Parameter estimation for the stochastic equation of second order

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Introduction

- Consider the following wave equation

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(t,\xi) &= bAu(t,\xi) - 2a\frac{\partial u}{\partial t}(t,\xi) + Q^{\frac{1}{2}}\dot{B}(t,\xi),\ (t,\xi) \in \mathbb{R}_+ \times D,\\
u(0,\xi) &= u_1(\xi), \quad \xi \in D,\\
\frac{\partial u}{\partial t}(0,\xi) &= u_2(\xi), \quad \xi \in D,\\
u(t,\xi) &= 0, \quad (t,\xi) \in \mathbb{R}_+ \times \partial D,
\end{aligned}
$$

  where $D \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary, $a > 0$, $b > 0$ are unknown parameters and the $\dot{B}(t,\xi)$ is the formal time derivative of the Brownian motion.

- Based on the observation of trajectory of process $\{X_t = (u(t,\cdot), \frac{\partial u}{\partial t}(t,\cdot))^\top, 0 \le t \le T\}$, the strong consistent estimators of parameters a and b will be proposed.

Assumptions

- Assume, that $\{e_n,\ n \in \mathbb{N}\}$ is the orthonormal basis in $L^2(D)$ and the operator $A : L^2(D) \to L^2(D)$ is such that
  (i) $Ae_n = -\alpha_n e_n$,
  (ii) $\exists \varepsilon > 0\, \forall n \in \mathbb{N} \quad \alpha_n > \varepsilon$,
  (iii) $\alpha_n \to \infty$.
- These assumptions cover the case, that if the set $D \subset \mathbb{R}^d$ is open, bounded and with a smooth boundary, then the operator $A = \Delta|_{\mathrm{Dom}(A)}$ and $\mathrm{Dom}(A) = H^2(D) \cap H_0^1(D)$.

Assumptions

- Assume, that the operator Q is positive nuclear operator in $L^2(D)$ with eigenvalues $\{\lambda_n,\ n \in \mathbb{N}\}$, i.e.
  (iv) $Qe_n = \lambda_n e_n$,
  (v) $\forall n \in \mathbb{N} \quad \lambda_n > 0$,
  (vi) $\sum_{n=1}^{\infty} \lambda_n < \infty$.
- We consider the diagonal case. That means, that the eigenvectors $\{e_n, n \in \mathbb{N}\}$ of the operator Q are the same as the eigenvectors of the operator A.

General setting

- This problem may be rewritten as an infinite dimensional stochastic differential equation

$$dX_t = \mathcal{A} X_t \, dt + \Phi \, dB_t, \tag{1}$$
$$X_0 = x_0 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

- To this aim, introduce the Hilbert space $\mathcal{H} = \mathrm{Dom}((-A)^{\frac{1}{2}}) \times L^2(D)$ endowed with the norm

$$\begin{aligned} \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_{\mathcal{H}}^2 &= \|x_1\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})} + \|x_2\|^2_{L^2(D)} \\ &= \left\| (-A)^{\frac{1}{2}} x_1 \right\|^2_{L^2(D)} + \|x_2\|^2_{L^2(D)}. \end{aligned} \tag{2}$$

General setting

- Define the linear operator $\mathcal{A}$:

$$\mathcal{A}x = \mathcal{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ bA & -2aI \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$\forall\, x \in \mathrm{Dom}(\mathcal{A}) = \mathrm{Dom}(A) \times \mathrm{Dom}((-A)^{\frac{1}{2}}).$$

- Also define the linear operator Φ in $\mathcal{H}$ as follows

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & Q^{\frac{1}{2}} \end{pmatrix}.$$

Semigroup $S(t)$

- Assume, that

$$\forall\, n \in \mathbb{N} \quad a^2 - b\alpha_n < 0. \tag{3}$$

- Under this assumption, the eigenvalues $\{l_n,\ n \in \mathbb{N}\}$ of the operator $\mathcal{A}$ equal to

$$l_n^{1,2} = -a \pm i\sqrt{b\alpha_n - a^2}$$

and the operator $\mathcal{A}$ generates a C_0–semigroup in $\mathcal{H}$, which has the following form.

Semigroup $S(t)$

Lemma

For all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H}$, *the semigroup* $S(t)$ *equals to*

$$S(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s_{11}(t) & s_{12}(t) \\ s_{21}(t) & s_{22}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where

$$\begin{aligned} s_{11}(t) &= e^{-at}\left(\cos(\beta t) + a\beta^{-1}\sin(\beta t)\right), \\ s_{12}(t) &= e^{-at}\beta^{-1}\sin(\beta t), \\ s_{21}(t) &= e^{-at}\left(-\beta - a^2\beta^{-1}\right)\sin(\beta t), \\ s_{22}(t) &= e^{-at}\beta^{-1}\left(-a\sin(\beta t) + \beta\cos(\beta t)\right). \end{aligned}$$

Semigroup $S(t)$

- The operator $\beta : L^2(D) \to L^2(D)$ in the previous formulae is defined by $\beta = (-bA - a^2 I)^{\frac{1}{2}}$.
- All operators are defined by their respective series. For example

$$\beta x = \sum_{n=1}^{\infty} \sqrt{b\alpha_n - a^2}\,\langle x, e_n\rangle\, e_n,$$

$$\sin(\beta t)x = \sum_{n=1}^{\infty} \sin\left(\sqrt{b\alpha_n - a^2}\, t\right)\langle x, e_n\rangle\, e_n,$$

where $x \in L^2(D)$ are from their respective domains.

