Duality for some classes of convex sets

A. B. ROMANOWSKA and A. Mućka
Faculty of Mathematics and Information Science, Warsaw University of Technology, 00-661 Warsaw, Poland

Dualities

There is a **full dual equivalence** or simply **full duality** between categories $\mathbf{A}$ and $\mathbf{X}$ if there are contravariant functors

$$D : \mathbf{A} \to \mathbf{X} \qquad \text{and} \qquad E : \mathbf{X} \to \mathbf{A}$$

with natural isomorphisms

$$e : DE \to Id_{\mathbf{A}} \qquad \text{and} \qquad \varepsilon : ED \to Id_{\mathbf{X}}.$$

In many cases the functors of the duality are represented by a **schizophrenic object**, The schizophrenic object T appears simultaneously as an object $\underline{\mathsf{T}}$ of $\mathbf{A}$ and as an object $\utilde{\mathsf{T}}$ in $\mathbf{X}$.

The functors D and E are defined on objects and morphisms by

$$
\begin{array}{lcll}
A & & \mathbf{A}(A, \underline{\mathsf{T}}) & fx : A \to B \to \underline{\mathsf{T}} \\
\downarrow f & \overset{D}{\mapsto} & \uparrow f^D & \qquad\qquad \uparrow \\
B & & \mathbf{A}(B, \underline{\mathsf{T}}) & \qquad x : B \to \underline{\mathsf{T}}.
\end{array}
$$

$$
\begin{array}{lcll}
X & & \mathbf{X}(X, \underset{\sim}{\mathsf{T}}) & \varphi\alpha : X \to Y \to \underset{\sim}{\mathsf{T}} \\
\downarrow \varphi & \overset{E}{\mapsto} & \uparrow \varphi^E & \qquad\qquad \uparrow \\
Y & & \mathbf{X}(Y, \underset{\sim}{\mathsf{T}}) & \qquad \alpha : Y \to \underset{\sim}{\mathsf{T}}
\end{array}
$$

The natural isomorphisms e and ε are given by the evaluations:

$$e_A : A \to A^{DE}; a \mapsto (e_A(a) : x \mapsto x(a)),$$

$$\varepsilon_X : X \to X^{ED}; x \mapsto (\varepsilon_X(x) : \alpha \mapsto \alpha(x)).$$

Examples

(a) Self-duality for finite-dimensional vector spaces over a field F with F as T,
(b) Pontryagin duality for abelian groups with the circle group as T.

"**Natural dualities**" concern the case when a schizophrenic objects is finite, and certain additional conditions hold.

A well-known example is provided by:
(c) Hofmann-Mislove-Stralka duality for semilattices (1974) given by

$$C : \mathbf{S} \to \mathbf{Z} \qquad \text{and} \qquad \mathbf{Z} \to \mathbf{S},$$

where $\mathbf{Z}$ is the category of bounded compact toplogical semilattices carrying Boolean topology.
The duality is natural with two element semilattice $\underline{\mathbf{2}} = (\{0 < 1\}, \wedge)$ as $\underline{\mathrm{T}}$ and $\underset{\sim}{\mathbf{2}} = (\{0 < 1\}, \wedge, \theta)$ as $\underset{\sim}{\mathrm{T}}$.

REAL AFFINE SPACES

Given a vector space A over a field $\mathbb{R}$
(more generally: a subfield R of $\mathbb{R}$).

An **affine space** A **over** R (or **affine** R-**space**)
is the algebra

$$\left(A, \sum_{i=1}^{n} x_i r_i \,\middle|\, \sum_{i=1}^{n} r_i = 1\right).$$

This algebra is equivalent to $(A, \underline{R})$, where

$$\underline{R} = \{\underline{r} \mid r \in R\}$$

and

$$xy\underline{r} = \underline{r}(x, y) = x(1-r) + yr.$$

The class $\underline{\underline{R}}$ of affine R-spaces is a variety.

CONVEX SETS and BARYCENTRIC ALGEBRAS

Let $I^o :=]0,1[= (0,1) \subset R$
(with R a subfield of $\mathbb{R}$).

Convex subsets of affine R-spaces are I^o-subreducts $(A, \underline{I}^o)$ of R-spaces.

A finitely generated convex set is called a **polytope**,
its interor is an **open polytope**,
and a convex set with a polytope as closure and containing its interior is called a **quasi-polytope**.

The class $\mathbf{C}$ of convex sets generates the variety $\mathbf{B}$ of **barycentric algebras** with $\mathbf{C}$ is a (minimal) subquasivariety of $\mathbf{B}$.

The only non-trivial subvariety of $\mathbf{B}$ is the variety $\mathbf{S}$ of semilattices.

MODES

An algebra (A, Ω) is a **mode** if it is

- **idempotent**:

  $$x...x\omega = x,$$

  for each n-ary $\omega \in \Omega$, and

- **entropic**:

  $$(x_{11}...x_{1n}\omega)...(x_{m1}...x_{mn}\omega)\,\varphi$$

  $$= (x_{11}...x_{m1}\varphi)...(x_{1n}...x_{mn}\varphi)\,\omega.$$

  for all $\omega, \varphi \in \Omega$.

Affine R-spaces and their subreducts (subalgebras of reducts) are modes. In particular, barycentric algebras are modes.

BARYCENTRIC ALGEBRAS, again

Theorem For barycentric algebras A and C

$$\mathbf{B}(A,C) \leq C^A,$$

hence it is a barycentric algebra.
If A and C are convex, then this algebra is convex, too.

