Robustní monitorování stability v modelu CAPM

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Contents

Introduction

CAPM (Capital Assets Pricing Model)

$$r_{tj} - r_{t,f} = \beta_j(r_{t,M} - r_{t,f}) + \varepsilon_{tj}, \; j = 1, \ldots, d, \; t = 1, \ldots N$$

r_{tj} -return of an asset j at time t
$r_{t,M}$ - return of the market portfolio at time t
$r_{t,f}$ - riskless security
$r_{tj} - r_{t,f}$ - access return (risk premium)
β_j - measure of risk of the asset j w.r.t. the market portfolio

reparametrized model

$$r_{tj} = \alpha_j + \beta_j r_{t,M} + \varepsilon_{tj}$$

Model with time varying betas:

$$r_{tj} = \alpha_{tj} + \beta_{tj} r_{t,M} + \varepsilon_{tj},\ j = 1, \ldots, d, t = 1, \ldots, N$$

or more generally,

$$\mathbf{r}_i = \boldsymbol{\alpha}_i + \boldsymbol{\beta}_i r_{i,M} + \boldsymbol{\varepsilon}_i, \quad i = 1, \ldots,$$

$\mathbf{r}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \boldsymbol{\varepsilon}_i$ are d-dimensional vectors

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Sequential monitoring:

We assume stable historical data of size m such that

$$\boldsymbol{\alpha}_1 = \ldots = \boldsymbol{\alpha}_m = \boldsymbol{\alpha}_0, \quad \boldsymbol{\beta}_1 = \ldots = \boldsymbol{\beta}_m = \boldsymbol{\beta}_0,$$

and with any new observation we want to decide whether the stability in parameters is violated or not.

Hypothesis testing problem:

$H_0 : \boldsymbol{\beta}_1 = \ldots = \boldsymbol{\beta}_0$
$H_1 : \boldsymbol{\beta}_0 = \boldsymbol{\beta}_1 = \ldots = \boldsymbol{\beta}_{m+k^*} \neq \boldsymbol{\beta}_{m+k^*+1} = \ldots$
where $k^* = k^*_m$ is an unknown change point.

The null hypothesis is rejected whenever for the first time

$$\widehat{Q}(k,m)/q_\gamma(k/m) \geq c_\alpha$$

where $\widehat{Q}$ is a test statistic, $q_\gamma(t)$, $t \in (0,\infty)$ is a boundary function and c_α is an appropriately chosen critical value.

The stopping rule:

$$\tau_m(\gamma) = \begin{cases} \inf\{1 \le k < mT + 1 : \widehat{Q}(k,m)/q_\gamma(k/m) \ge c_\alpha\}, \\ \infty, \ \text{if } \widehat{Q}(k,m)/q_\gamma(k/m) < c_\alpha \quad \forall\, 1 \le k < mT + 1. \end{cases}$$

(closed-end procedure)

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(closed-end procedure)

The constant $c = c_\alpha$ is chosen such that

$$\lim_{m\to\infty} P\big(\tau_m < \infty | H_0\big) \;=\; \alpha,$$

$$\lim_{m\to\infty} P\big(\tau_m < \infty | H_1\big) \;=\; 1,$$

Sequential monitoring in linear models

- Chu, Stinchcombe and White, 1996
- Horváth, Hušková, Kokoszka, Steinebach, 2004
- Aue, Hörmann, Horváth, Hušková, and Steinebach, 2011 for CAPM
- Koubková 2006 (monitoring, L1-norm)

Robust procedures in linear models:

- Wu (2007) robust estimators, dependent variables
- Koenker, Portnoy, 1990 - multivariate, independent
- Bai et al. 1990, 1992 - multivariate, independent
- Chan, Lakonischok (1992), Genton and Ronchetti (2008) - empirical studies in CAPM

