Finite characterizability of equational classes of threshold functions

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Minors

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f is obtained from g by

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Minors are also known as
simple minors,
subfunctions,
polymers,
$\dots$

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Examples of minor-closed classes:

- all clones
- monotone decreasing functions
- supermodular functions
- threshold functions
- ...

Relational constraints

A *relational constraint* is a pair (R, S) of relations $R \subseteq A^r$, $S \subseteq B^r$.

A function $f\colon A^n \to B$ *preserves* a relational constraint (R, S), denoted $f \triangleright (R, S)$, if for all $\mathbf{a}^1, \ldots, \mathbf{a}^n \in R$, we have $f(\mathbf{a}^1, \ldots, \mathbf{a}^n) \in S$.

Minors and relational constraints

The preservation relation induces a Galois connection between functions and relations.

For any set $\mathcal{Q}$ of relational constraints, and
for any set $\mathcal{F}$ of functions,

$$\mathrm{cPol}(\mathcal{Q}) = \{f \in \mathcal{F} : f \triangleright (R, S) \text{ for every } (R, S) \in \mathcal{Q}\},$$
$$\mathrm{cInv}(\mathcal{F}) = \{(R, S) \in \mathcal{Q} : f \triangleright (R, S) \text{ for every } f \in \mathcal{F}\}.$$

Theorem (Pippenger 2002)

The Galois closed classes of functions are exactly the classes closed under taking minors.

Minors and relational constraints

Theorem (Pippenger 2002; Ekin, Foldes, Hammer, Hellerstein 2000)

Let $\mathcal{F}$ be a class of functions. The following are equivalent.

1. *$\mathcal{F}$ is closed under taking minors.*
2. *$\mathcal{F}$ is characterizable by relational constraints.*
3. *$\mathcal{F}$ is of the form*

$$\text{forbid}(\mathcal{A}) := \{f : g \not\leq f \text{ for all } g \in \mathcal{A}\} = \overline{\uparrow\mathcal{A}},$$

for some antichain $\mathcal{A}$ (with respect to the minor relation $\leq$).

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   for some antichain $\mathcal{A}$ (with respect to the minor relation $\leq$).
4. *$\mathcal{F}$ is defined by functional equations of a certain type.*

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Minors and relational constraints

Theorem (Pippenger 2002; Ekin, Foldes, Hammer, Hellerstein 2000)

Let $\mathcal{F}$ be a class of functions. The following are equivalent.

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Threshold functions

A Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is *threshold* if there exist $w_1, \dots, w_n, t \in \mathbb{R}$ such that

$$f(x_1, \dots, x_n) = 1 \qquad \iff \qquad \sum_{i=1}^{n} w_i x_i \geq t.$$

In other words, f is threshold if there exists a hyperplane in $\mathbb{R}^n$ that separates the true points of f from the false points of f.

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Theorem (Hellerstein 2001)

The class of threshold functions, while characterizable by relational constraints, is not finitely characterizable.

Threshold functions

Problem

Is the class of "majority games", i.e., self-dual monotone threshold functions finitely characterizable by relational constraints?

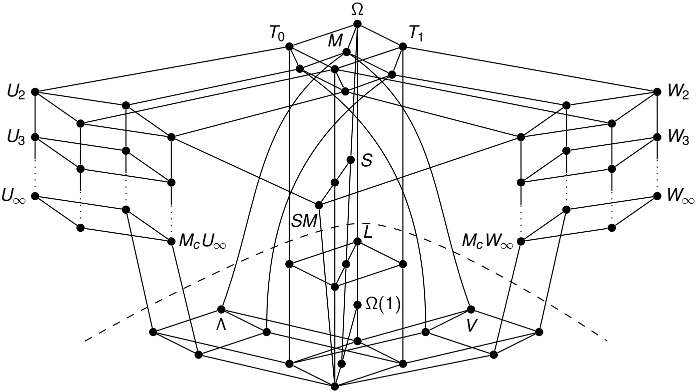
Threshold functions

$T =$ the class of threshold functions

Problem

Which clones C on $\{0, 1\}$ have the property that $C \cap T$ is finitely characterizable by relational constraints?