Covariance operator $Q_\infty^{(a,b)}$

- The equation (1) is a linear equation $\Rightarrow$ there exists a mild solution X_t.
- The semigroup $S(t)$ is exponentially stable $\Rightarrow$ there exists an invariant measure $\mu_\infty^{(a,b)}$, which fullfils $\mu_\infty^{(a,b)} = N\left(0, Q_\infty^{(a,b)}\right)$.
- The covariance operator $Q_\infty^{(a,b)}$ of the limit measure $\mu_\infty^{(a,b)}$ satisfies

$$Q_\infty^{(a,b)} = \int_0^\infty S(t)\Phi\Phi^* S^*(t)\,dt. \tag{4}$$

Covariance operator $Q_\infty^{(a,b)}$

- The computation yields

$$Q_\infty^{(a,b)} = \int_0^\infty \begin{pmatrix} q_{11}(t) & q_{12}(t) \\ q_{21}(t) & q_{22}(t) \end{pmatrix} dt, \\ = \begin{pmatrix} \frac{1}{4ab} Q(-A)^{-1} & 0 \\ 0 & \frac{1}{4a} Q \end{pmatrix}, \tag{5}$$

where

$$q_{11}(t) = e^{-2at} \beta^{-1} \sin(\beta t) Q \beta^{-1} \sin(\beta t),$$
$$q_{12}(t) = e^{-2at} \beta^{-1} \sin(\beta t) Q \beta^{-1} \left(-a \sin(\beta t) + \beta \cos(\beta t)\right),$$
$$q_{21}(t) = e^{-2at} \beta^{-1} \left(-a \sin(\beta t) + \beta \cos(\beta t)\right) Q \beta^{-1} \sin(\beta t),$$
$$q_{22}(t) = e^{-2at} \beta^{-1} \left(-a \sin(\beta t) + \beta \cos(\beta t)\right) Q \beta^{-1} \times \\ \times \left(-a \sin(\beta t) + \beta \cos(\beta t)\right).$$

Estimators of parameters

- According to [Maslowski, Pospíšil], some Birkohoff–type ergodic theorem may be applied. Namely

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \|X_t^{x_0}\|_{\mathcal{H}}^2 \, dt = \int_{\mathcal{H}} \|y\|_{\mathcal{H}}^2 \, d\mu_\infty^{(a,b)}(y) \tag{6}$$

$$= \mathrm{Tr}\ Q_\infty^{(a,b)}, \tag{7}$$

  for any initial contidion $x_0 \in \mathcal{H}$.

- From the expression (5), Tr $Q_\infty^{(a,b)}$ equals to

$$\mathrm{Tr}\ Q_\infty^{(a,b)} = \frac{1}{4ab} \mathrm{Tr}\ Q(-A)^{-1} + \frac{1}{4a} \mathrm{Tr}\ Q \tag{8}$$

$$= \frac{1}{4ab} \sum_{n=1}^{\infty} \frac{\lambda_n}{\alpha_n} + \frac{1}{4a} \sum_{n=1}^{\infty} \lambda_n. \tag{9}$$

Estimators of parameters

- If we denote $Y_T := \frac{1}{T}\int_0^T \|X_t^{x_0}\|_{\mathcal{H}}^2\, dt$, then (based on (7) and (8)) some strongly constistent estimators of parameters a and b may be proposed.
- If the true value of the parameter b is known, then the strongly consistent estimator of the parameter a is

$$\hat{a}_T = \frac{1}{4Y_T}\left(\frac{1}{b}\,\mathrm{Tr}\; Q(-A)^{-1} + \mathrm{Tr}\; Q\right). \qquad (10)$$

- If the true value of the parameter a is known, then the strongly consistent estimator of the parameter b is

$$\hat{b}_T = \frac{\mathrm{Tr}\; Q(-A)^{-1}}{4aY_T - \mathrm{Tr}\; Q}. \qquad (11)$$

What if $a^2 - b\alpha_n \geq 0$?

- If
$$\exists\, n \in \mathbb{N} \quad a^2 - b\alpha_n > 0, \tag{12}$$
or
$$\exists\, n \in \mathbb{N} \quad a^2 - b\alpha_n = 0, \tag{13}$$
then the eigenvalues of the operator $\mathcal{A}$ are different and the semigroup $S(t)$ has different forms. But the covariance operator $Q_\infty^{(a,b)}$ will remain the same.

Asymptotic normality

- Are the estimators asymptotic normal?
- $$\sqrt{T}\left(\hat{a}_T - a\right) \stackrel{?}{\to} Z_1 \sim N(0, V_1),$$
$$\sqrt{T}\left(\hat{b}_T - b\right) \stackrel{?}{\to} Z_2 \sim N(0, V_2).$$
- If so, we could make interval estimators very easily.
- We could test some hypotheses about the parameters.

Possible future extensions

- We could consider a non–diagonal case.
- We could consider another operator A.
- We could consider B_t^H instead of B_t.

References

- G. Da Prato, J. Zabczyk: *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications 45, Cambridge University Press, 2008.
- T. E. Duncan, B. Maslowski, B. Pasik–Duncan: *Linear stochastic equations in a Hilbert space with a fractional Brownian motion*, International Series in Operations Research & Management Science, Vol. 94, 201–222, Springer Verlag, 2006.
- B. Maslowski, J. Pospíšil: *Ergodicity and parameter estimates for infinite–dimensional fractional Ornstein–Uhlenbeck process*, Applied Mathematics and Optimization, Vol. 57, Issue 3, 401–429, 2007.