Homomorphisms to the Clone of Projections

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CNRS / LIX, Ecole Polytechnique

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New parts on joint work with Michael Pinsker and András Pongracz.

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  $\mathcal{C} = \mathrm{Pol}(\Gamma)$ for some relational structure Γ with domain D.
- with respect to this topology, composition in $\mathcal{D}$ is continuous.
- $\mathcal{C}$ is a topological clone: an abstract clone $\mathbf{C}$ (viewed as a multi-sorted algebra) together with a topology under which composition is continuous.

Clone Homomorphisms

Let $\mathcal{C}, \mathcal{D}$ be two clones. For $i \leq k \in \mathbb{N}$, write π_i^k for the i-th k-ary projection. Then a mapping ξ from $\mathcal{C}$ to $\mathcal{D}$ is a clone homomorphism if

- $\pi_i^k \in \mathcal{C}$ is mapped to $\pi_i^k \in \mathcal{D}$, and
- for all n-ary $f \in \mathcal{C}$ and all m-ary $g_1, \dots, g_k \in \mathcal{C}$:
  $\xi(f(g_1, \dots, g_k)) = \xi(f)(\xi(g_1), \dots, \xi(g_k))$

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Not the topic of this talk ...

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Open in general.

Constraint Satisfaction Problems

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Γ also called the template.

Definition 2 (CSP).

$\mathrm{CSP}(\Gamma)$ is the computational problem to decide whether a given finite τ-structure homomorphically maps to Γ.

Constraint Satisfaction Problems

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Definition 2 (CSP).

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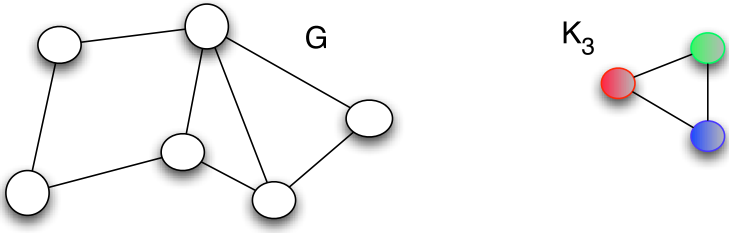
Example: 3-colorability is CSP(K_3) where $K_3 := (\{0, 1, 2\}; \neq)$



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Strongest evidence comes from the so-called universal algebraic approach.

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- Fraïssé-limits of classes of structures with finite relational signature are ω-categorical.
  These limits Γ are homogeneous: any isomorphism between finite substructures of Γ can be extended to an automorphism of Γ.

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Call $\mathcal{C}$ oligomorphic if the invertible elements of $\mathcal{C}^{(1)}$ form an oligomorphic permutation group.

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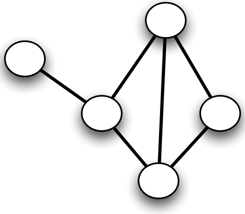

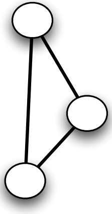

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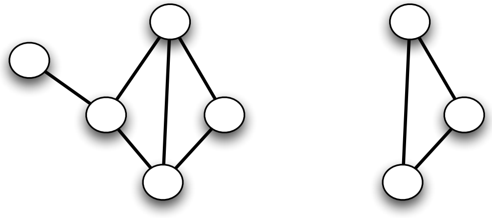

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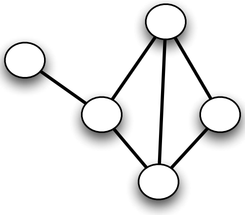

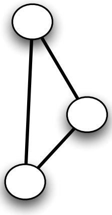

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- In a finite core Γ, all orbits of n-tuples are subalgebras of $\mathrm{Pol}(\Gamma)^n$.
- When Γ is a finite core, and $c_1, \ldots, c_n$ are elements of Γ, then $\mathrm{CSP}(\Gamma)$ and $\mathrm{CSP}(\Gamma, \{c_1\}, \ldots, \{c_n\})$ are Ptime-equivalent.

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For $c_1, \dots, c_n$ from Δ, $\mathrm{CSP}(\Delta)$ and $\mathrm{CSP}(\Delta, \{c_1\}, \dots, \{c_n\})$ are Ptime equivalent.