Sequential robust monitoring in CAPM

$$r_{ij} = \alpha_j^0 + \beta_j^0 \widetilde{r}_{iM} + (\alpha_j^1 + \beta_j^1 \widetilde{r}_{iM}) I\{i > m + k^*\} + \varepsilon_{ij}, i = 1, 2, \ldots,$$

k^* - change point, $\alpha_j^0, \beta_j^0, \alpha_j^1, \beta_j^1$, $j = 1, \ldots, d$ unknown parameters

$$\widetilde{r}_{iM} = r_{i,M} - \overline{r}_M, \quad \overline{r}_M = \frac{1}{m} \sum_{i=1}^{m} r_{i,M}.$$

$M-$ estimators $\widehat{\alpha}_{jm}, \widehat{\beta}_{jm}$ of α_j^o, β_j^o, based on the training sample:

$$\min \sum_{i=1}^{m} \rho_j (r_{ij} - a_j - b_j \widetilde{r}_{iM})$$

ρ_j are convex loss functions with the derivatives $\psi_j, j = 1, \ldots, d$

M-residuals

$$\boldsymbol{\psi}(\widehat{\boldsymbol{\varepsilon}}_i) = (\psi_1(\widehat{\varepsilon}_{i1}), \ldots, \psi_d(\widehat{\varepsilon}_{id}))^T$$

with

$$\widehat{\varepsilon}_{ij} = r_{ij} - \widehat{\alpha}_{jm}(\psi) - \widetilde{r}_{iM} \widehat{\beta}_{jm}(\psi)$$

A test statistic based on first $m + k$ observations

$$\widehat{Q}(k,m) = \Big(\frac{1}{\sqrt{m}}\sum_{i=m+1}^{m+k}\widetilde{r}_{iM}\psi(\widehat{\varepsilon}_i)\Big)^T \mathbf{\Sigma}_m^{-1}\Big(\frac{1}{\sqrt{m}}\sum_{i=m+1}^{m+k}\widetilde{r}_{iM}\psi(\widehat{\varepsilon}_i)\Big),$$

where the matrix $\mathbf{\Sigma}_m$ is an estimator of the asymptotic variance

$$\lim_{m\to\infty} var\Big\{\frac{1}{\sqrt{m}}\sum_{i=1}^{m}(r_{i,M} - Er_{i,M})\psi(\varepsilon_i)\Big\}$$

based on the first m observations

Typical score functions $\psi(x) = \rho'(x)$

- $\psi(x) = x$, $x \in R^1$ $(\rho(x) = x^2)$ - least squares estimators and L_2 residuals
- $\psi(x) = \operatorname{sign} x$, $x \in R^1$ $(\rho(x) = |x|)$ - L_1 estimators and L_1 residuals
- Huber

$$\psi(x) = \begin{cases} |x| & |x| \leq K \\ K \operatorname{sign} x & |x| > K \end{cases}$$

for $x \in R^1$ and some $K > 0$

Assumptions on score functions ψ_j's

1. ψ_j are monotone functions,
2. functions $\lambda_j(t) = -\int \psi_j(x-t)dF_j(x),\ t \in \mathbb{R}$, satisfy $\lambda_j(0) = 0,\ \lambda_j'(0) > 0,\ \lambda_j'(t)$ exists in a neighborhood od 0 and is Lipschitz in neighborhood of 0 for $|t| \leq c_o$ for some $c_o > 0$.
3. $\int |\psi_j(t)|^{2+\Delta} dF_j(t) < \infty$ for some $\Delta > 0$ and

$$\int |\psi_j(x+t_2) - \psi_j(x+t_1)|^2 dF_j(x) \leq C_0 |t_2 - t_1|^{\kappa},$$

for some $1 \leq \kappa \leq 2,\ c_o > 0,\ C_0 > 0$

F_j distribution function of ε_{ij}

Assumptions on regressors

- For any $i \in \mathbb{Z}$, $r_{i,M} = h(\boldsymbol{\xi}_i, \boldsymbol{\xi}_{i-1}, \ldots)$, where h is measurable, $\{\boldsymbol{\xi}_i\}_i$ is a sequence of i.i.d. random vectors and $E|r_{0M}|^{2+\Delta} < \infty$ for some $\Delta > 0$.
- For all $i \in \mathbb{Z}$,

$$\sum_{L=1}^{\infty} \|r_{iM} - r_{iM}^{(L)}\|_2 < \infty$$

where

$$r_{iM}^{(L)} = h(\boldsymbol{\xi}_i, \boldsymbol{\xi}_{i-1}, \ldots \boldsymbol{\xi}_{i-L+1}, \boldsymbol{\xi}_{i-L}^{(L)}, \boldsymbol{\xi}_{i-L-1}^{(L)}, \ldots),$$

