Simultánne testovanie strednej hodnoty a variancie normálneho rozdelenia

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Obsah

$X_1, X_2, \ldots, X_n$ je náhodný výber z normálneho rozdelenia so neznámou strednou hodnotou μ a s neznámou varianciou σ^2

- ozn. *parametrický priestor*

$$\Theta = \{\theta = (\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 > 0\}$$

Simultánne testovanie strednej hodnoty a variancie $N(\mu, \sigma^2)$

$$H_0 : (\mu, \sigma^2) = (\mu_0, \sigma_0^2) \in \Theta_0 \quad \text{proti} \quad H_1 : (\mu, \sigma^2) \in \Theta - \Theta_0 = \Theta_1$$

- hladina významnosti testu α ($\in (0, 1)$)

$$\bar{X} \sim N(\mu_0, \sigma_0^2/n), \quad \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$$

Prehľad literatúry

Mood, A. M.: Introduction to the Theory of Statistics, *New York, McGraw-Hill*, 1950.

Wilks, S. S.: Mathematical Statistics, *New York, John Wiley and Sons*, 1962.

Arnold, B.C. - Shavelle, R. M.: Joint confidence sets for the mean and variance of a normal distribution, *The American Statistician 52*, 1998, 133–140.

Choudhari, P. - Kundu, D. - Misra, N.: Likelihood ratio test for simultaneous testing of the mean and variance of a normal distribution, *Journal of Statistical Computation and Simulation 71*, 2001, 313–333.

Arnold a Shavelle (1998)

LSR1 (Large Sample Ragion)

$$\mathcal{R} = \left\{ (\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{\sigma^2} + \frac{n}{2\sigma^4}(S^2 - \sigma^2)^2 \leq \chi_2^2(1 - \alpha) \right\}$$

LSR2

$$\mathcal{R} = \left\{ (\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{S^2} + \frac{n}{2S^4}\left(S^2 - \sigma^2\right)^2 \leq \chi_2^2(1 - \alpha)^* \right\}$$

$$^*2F_{2,n-2}(1 - \alpha)$$

Asymptotické rozdelenie $\sqrt{n}(S^2 - \sigma_0^2) \approx N(0, 2\sigma_0^2)$

Modifikovaný Arnold a Shavelle a navrhnutá LSR3

$$\mathcal{R}_{LSR1} = \left\{(\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{\sigma^2} + \frac{n}{2\sigma^4}(S^2 - \sigma^2)^2 \leq c_{21}(1-\alpha)\right\}$$

$$\mathcal{R}_{LSR2} = \left\{(\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{S^2} + \frac{n}{2S^4}\left(S^2 - \sigma^2\right)^2 \leq c_{22}(1-\alpha)\right\}$$

$$\mathcal{R}_{LSR3} = \left\{(\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{\sigma^2} + \left(\sqrt{\frac{2(n-1)S^2}{\sigma^2}} - \sqrt{2v-1}\right)^2 \leq c_{23}(1-\alpha)\right\}$$

Asymptotické rozdelenie[1] $\sqrt{2\chi_v^2} \approx N(\sqrt{2v-1}, 1)$

[1] Fisher, R. A. (1928), *Statistical Methods for Research Workers*, 2nd Edition, pp. 96-97.

Mood, Wilks a LRT

Mood (1950)

$$\mathcal{R} = \{(\mu, \sigma^2) : \sqrt{n}|\bar{X} - \mu|/\sigma \leq u(1 - \alpha_1/2) \\ \wedge \ \ \chi^2_{n-1}(\alpha_2/2) \leq (n-1)S^2/\sigma^2 \leq \chi^2_{n-1}(1 - \alpha_2/2)\}$$

- $1 - \alpha = (1 - \alpha_1)(1 - \alpha_2)$ a $\alpha_1 = \alpha_2 = 1 - \sqrt{1 - \alpha}$