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$\mathsf{HSP}_{\mathsf{fin}}(\mathbf{A})$: pseudovariety generated by $\mathbf{A}$.

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Theorem.

Let Γ be an ω-categorical structure and $\mathbf{A}$ an algebra with $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\Gamma)$. Let Δ be a relational structure and $\mathbf{B} \in \mathrm{HSP}_{\mathrm{fin}}(\mathbf{A})$ such that $\mathrm{Clo}(\mathbf{B}) \subseteq \mathrm{Pol}(\Delta)$. Then there is a Ptime reduction from $\mathrm{CSP}(\Delta)$ to $\mathrm{CSP}(\Gamma)$.

Topological Birkhoff

Theorem (MB+Pinsker'13).

Let $\mathcal{C}$ and $\mathcal{D}$ be two closed oligomorphic subclones of $\mathcal{O}$. Equivalent:

- $\mathcal{C}$ and $\mathcal{D}$ are isomorphic via a homeomorphism;
- there are algebras $\mathbf{A}$, $\mathbf{B}$ such that $\mathcal{C} = \text{Clo}(\mathbf{A})$, $\mathcal{C} = \text{Clo}(\mathbf{B})$, and $\text{HSP}_{\text{fin}}(\mathbf{A}) = \text{HSP}_{\text{fin}}(\mathbf{B})$.

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Let $\mathcal{C}$ be a closed oligomorphic subclone of $\mathcal{O}$, and let $\mathcal{D}$ be a clone over a finite set. The following are equivalent:

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Conjecture 1 (Bulatov+Jeavons+Krokhin'05).

Let Γ be such that $\mathcal{C} := \mathrm{Pol}(\Gamma)$ satisfies the conditions from the previous theorem. Then CSP(Γ) is in P.

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Consecutives of the previous results:

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Is there also $\mathbf{B}' \in \text{HSP}_{\text{fin}}(\mathbf{A})$ such that $\text{Clo}(\mathbf{B}') = \mathbf{P}$?

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  Hence, there is no homomorphism from $\mathbf{A}$ to $\mathbf{P}$. $\square$

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Equivalently: there are algebras $\mathbf{A}$, $\mathbf{B}$ such that

- $\mathsf{Clo}(\mathbf{A})$ is oligomorphic,
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- $\mathbf{B} \in \mathsf{HSP}(\mathbf{A}) \setminus \mathsf{HSP}_{\mathsf{fin}}(\mathbf{A})$.

The Example

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3. Let Γ be the structure with the same domain as Δ that contains for all n the $3n$-ary relation S_2

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4. For each n, every $f \in \mathrm{Pol}(\Gamma)$ acts as an essentially unary projection $\mu_n(f)$ on the equivalence classes of E_n.
5. Let $\mathcal{F}$ be a non-principal ultrafilter on ω. Define $\xi\colon \mathrm{Pol}(\Gamma) \to \mathbf{P}$ by $\xi(f) = \pi^i_k$ if $U_f := \{n : \mu_n(f) = \pi^k_i\} \in \mathcal{F}$.

Canonical Clones

Let Γ a structure with domain D.

Definition

A function $f\colon D^n \to D$ is canonical (wrt. Γ) if for every $m \in \mathbb{N}$, all $t_1, \dots, t_n \in D^m$, and all $\alpha_1, \dots, \alpha_n \in \mathrm{Aut}(\Gamma)$ there exists a $\beta \in \mathrm{Aut}(\Gamma)$ such that

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Conjecture 2.

Suppose that Γ is an ω-categorical model-complete core. Then there exists a finite tuple t such that either Pol(Γ, t) has a homomorphism to $\mathbf{P}$, or Pol(Γ) contains a weak near unanimity modulo unary polymorphisms.

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Theorem.

There are oligomorphic clones $\mathcal{C}$ such that there is no homomorphism to $\mathbf{P}$, but $\mathcal{C}$ does not contain cyclic operations modulo unary operations in $\mathcal{C}$.

Conclusion

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permutation group	topological group	group
transformation monoid	topological monoid	monoid
function clone	topological clone	clone

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