$\boldsymbol{\xi}_{i-L}^{(L)}, \boldsymbol{\xi}_{i-L-1}^{(L)}, \ldots$ are i.i.d. with the same distribution as $\boldsymbol{\xi}_i$ independent of $\{\boldsymbol{\xi}_i\}_i$

$\Rightarrow \{r_{i,M}^{(L)}\}$ is $L-$ dependent, $r_{i,M}^{(L)} \overset{\mathcal{D}}{=} r_{iM} \forall i \in \mathbb{Z}$.

Assumptions on errors

- For any $i \in \mathbb{Z}$, $\varepsilon_i = \mathbf{g}(\zeta_i, \zeta_{i-1}, \ldots)$, where $\mathbf{g}$ is measurable, $\{\zeta_i\}_i$ is sequence of i.i.d. random vectors
- For all $i \in \mathbb{Z}$,

$$\sum_{L=1}^{\infty} \|\psi(\varepsilon_i) - \psi(\varepsilon_i^{(L)})\|_2 < \infty$$

$$\sum_{L=1}^{\infty} \sup_{|\mathbf{a}| \le a_0} \|\psi(\varepsilon_i - \mathbf{a}) - \psi(\varepsilon_i^{(L)} - \mathbf{a})\|_2 < \infty$$

for some $a_0 > 0$, where

$$\varepsilon_i^{(L)} = \mathbf{g}(\zeta_i, \zeta_{i-1}, \ldots \zeta_{i-L+1}, \zeta_{i-L}^{(L)}, \zeta_{i-L-1}^{(L)}, \ldots)$$

$\zeta_{i-L}^{(L)}, \zeta_{i-L-1}^{(L)}, \ldots$ are i.i.d., independent of $\{\zeta_i\}_i$, with the same distribution as ζ_i.

Model under the null hypothesis

Theorem. Let the above assumptions be satisfied and

$$\boldsymbol{\Sigma}_m - \boldsymbol{\Sigma} = o_P(1) \; as \; m \to \infty.$$

Then under the null hypothesis as $m \to \infty$

$$\max_{1 \le k \le mT} \frac{\widehat{Q}(k,m)}{q_\gamma^2(k/m)} \xrightarrow{\mathcal{D}} \sup_{0<t<T/(T+1)} \frac{\sum_{j=1}^{d} W_j^2(t)}{t^{2\gamma}},$$

where $\{W_j(t),\ t \in (0,1)\}$, $j = 1, \dots, d$ are independent Brownian motions and

$$q_\gamma(t) = (1+t)\,(t/(t+1))^\gamma,\ t \in (0,\infty),\ \gamma \in [0, 1/2)$$

Critical values c_α satisfies

$$P\left(\sup_{0<t<T/(T+1)} \frac{\sum_{j=1}^{d} W_j^2(t)}{t^{2\gamma}} \geq c_\alpha\right) = \alpha.$$

The explicit form of the limit distribution is unknown

Model under local alternatives:

$$r_{ij} = \alpha_j^0 + \beta_j^0 \tilde{r}_{iM} + (\alpha_j^1 + \beta_j^1 \tilde{r}_{iM})\delta_m I\{i > m + k^*\} + \varepsilon_{ij}$$

$\delta_m \to 0$ and $k^* < Tm + 1$
Theorem (consistency): When $\delta_m \to 0$, $|\delta_m| m^{1/2} \to \infty$, $\beta_j^1 \neq 0$ for at least one j and $k^* = \lfloor ms \rfloor$, $0 < s < T$, as $m \to \infty$

$$max_{1 \leq k \leq mT} \frac{\widehat{Q}(k,m)}{q_\gamma(k/m)} \to \infty, \quad in \quad probability.$$