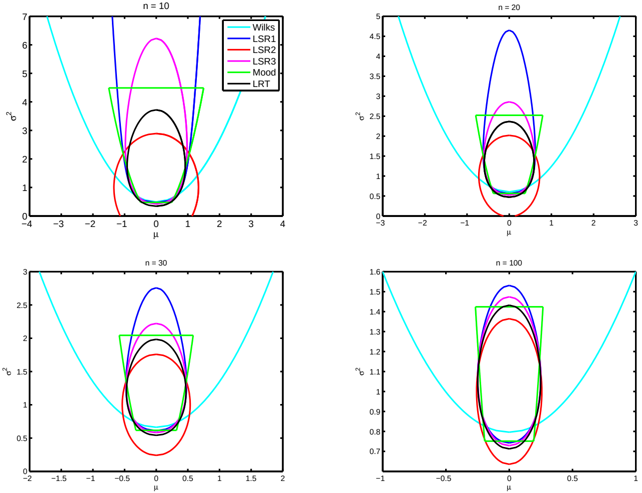
Wilks (1962)

$$\mathcal{R} = \left\{(\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{\sigma^2} + \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_n(1 - \alpha)\right\}$$

Choudhari–Kundu–Misra (2001) - LRT

$$\mathcal{R} = \left\{(\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{\sigma^2} + \frac{(n-1)S^2}{\sigma^2} - n \ln\left(\frac{(n-1)S^2}{n\sigma^2}\right) - n \leq d_n(1 - \alpha)\right\}$$

Tvary oblastí spoľahlivosti pre $\alpha = 0,05$ a rôzne rozsahy výberu



Porovnávacia štúdia

Porovnávame:

- obsahy: LSR1, LSR2, LSR3, Mood, LRT
(pre $\alpha \in \{0,10; 0,05; 0,01\}$ a $n \in \{10, 20, 30, 100\}$)
$obsah = [\,] \times S^3$

- sily testov: Wilks, LSR1, LSR2, LSR3, Mood, LRT
(pre $\alpha = 0,05$ a $n \in \{10, 20, 30, 100\}$)

Silofunkcie testov závisia od $(\mu_1 - \mu_0)^2$, σ_1^2/σ_0^2

- uvažovali sme $\mu_0 = 0$ a $\sigma_0^2 = 1$
- meníme μ_1 pri pevnom $\sigma_1^2 = 1$
- meníme σ_1^2 pri pevnom $\mu_1 = 0$
- meníme μ_1 a zároveň meníme aj σ_1^2

Obsahy $(1-\alpha)$-spoľahlivých oblastí

Metóda	LSR1	LSR2	LSR3	Mood	Mood*	LRT
n			$\alpha = 0,10$			
10	36,1507	4,1982	5,1218	4,6742	3,2702	3,2079
20	2,0077	1,4259	1,5242	1,5412	1,3104	1,2616
30	1,0185	0,8402	0,8802	0,9059	0,8088	0,7819
100	0,2275	0,2158	0,2200	0,2300	0,2224	0,2129

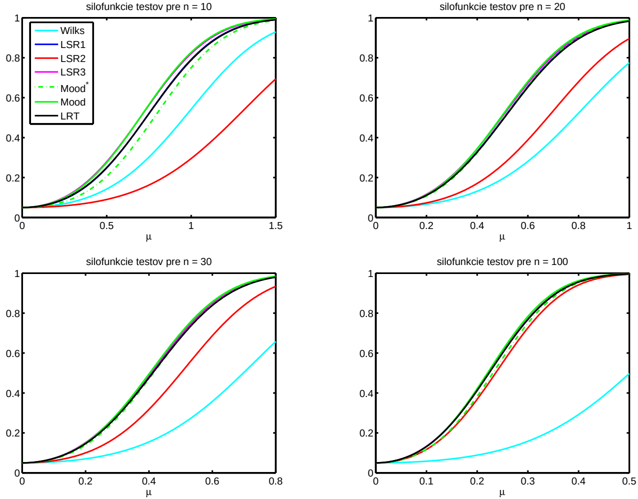
Mood*

$$\mathcal{R} = \{(\mu, \sigma^2) : \sqrt{n}|\bar{X} - \mu|/\sigma \leq u(1-\alpha_1/2) \\ \wedge \quad \chi^2_{n-1}(\delta) \leq (n-1)S^2/\sigma^2 \leq \chi^2_{n-1}(1-\alpha_2+\delta)\}$$

