Detecting atoms in deconvolution

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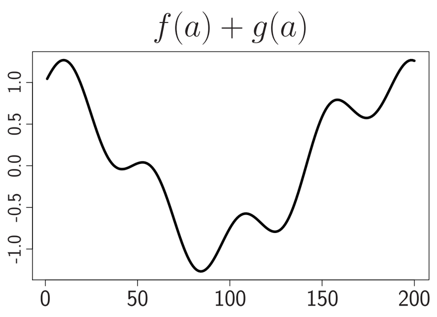
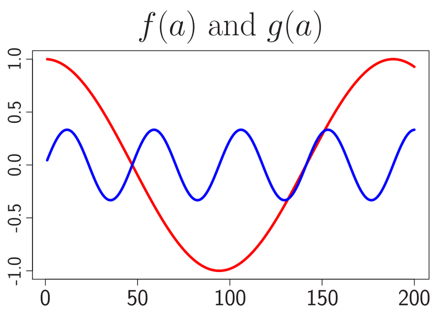
SUMMARY

We introduce the atomic deconvolution problem and we propose the estimator for an atom location and give its asymptotic distribution. We show the asymptotic distribution for the r-th derivative of a density estimator in the ordinary deconvolution problem.

ATOMIC DECONVOLUTION

In the ordinary deconvolution problem one wants to estimate the distribution of Y from the given sample of i.i.d. random variables $X_1, \ldots, X_n$, where $X_i = Y_i + Z_i$, and where Z_i is supposed to be an error term with a known distribution. In the ordinary deconvolution it is usually assumed that Y has a continuous density. In the atomic deconvolution we suppose that the distribution of Y has an atom at a_0.





Example

As example of the atomic deconvolution models we can consider models where the direct data are censored, i.e.

$$Y_i = \begin{cases} a_0, & \text{if censored,} \\ V, & \text{if not censored,} \end{cases}$$

where V is a random variable with density function f. The censoring is with the probability p_0. For direct data you will see the number of equal values a_0 appearing. So it is clear that something is "wrong". If the data can not be directly observed, e.g. if there is an other term Z added, the measuring error, you do not see the values a_0. In this case the censoring is disguised. Then the use of deconvolution methods is necessary. We suppose that the error term Z has the standard normal distribution.

Choice of kernel

It is common in the ordinary deconvolution that conditions are stated for $\phi_w(t)$, the Fourier transformation of the kernel function $w(x)$, rather then for kernel function itself. We use the class of kernels for which, as $t \to 0$, holds

Condition on the kernel function

$$\phi_w(1-t) = At^\alpha + o(t^\alpha),$$

for some constants A and $\alpha \geq 0$.

Kernel estimator

If Y has the density f, which corresponds to the standard deconvolution problem, the r-th derivative of the ordinary deconvolution kernel estimator would be

Derivative of the ordinary kernel deconvolution estimator

$$\hat{f}_{nh}^{(r)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-it)^r e^{-itx} \phi_{\mathrm{emp}}(t) \phi_w(ht) \frac{1}{\phi_Z(t)}\, dt.$$

In our setting, because of the atom, unless $p_0 = 0$, the random variable Y does not have a density. However, we have the asymptotic properties of the estimator $\hat{f}_{nh}^{(r)}$.

Asymptotic distribution of the estimator

$$\frac{\sqrt{n}}{h^{1-r+2\alpha} e^{\frac{1}{2h^2}}} \left(\hat{f}_{nh}^{(r)}(x) - \mathrm{E}[\hat{f}_{nh}^{(r)}(x)] \right) \xrightarrow{\mathcal{D}} N\left(0, \frac{A^2\left(\Gamma(\alpha+1)\right)^2}{2\pi^2}\right).$$

DETECTING AN ATOM

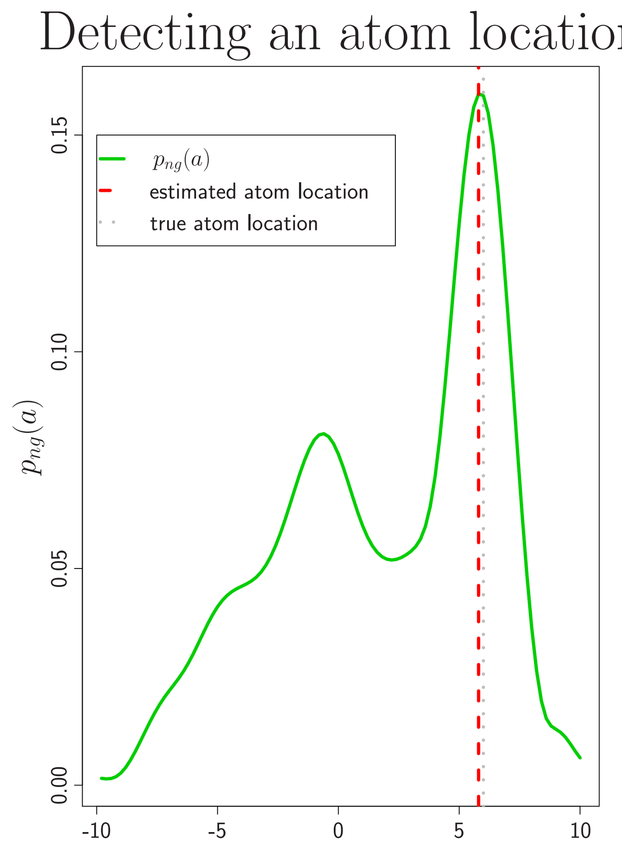
Important task is to estimate the location of an atom a_0 when both the density of V and the atom size p_0 are unknown. As the atom location estimator we propose A_{ng}.

Atom location estimator

$$A_{ng} = \arg\max_a \left| p_{ng}(a) \right|,$$

where $p_{ng}(a)$ can be interpreted as an estimator of the atom size if the atom is at point a. We define $p_{ng}(a)$ as multiple of $\hat{f}_{nh}(a)$ using the same kernel bandwidth.

Detecting an atom location



Definition of $p_{ng}(a)$

$$p_{ng}(a) = \pi g \hat{f}_{ng}(a).$$

Main result

Assuming suitable choice of kernel function and appropriate choice of convergence rate of bandwidth we can show the asymptotic distribution of the atom location estimator.

Asymptotic distribution of the atom location estimator

$$\frac{\sqrt{n}}{g^{3+2\alpha} e^{\frac{1}{2g^2}}} \left(A_{ng} - a_0 + \frac{g^2 \mathrm{E} p'_{ng}(a_0)}{p \pi k''(0)} \right) \xrightarrow{\mathcal{D}} N\left(0, \frac{A^2\left(\Gamma(\alpha+1)\right)}{2p^2\pi^2\left(k''(0)\right)^2}\right).$$

FUTURE DEVELOPMENT

Complexity: How we should measure the difficulty of atomic deconvolution problem? The signal-to-noise ratio is the sufficient measure of complexity in the ordinary deconvolution problem. Now we also have to consider the atom size and the atom location.

More atoms: How we should proceed when two or more atoms are considered?

Sensitivity: When is the indication of the atom significant? How small atom size, related to the sample size, can we indicate? We must consider a finite sample!

Bandwidth: What is the optimal choice of the kernel bandwidth?

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