Estimator of asymptotic variance matrix $\boldsymbol{\Sigma}$

$$\boldsymbol{\Sigma} = \sum_{i=-\infty}^{\infty} E[(r_{0M} - Er_{0M})(r_{iM} - Er_{iM})\psi(\varepsilon_0)\psi(\varepsilon_i)^T]$$

Bartlett -type estimator

$$\boldsymbol{\Sigma}_m = \sum_{|k|\leq q} \omega_q(k)\widehat{\boldsymbol{\Gamma}}_k$$

$$\widehat{\boldsymbol{\Gamma}}_k = \begin{cases} \frac{1}{m}\sum_{j=1}^{m-k} \widetilde{r}_{jM}\widetilde{r}_{j+k,M}\psi(\widehat{\varepsilon}_j)\psi(\widehat{\varepsilon}_{j+k})^T & k \geq 0 \\ \widehat{\boldsymbol{\Gamma}}_{|k|}^T & k < 0 \end{cases}$$

$$\omega_q(x) = \big(1 - \frac{|x|}{q}\big) I\{|x| \leq q\}$$

$q(m) \to \infty$ as $m \to \infty$, $q(m)/m \to 0$

Numerical results

Simulations

- Critical values computed from simulated limit distribution
- Empirical quantiles of sample values of test statistic by Monte Carlo method
- Empirical quantiles of sample values of test statistic generated by pair bootstrap from historical period
- Empirical level of test
- Empirical power

	m \ γ	L_2 0	L_2 0.25	L_2 0.45	Huber 0	Huber 0.25	Huber 0.45	L_1 0	L_1 0.25	L_1 0.45
$r_{iM} \sim$AR(1)	100	8.2	9.6	8.7	7.3	8.5	7.2	7.0	8.1	5.9
	200	6.1	7.1	6.6	5.6	6.6	5.4	5.1	6.2	4.5
	400	5.6	6.5	5.7	4.8	5.9	5.1	4.7	5.7	5.0
$\varepsilon_i \sim$VAR(1)	100	7.5	9.6	9.8	5.9	7.2	6.0	3.8	5.0	3.4
	200	5.7	6.9	6.9	5.1	6.6	5.2	4.3	5.2	3.8
	400	4.9	5.6	6.6	4.3	5.2	5.3	3.9	4.6	3.8
$r_{iM} \sim$AR(1), $\varepsilon_i \sim$VAR(1)	100	12.1	14.0	12.4	8.9	10.7	8.7	6.4	7.7	6.1
	200	7.2	8.6	8.5	6.6	7.6	6.2	5.1	6.7	5.6
	400	5.1	6.8	7.3	5.0	6.6	6.1	4.3	5.5	5.0

Table: Empirical sizes for 5% level, $T = 10$, dependent observations.

	m \ γ	L_2 0	L_2 0.25	L_2 0.45	Huber 0	Huber 0.25	Huber 0.45	L_1 0	L_1 0.25	L_1 0.45
N	100	7.4	8.1	6.9	5.9	6.7	5.3	3.8	4.9	3.6
	200	4.7	5.9	5.0	4.4	5.3	4.3	4.3	5.0	3.9
	400	4.3	5.0	4.6	4.5	5.1	4.4	3.2	4.5	4.1
t_4	100	6.2	8.5	9.3	5.7	7.6	6.5	5.2	6.7	6.5
	200	6.5	8.5	9.5	5.0	6.7	4.4	4.5	5.4	6.4
	400	3.7	5.3	6.3	4.2	5.4	3.9	3.8	4.3	5.2
t_1	100	62.6	65.6	64.2	5.5	6.5	4.2	4.2	5.1	5.6
	200	65.9	68.5	67.9	5.3	6.0	4.6	3.9	4.0	5.6
	400	62.6	66.1	65.4	4.4	4.8	4.1	3.7	4.6	5.5

Table: Empirical sizes for 5% level, $T = 10$, different error distribution.

	m \ γ	L_2 0	L_2 0.25	L_2 0.45	Huber 0	Huber 0.25	Huber 0.45	L_1 0	L_1 0.25	L_1 0.45
N	100	30	20	15	41	28	22	62	45	39
	200	38	22	15	50	31	22	73	47	35
	400	40	27	16	66	36	22	92	54	34
t_4	100	38	25	20	41	27	22	63	44	38
	200	48	29	20	50	31	22	73	47	34
	400	64	35	21	65	37	22	92	54	35
t_1	100	ND	ND	186	71	50	42	90	66	60
	200	ND	348	301	79	51	36	99	66	52
	400	ND	ND	766	100	59	36	120	73	48

Table: Medians of detection delays, $k^* = 10$.