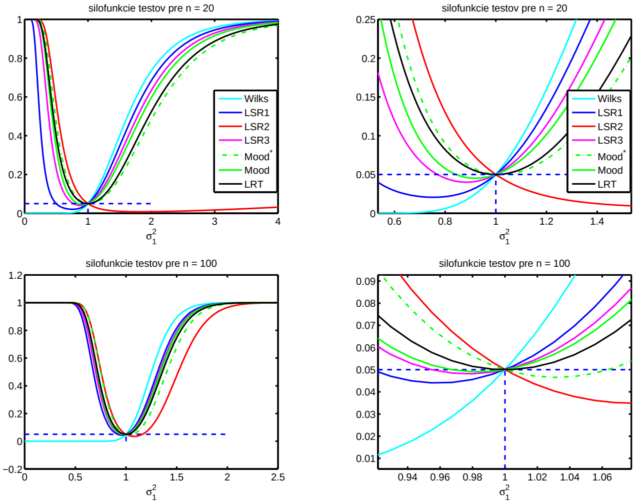
Obsahy $(1-\alpha)$-spoľahlivých oblastí

Metóda	LSR1	LSR2	LSR3	Mood	Mood*	LRT
n			$\alpha = 0,05$			
10	-	6,4583	8,8844	7,3943	5,1747	4,8910
20	3,8435	2,2946	2,2086	2,2091	1,9069	1,766
30	1,5887	1,2771	1,2236	1,2606	1,1291	1,0668
100	0,3075	0,2949	0,2914	0,3078	0,2995	0,2808
			$\alpha = 0,01$			
10	-	17,881	32,57	18,028	12,617	10,959
20	-	4,8945	4,4771	4,2564	3,5908	3,224
30	4,8396	2,7298	2,2175	2,263	2,0225	1,8329
100	0,5265	0,5246	0,4676	0,5034	0,4868	0,4458

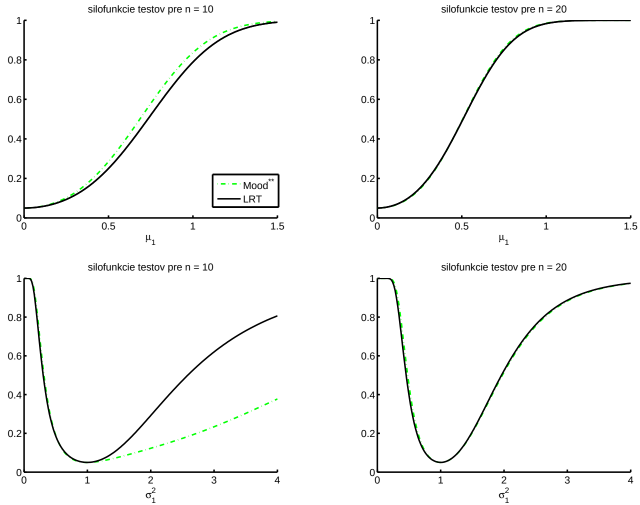
Silofunkcie pre rôzne μ, ak $\alpha = 0,05$; $\sigma_0^2 = \sigma_1^2 = 1$