Note that for $r \in I^o$ and $f, g \in C^A$:

$$\underline{r}(f,g)(x) = \underline{r}(f(x), g(x)).$$

Theorem Each barycentric algebra A is a semilattice sum $\bigcup_{s \in S} A_s$ of open convex sets A_s over its semilattice replica S.

Theorem Each barycentric algebra $A = \bigcup_{s \in S} A_s$ is a subalgebra of a Płonka sum $\sum_{s \in S} E_s$ of convex extensions E_s of A_s over its semilattices replica S.
The semilattice replica S of A is the semilattice of its principal walls.

Corollary If $A = \bigcup_{s \in S} A_s$ is a quasi-polytope, then

$$E_s = W_s = \bigcup_{t \geq s} A_t,$$

they are walls of A, and there are the following embeddings

$$A = \bigcup_{s \in S} A_s \hookrightarrow \sum_{s \in S} W_s \hookrightarrow \sum_{s \in S} \overline{W}_s = \sum_{s \in S} \overline{A}_s.$$

Note that $\sum_{s \in S} \overline{A}_s$ is an injective Płonka sum of polytopes.

...and again

An alternative description of barycentric algebras is given by algebras $(A, \underline{I})$ with

$$xy\underline{0} = x = yx\underline{1}.$$

Note that

$$(A, \underline{I}) \simeq (A, \underline{I}^o, \star = \underline{0})$$

.

The corresponding variety of such algebras is $\mathbf{B}^\star$ and the quasivariety of convex sets $\mathbf{C}^\star$. Note that

$$\mathbf{B} \simeq \mathbf{B}^\star \text{ and } \mathbf{C} \simeq \mathbf{C}^\star.$$

Note! Convex sets, as algebras in $\mathbf{C}^\star$, have no non-trivial decomposition as a sum of convex sets over their semilattice replicas.

Our strategy

to find a duality for the class $\mathbf{QP}$ of quasi-polytopes is based on

- the embedding of quasi-polytopes into Płonka sums of polytopes and

- the following known dualities:

- $C : \mathbf{S} \to \mathbf{Z}$ and $F : \mathbf{Z} \to \mathbf{S}$,

for semilattices;

- $D : \mathbf{P} \to \widehat{\mathbf{P}}$ and $E : \widehat{\mathbf{P}} \to \mathbf{P}$,

for polytopes (Romanowska, Ślusarski, Smith, 2009), based on the schizophrenic object I.

The duality for polytopes is used to find a duality

- $$D^o : \mathbf{P}^o \to \widehat{\mathbf{P}}^o \qquad \text{and} \qquad E^o : \widehat{\mathbf{P}}^o \to \mathbf{P}^o,$$

for open polytopes, based on the schizophrenic object I^o;

The next step is to find a duality

- $$\bar{D} : \mathbf{PP} \to \widehat{\mathbf{PP}} \qquad \text{and} \qquad \bar{E} : \widehat{\mathbf{PP}} \to \mathbf{PP},$$

for injective Płonka sums of polytopes, based on the schizophrenic object I^∞, the Płonka sum of I and one element algebra $\{\infty\}$. The category $\widehat{\mathbf{PP}}$ consists of certain Płonka sums with constants, considered as members of $\mathbf{B}^*$.

Duality for injective Płonka sums in PP

Theorem Let $A = \sum_{s \in S} A_s$ be a member of $\mathbf{PP}$. Then the barycentric algebra $A\bar{D}$ is isomorphic to the Płonka sum $\sum A_s D$ of the first duals $A_s D$ of the polytopes A_s over the first dual SC of S.

Theorem Let $A = \sum_{s \in S} A_s$ be a member of $\mathbf{PP}$. Then the barycentric algebra $A\bar{D}\bar{E}$ is isomorphic to the Płonka sum $\sum A_s DE$ of the second duals $A_s DE$ of the polytopes A_s over the second dual SCF of S, and hence is isomorphic to $A = \sum_{s \in S} A_s$.

Corollary There is a full duality between the categories $\mathbf{PP}$ and $\widehat{\mathbf{PP}}$ given by the schizophrenic object I^{∞}.

Duality for quasi-polytopes

Let $\mathbf{QP}$ be the class of quasi-polytopes. The duality for for $\mathbf{QP}$ is given by

$$\widetilde{D} : \mathbf{QP} \to \widehat{\mathbf{QP}} \qquad \text{and} \qquad \widetilde{E} : \widehat{\mathbf{QP}} \to \mathbf{QP},$$

where $\widehat{\mathbf{QP}}$ is the class of corresponding representation spaces. It is based on the schizophrenic object $(I^o)^\infty$.

Theorem Let $A = \bigcup_{s \in S} A_s$ be a quasi-polytope. Then the barycentric algebra $A\widetilde{D}$ of homomorphisms from A to $(I^o)^\infty$ is isomorphic to the Płonka sum $\sum(A_s D^o)$ of the first duals $A_s D^o$ of the open polytopes A_s over the first dual SC of S.

Theorem Let $A = \bigcup_{s \in S} A_s$ be a quasi-polytope. Then the barycentric algebra $A\widetilde{D}\widetilde{E}$ of homomorphisms from $A\widetilde{D}$ to $(I^o)^\infty$ is isomorphic to the semilattice sum $\bigcup(A_s D^o E^o)$ of the second duals $A_s D^o E^o$ of the open polytopes A_s over the second dual SCF of S.

Main Theorem There is a full duality between the categories $\mathbf{QP}$ and $\widehat{\mathbf{QP}}$ given by the schizophrenic object $(I^o)^\infty$.