al.'s sense (four DEA problems computed)
  - the index can be decomposed into two components: an *efficiency change* (the ratio of the technical efficiencies in two time periods), and a *technical change* (the shift of the frontier between two time periods)
  - input-oriented DEA model with CRS used to calculate the input distances
- further studies: using different distance functions / DEA models to calculate the index, e. g.
  - Chung, Färe, Grosskopf (1997), Oh (2010): (local and global) Malmquist-Luenberger index (using directional distance)
  - Asmild, Baležentis, Hougaard (2016): multi-directional efficiency
  - Boussemart, Briec et al. (2009): (generalized) α-returns to scale

- define an *input (output) distance function* $D(x_0, y_0)$ for the DMU_0 as the inverse of the optimal value of the input (output) based DEA problem under technology PPS:

$$D(x_0, y_0) := \frac{1}{\theta^*}$$

- take these distance for two different time periods t, $t+1$
- define the *Malmquist index* as

$$M_t^{t+1} := \sqrt{\frac{D^t(x_0^{t+1}, y_0^{t+1})}{D^t(x_0^t, y_0^t)} \cdot \frac{D^{t+1}(x_0^{t+1}, y_0^{t+1})}{D^{t+1}(x^t, y^t)}}$$

- two components of the Malmquist index:

$$M_t^{t+1} := EC_t^{t+1} \cdot TC_t^{t+1}$$

  - efficiency change $EC_t^{t+1} := \dfrac{D^t(x_0^{t+1}, y_0^{t+1})}{D^t(x_0^t, y_0^t)}$
  - technology change $TC_t^{t+1} := \sqrt{\dfrac{D^t(x_0^{t+1}, y_0^{t+1})}{D^{t+1}(x_0^{t+1}, y_0^{t+1})} \cdot \dfrac{D^t(x^t, y^t)}{D^{t+1}(x^t, y^t)}}$

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