Silofunkcie pre rôzne σ_1^2, ak $\alpha = 0,05$; $\mu_0 = \mu_1 = 0$



Silofunkcie LRT a Mood**, $\alpha = 0.05$



Mood ('optimálne') $\alpha_1, \alpha_2, \delta$ pre $n = 20$

	$\alpha = 0,10$			$\alpha = 0,05$		
	α_1	α_2	δ	α_1	α_2	δ
Mood	,0513	,0513	,0257	,0253	,0253	,0127
Mood*	,0428	,0598	,0532	,0168	,0338	,0325
Mood**	,0446	,0580	,0447	,0215	,0291	,0231

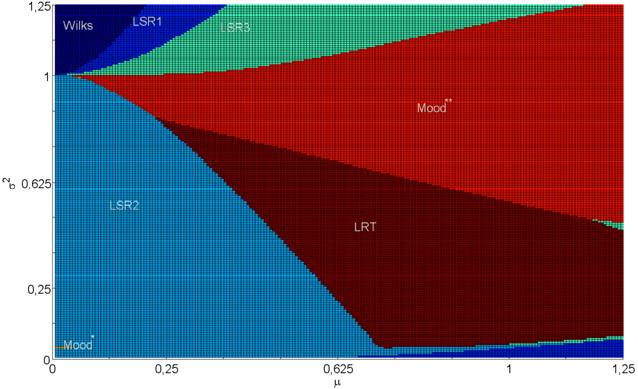
$$1 - \alpha = (1 - \alpha_1)(1 - \alpha_2)$$

Mood $\quad \alpha_1 = \alpha_2 = 1 - \sqrt{1 - \alpha}$

Mood*, Mood**

$$\mathcal{R} = \{(\mu, \sigma^2) : \sqrt{n}|\bar{X} - \mu|/\sigma \leq u(1 - \alpha_1/2) \wedge \quad \chi^2_{n-1}(\delta) \leq (n-1)S^2/\sigma^2 \leq \chi^2_{n-1}(1 - \alpha_2 + \delta)\}$$

Silofunkcie pre rôzne μ_1, σ_1^2 pre $\alpha = 0,05; n = 10$



Iná alternatíva

$X_1, X_2, \ldots, X_n$ je náhodný výber z normálneho rozdelenia so neznámou strednou hodnotou μ a s neznámou varianciou σ^2

$$H_0 : \mu = \mu_0 \ \wedge \ \sigma^2 = \sigma_0^2 \ \text{ proti } \ H_1 : \mu \neq \mu_0 \ \vee \sigma^2 < \sigma_0^2$$

Presné testy, presné oblasti spoľahlivosti

Upravený Mood

$$\mathcal{R} = \{(\mu, \sigma^2) : \sqrt{n}|\bar{X} - \mu|/\sigma \leq u(1 - \alpha_1/2) \\ \wedge \quad (n-1)S^2/\sigma^2 \geq \chi^2_{n-1}(\alpha_2)\}$$

- $1 - \alpha = (1 - \alpha_1)(1 - \alpha_2)$ a $\alpha_1 = \alpha_2 = 1 - \sqrt{1 - \alpha}$

Návrh upraviť LRT

$$\mathcal{R} = \left\{(\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{\sigma^2} + \frac{(n-1)S^2}{\sigma^2} - n \ln\left(\frac{(n-1)S^2}{n\sigma^2}\right) - n \leq d_n(1 - \alpha)\right\}$$

Presné testy, presné oblasti spoľahlivosti

Upravený Mood

$$\mathcal{R} = \{(\mu, \sigma^2) : \sqrt{n}|\bar{X} - \mu|/\sigma \leq u(1 - \alpha_1/2) \\ \wedge \quad (n-1)S^2/\sigma^2 \geq \chi^2_{n-1}(\alpha_2)\}$$

- $1 - \alpha = (1 - \alpha_1)(1 - \alpha_2)$ a $\alpha_1 = \alpha_2 = 1 - \sqrt{1 - \alpha}$