Post's Lattice and threshold functions



$$\Lambda \subseteq T, \qquad V \subseteq T, \qquad L \cap T = \Omega(1).$$

Threshold functions and asummability

A Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is *k-asummable* if for any $m \in \{2,\dots,k\}$ and for all $\mathbf{a}_1,\dots,\mathbf{a}_m \in f^{-1}(0)$ and $\mathbf{b}_1,\dots,\mathbf{b}_m \in f^{-1}(1)$, it holds that

$$\mathbf{a}_1 + \dots + \mathbf{a}_m \neq \mathbf{b}_1 + \dots + \mathbf{b}_m$$

(standard vector addition in $\mathbb{R}^n$).

A Boolean function f is *asummable* if it is k-asummable for all $k \geq 2$.

Theorem (Chow 1961, Elgot 1961, Muroga 1971)

A Boolean function is threshold if and only if it is asummable.

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Asummability and relational constraints

For $m \geq 1$, define the $2m$-ary relational constraint B_m as

$$R(B_m) := \{(x_1, \dots, x_{2m}) \in \{0,1\}^{2m} : \sum_{i=1}^{m} x_i = \sum_{i=m+1}^{2m} x_i\},$$

$$S(B_m) := \{0,1\}^{2m} \setminus \{(\underbrace{0, \dots, 0}_{m}, \underbrace{1, \dots, 1}_{m}), (\underbrace{1, \dots, 1}_{m}, \underbrace{0, \dots, 0}_{m})\}.$$

$f \triangleright B_m$ if and only if for all $\mathbf{a}_1, \dots, \mathbf{a}_m \in f^{-1}(0)$ and $\mathbf{b}_1, \dots, \mathbf{b}_m \in f^{-1}(1)$, it holds that

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Thus f is k-asummable iff $f \in \mathrm{cPol}(\mathcal{A}_k)$, where $\mathcal{A}_k := \{B_m : 2 \leq m \leq k\}$.

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Corollary

Let $f \colon \{0,1\}^n \to \{0,1\}$. The following are equivalent:

1. *f is a threshold function.*
2. *$f \in \bigcap_{k \geq 2} \mathrm{cPol}(\mathcal{A}_k)$.*
3. *$f \in \mathrm{cPol}(\{B_m : m \geq 2\})$.*

Asummability and relational constraints

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$\mathcal{A}_k \subseteq \mathcal{A}_k \cup \{B_{k+1}\} = \mathcal{A}_{k+1} \quad \Longrightarrow \quad \mathrm{cPol}(\mathcal{A}_{k+1}) \subseteq \mathrm{cPol}(\mathcal{A}_k)$
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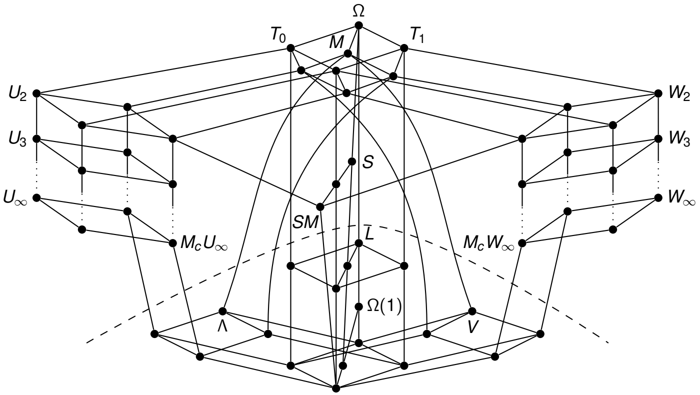
for all $k \geq 2$

Theorem (Taylor, Zwicker 1995)

For all $k \geq 2$, there exists a function that is k-asummable but not $(k+1)$-asummable, i.e., $\mathrm{cPol}(\mathcal{A}_{k+1}) \subsetneq \mathrm{cPol}(\mathcal{A}_k)$.

Finite characterizability of $C \cap T$



Theorem

Let C be a clone on $\{0, 1\}$. Then $C \cap T$ is finitely characterizable by relational constraints if and only if C is contained in one of the clones L, V, Λ.

Proof idea

Theorem (Taylor, Zwicker 1995)

For all $k \geq 2$, $\mathrm{cPol}(\mathcal{A}_{k+1}) \subsetneq \mathrm{cPol}(\mathcal{A}_k)$.