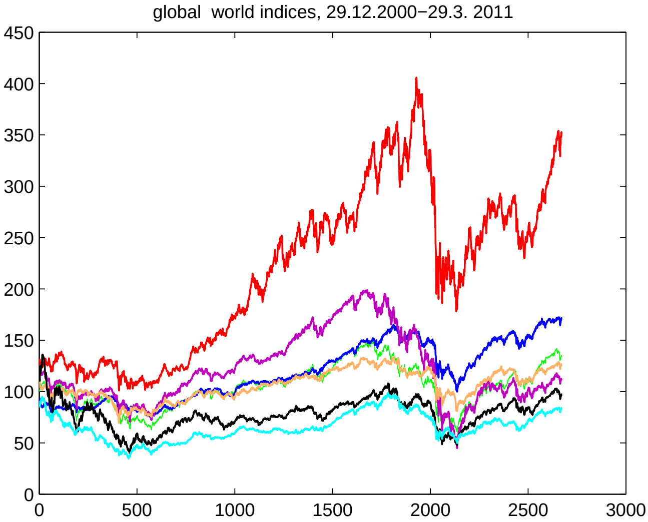
	L_2			Huber			L_1		
m $\backslash\gamma$	0	0.25	0.45	0	0.25	0.45	0	0.25	0.45
100	41.0	47.5	55.0	98	99	98	97	98	97
200	44.4	51.8	53.5	100	100	100	100	100	100
400	41.4	48.2	57.0	100	100	100	100	100	100

Table: Empirical power of the test (in %) for t_1 errors

Application to real data

Data: not traded indices computed from sector data of global world economy, serve as benchmarks for investors:

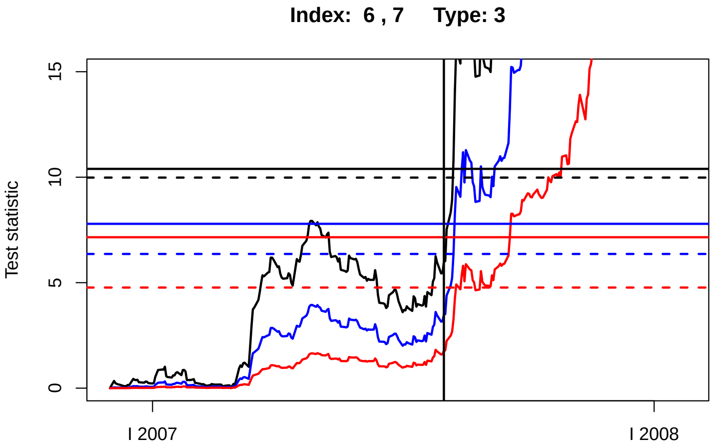
- World Consumer Disretionary (3)
  World Consumer Staples (4)
  World Energy (5)
  World Financials (6)
  World Health Care (7)
  Information Technology (9)
  Telecommunication Services (11)
- Market portfolio: MSCI World Daily Index (NDDUWI)
- Risk-free asset: S&P 3M US Treasury Bill
- Training period: 31.12. 2004-30.11. 2006 (m=500)
- Monitoring period: 1.12.2006-1.10. 2010


global world indices, 29.12.2000−29.3. 2011
450
400
350
300
250
200
150
100
50
0
0
500
1000
1500
2000
2500
3000