Návrh upraviť LRT

$$\mathcal{R} = \left\{(\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{\sigma^2} + \frac{(n-1)S^2}{\sigma^2} - n \ln\left(\frac{(n-1)S^2}{n\sigma^2}\right) - n \leq d_n(1 - \alpha)\right\}$$

Presné testy, presné oblasti spoľahlivosti

Upravený Mood

$$\mathcal{R} = \{(\mu, \sigma^2) : \sqrt{n}|\bar{X} - \mu|/\sigma \leq u(1 - \alpha_1/2) \\ \wedge \ (n-1)S^2/\sigma^2 \geq \chi^2_{n-1}(\alpha_2)\}$$

- $1 - \alpha = (1 - \alpha_1)(1 - \alpha_2)$ a $\alpha_1 = \alpha_2 = 1 - \sqrt{1 - \alpha}$

Návrh upraviť LRT

$$\mathcal{R} = \left\{(\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{\sigma^2} + \frac{(n-1)S^2}{\sigma^2} - n \ln\left(\frac{(n-1)S^2}{n\sigma^2}\right) - n \leq d_n(1 - \alpha)\right\}$$

Presné testy, presné oblasti spoľahlivosti

Upravený Mood

$$\mathcal{R} = \{(\mu, \sigma^2) : \sqrt{n}|\bar{X} - \mu|/\sigma \leq u(1 - \alpha_1/2) \\ \wedge \quad (n-1)S^2/\sigma^2 \geq \chi^2_{n-1}(\alpha_2)\}$$

- $1 - \alpha = (1 - \alpha_1)(1 - \alpha_2)$ a $\alpha_1 = \alpha_2 = 1 - \sqrt{1 - \alpha}$

Návrh upraviť LRT

$$\mathcal{R} = \left\{(\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{\sigma^2} + \frac{(n-1)S^2}{\sigma^2} - n \ln\left(\frac{(n-1)S^2}{n\sigma^2}\right) - n \leq d_n^*(1 - \alpha)\right\}$$

Presné testy, presné oblasti spoľahlivosti

Upravený Mood

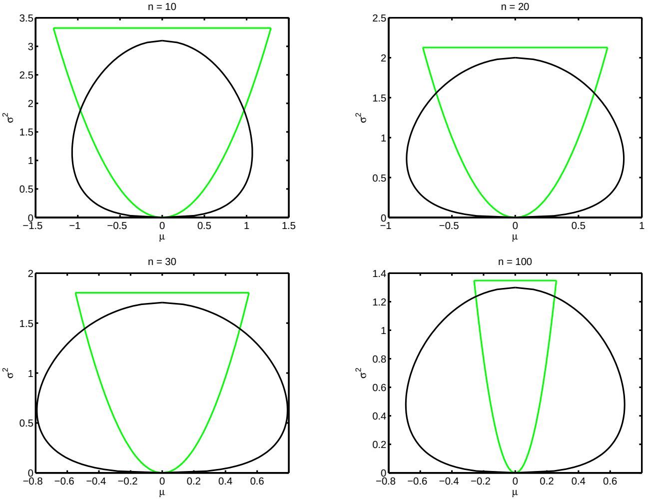
$$\mathcal{R} = \{(\mu, \sigma^2) : \sqrt{n}|\bar{X} - \mu|/\sigma \leq u(1 - \alpha_1/2) \\ \wedge \ (n-1)S^2/\sigma^2 \geq \chi^2_{n-1}(\alpha_2)\}$$

- $1 - \alpha = (1 - \alpha_1)(1 - \alpha_2)$ a $\alpha_1 = \alpha_2 = 1 - \sqrt{1 - \alpha}$

Návrh upraviť LRT

$$\mathcal{R} = \left\{(\mu, \sigma^2) : \frac{n(\bar{X} - \mu)^2}{\sigma^2} - n \ln\left(\frac{(n-1)S^2}{\sigma^2}\right) \leq d_n^*(1 - \alpha)\right\}$$