Thus, for every $k \geq 2$, there exists a function f_k such that $f_k \in \mathrm{cPol}(B_\ell)$ for all $\ell \in \{2, \ldots, k\}$ and $f_k \notin \mathrm{cPol}(B_{k+1})$.

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Lemma

Let $f \colon \{0,1\}^n \to \{0,1\}$, *and let* $C \in \{SM, M_cU_\infty, M_cW_\infty\}$. *There exists a Boolean function* $G_C(f)$ *that satisfies the following conditions:*

1. $G_c(f) \in C$,
2. *for all* $m \geq 2$, $G_C(f) \in \mathrm{cPol}\, B_m$ *if and only if* $f \in \mathrm{cPol}\, B_m$.

Thus, for every $k \geq 2$, there exists a function f_k^C such that
$f_k^C \in C$,
$f_k^C \in \mathrm{cPol}(B_\ell)$ for all $\ell \in \{2, \ldots, k\}$ and $f_k^C \notin \mathrm{cPol}(B_{k+1})$.

Proof idea

Suppose, on the contrary, that $C \cap T$ is finitely characterizable.

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Theorem (Pippenger; Ekin, Foldes, Hammer, Hellerstein)

Let $\mathcal{F}$ be a class of functions. The following are equivalent.

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Let $\mathcal{A}$ be a finite antichain such that $C \cap T = \mathrm{forbid}(\mathcal{A})$.
Each one of the functions f_k^C has a minor in $\mathcal{A}$.
Since $\mathcal{A}$ is finite, there is $g \in \mathcal{A}$ and an infinite set $S \subseteq \mathbb{N}$ such that $g \leq f_k^C$ for all $k \in S$.
Since g is not threshold, there exists $p \in \mathbb{N}$ such that $g \notin \mathrm{cPol}\, B_p$.
Since S is infinite, there is $q \in S$ such that $p \leq q$.
We have $g \leq f_q$ and $f_q \in \mathrm{cPol}\, B_p$.
Since $\mathrm{cPol}\, B_p$ is closed under taking minors, we have $g \in \mathrm{cPol}\, B_p$. $\lightning$

Constructions

Lemma

Let $f\colon \{0,1\}^n \to \{0,1\}$*, and let* $C \in \{SM, M_c U_\infty, M_c W_\infty\}$*. There exists a Boolean function* $G_C(f)$ *that satisfies the following conditions:*

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2. *for all* $m \geq 2$, $G_C(f) \in \operatorname{cPol} B_m$ *if and only if* $f \in \operatorname{cPol} B_m$.

Constructions

$f\colon \{0,1\}^n \to \{0,1\}$

$G_S(f)(x_1, \ldots, x_{n+1}) := (x_{n+1} \wedge f(x_1, \ldots, x_n)) \vee \overline{x}_{n+1} \wedge f^{\mathrm{d}}(x_1, \ldots, x_n))$

$G_{M_c}(f)\colon \{0,1\}^{2n} \to \{0,1\}$

- If $w(\mathbf{x}) < n$, then $G_{M_c}(f)(\mathbf{x}) = 0$.
- If $w(\mathbf{x}) > n$, then $G_{M_c}(f)(\mathbf{x}) = 1$.
- If $\mathbf{x} = (\mathbf{a}, \overline{\mathbf{a}})$, then $G_{M_c}(f)(\mathbf{x}) = f(\mathbf{a})$.
- If $w(\mathbf{x}) = n$ and there exists $i \in \{1, \ldots, n\}$ such that $x_i = x_{n+1}$ and $x_j \neq x_{n+j}$ for all $j < i$, then $G_{M_c}(f)(\mathbf{x}) = x_i$.

$G_{SM}(f) := G_{M_c}(G_S(f))$

$G_{U_\infty}(f)(x_1, \ldots, x_{n+1}) := x_{n+1} \wedge f(x_1, \ldots, x_n)$

$G_{M_c U_\infty}(f) := G_{U_\infty}(G_{M_c}(f))$

$G_{M_c W_\infty}(f) := G_{M_c U_\infty}(f)^{\mathrm{d}}$

The end

Děkuji.

Kiitos.

Merci.

Obrigado.

Thank you.