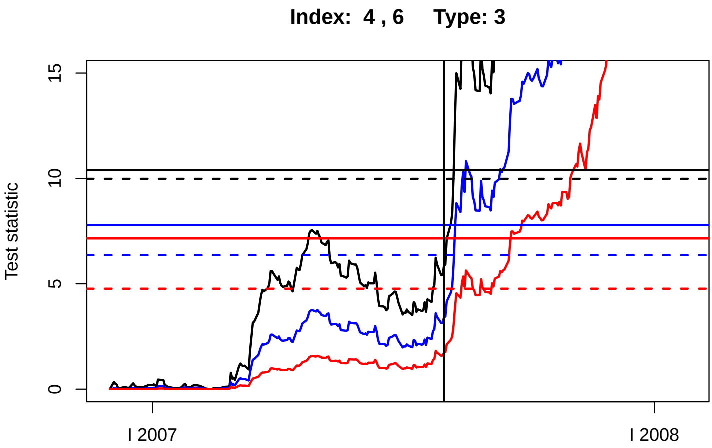
	L_2		L_1		Huber	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
3	0.00034	1.00939	0.00042	1.03095	0.00036	1.01308
4	0.00043	0.68471	0.00036	0.69194	0.00042	0.68609
5	0.00087	1.30334	0.00155	1.30181	0.00118	1.29619
6	0.00058	1.00013	0.00054	1.00746	0.00054	0.99581
7	0.00035	0.71222	0.00020	0.73012	0.00022	0.73045
9	0.00027	1.09566	0.00056	1.10835	0.00030	1.09610
11	0.00026	0.88648	0.00014	0.89396	0.00023	0.89153

Table: Estimators of α, β, historical period 31.12. 2004-30.11. 2006:

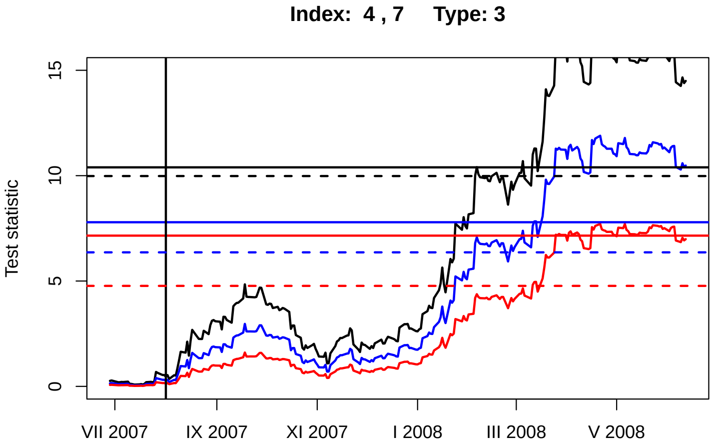
4-Consumer Staples, 6-Financial, 7-Health Care



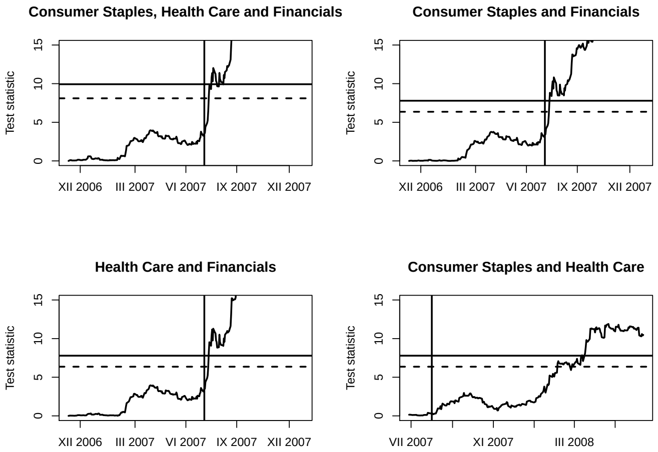
Financials (6), Health Care (7): red - $\gamma = 0$, blue - $\gamma = 0.25$, black - $\gamma = 0.45$
solid line - asymptotic critical values, dashed - closed end



Consumer Staples (4), Financials (6)



Consumer Staples (4), Health Care (7)


Consumer Staples, Health Care and Financials
Consumer Staples and Financials
Health Care and Financials
Consumer Staples and Health Care
Test statistic
0
5
10
15
XII 2006
III 2007
VI 2007
IX 2007
XII 2007
VII 2007
XI 2007
III 2008

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Thank you for your attention