Tvary oblastí spoľahlivosti pre $\alpha = 0,05$ a rôzne rozsahy výberu



Porovnávacia štúdia

Porovnávame:

- obsahy:
(pre $\alpha \in \{0,10; 0,05; 0,01\}$ a $n \in \{10, 20, 30, 100\}$)
$obsah = [\,] \times S^3$

- silofunkcie
(pre $\alpha = 0,05$ a $n \in \{10, 20\}$)

Silofunkcie testov závisia od $(\mu_1 - \mu_0)^2$, σ_1^2/σ_0^2

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- meníme μ_1 pri pevnom $\sigma_1^2 = 1$
- meníme σ_1^2 pri pevnom $\mu_1 = 0$
- meníme μ_1 a zároveň meníme aj σ_1^2

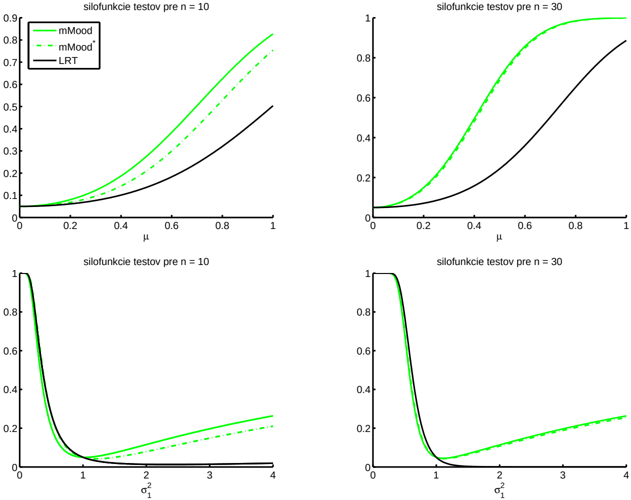
Obsahy $(1-\alpha)$-spoľahlivých oblastí

Metóda	mMood	mLRT	mMood	mLRT	mMood	mLRT
n	$\alpha = 0,10$		$\alpha = 0,05$		$\alpha = 0,01$	
10	3,6148	3,6997	5,7058	5,2645	13,9660	10,8755
20	1,4842	2,1979	2,0706	2,7300	3,8678	4,2086
30	0,9884	1,8170	1,3192	2,1489	2,2436	2,9915
100	0,3774	1,3223	0,4669	1,4291	0,6793	1,6792
10	3,4181		5,3218		12,7547	
20	1,4753		2,0463		3,7814	
30	0,9884		1,3172		2,2275	
100	0,3689		0,4613		0,6774	

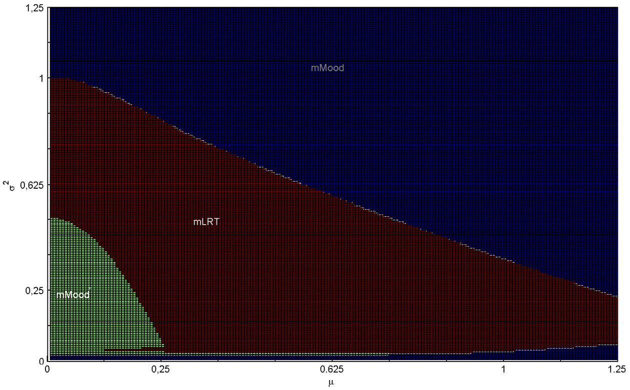
mMood*

$$\mathcal{R} = \left\{ (\mu, \sigma^2) : \frac{\sqrt{n}|\bar{X} - \mu|}{\sigma} \leq u(1-\alpha_1/2) \ \wedge \ \frac{(n-1)S^2}{\sigma^2} \geq \chi^2_{n-1}(\alpha_2) \right\}$$

Silofunkcie pre $\alpha = 0,05$



Silofunkcie pre rôzne μ_1, σ_1^2 pre $\alpha = 0,05; n = 10$



Ďakujem za pozornosť!