Operator Properties of CP Varieties With Strongly Definable Principal Congruences

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Overview

Motivation

- Describe the partially ordered monoid of operators generated by the operators H, S and P_f for the variety $\mathcal{R}_c$ of commutative rings.

Operators on Classes of Algebras

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- The most common operators in universal algebra are

$$I,\ H,\ S,\ P,\ P_{fin},\ P_s,\ P_f,\ P_u.$$

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- We also compare operators using the equality relation and the partial ordering
  - $V = HSP$, $SH \leq HS$, $SHPS \leq HSP$ etc.
- These properties naturally raise the following questions: given a set of operators σ, how many essentially different operators can one get as composites of operators from σ and how are the composites ordered?

Partially Ordered Monoids

Definition

A *partially ordered monoid* (briefly *po-monoid*) is an ordered quadruple $\mathbf{M} = \langle M, \cdot, 1, \leq \rangle$ such that

- $\langle M,\ \cdot,\ 1\rangle$ is a monoid,
- $\langle M,\ \leq\rangle$ is a partially ordered set,
- $(x \leq y \quad \wedge \quad u \leq v) \rightarrow xu \leq yv$ for all $x, y, u, v \in M$.

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  - Pigozzi described the po-monoid generated by H, S and P
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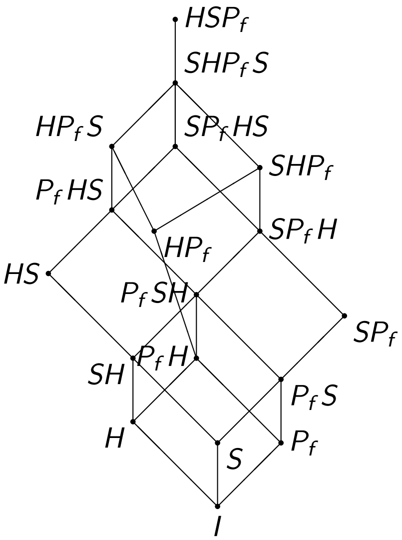
  - Tasić described the po-monoid generated by H, S, P and P_s

The Po-Monoid Generated by H, S and P_f

Theorem

The po-monoid $\mathcal{M}_f = (M_f, \cdot, I, \leq)$ generated by H, S and P_f has 18 elements and the corresponding ordering is given by the following diagram

The Po-Monoid Generated by H, S and P_f



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- $\mathcal{M}_f[\mathcal{V}]$ is a homomorphic image of $\mathcal{M}_f$.

The Po-Monoid Generated by H, S and P_f

Theorem

Let $\mathcal{V}$ be a variety. A necessary and sufficient condition for $\mathcal{M}_f[\mathcal{V}]$ to be isomorphic to $\mathcal{M}_f$ is that there exists classes $\mathcal{K}_1$, $\mathcal{K}_2 \subseteq \mathcal{V}$ satisfying:

$$HSP_f(\mathcal{K}_1) \not\subseteq SHP_fS(\mathcal{K}_1), \tag{1}$$

$$HP_f(\mathcal{K}_2) \not\subseteq SP_fHS(\mathcal{K}_2). \tag{2}$$

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- Using the equalities $HSP = HSP_f$ and $SHPS = SHP_fS$ we actually showed the non-inclusion (1).
- Unfortunately one cannot find a class $\mathcal{K} \subseteq \mathcal{R}_c$ that will show the non-inclusion (2).
- Namely, we will show that for every class $\mathcal{K} \subseteq \mathcal{R}_c$ we have $HP_f(\mathcal{K}) \subseteq SP_fHS(\mathcal{K})$, in fact, $HP_f \leq SP_fH$ when restricted to $\mathcal{R}_c$.

Motivation

- $\mathcal{R}_c$ is CP and for any $\mathbf{R} \in \mathcal{R}_c$, and a, b, c_1, $d_1, \ldots c_k$, $d_k \in R$ we have

$$(a, b) \in \mathrm{Cg}((c_1, d_1), \ldots, (c_k, d_k)) \leftrightarrow \exists e_1 \ldots \exists e_k (a - b = \sum_{i=1}^{k} e_i (c_i - d_i))$$

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- It will turn out that these two properties are the main reason why $\mathcal{R}_c$ satisfies $HP_f \leq SP_f H$.

Definitions

Definition

Let $\mathcal{F}$ be a type of algebras and $s_i(x, y, z_1, w_1, \ldots, z_k, w_k, \vec{u})$, $t_i(x, y, z_1, w_1, \ldots, z_k, w_k, \vec{u})$ be terms of type $\mathcal{F}$. A *k-finitely generated congruence formula* (k-FGCF) of the type $\mathcal{F}$ is a formula $\pi^k(x, y, z_1, w_1, \ldots, z_k, w_k)$ of the form

$$\exists \vec{u} \left(\bigwedge_{i=1}^{n} s_i(x, y, z_1, w_1, \ldots, z_k, w_k, \vec{u}) \approx t_i(x, y, z_1, w_1, \ldots, z_k, w_k, \vec{u}) \right),$$

such that for every algebra $\mathbf{A}$ of type $\mathcal{F}$ and every a, b, c_1, $d_1 \ldots$, c_k, $d_k \in A$ we have:

If $\mathbf{A} \models \pi^k(a, b, c_1, d_1, \ldots, c_k, d_k)$, then $(a, b) \in Cg^{\mathbf{A}}((c_1, d_1), \ldots, (c_k, d_k))$.

Definitions

Remark 1

For every algebra $\mathbf{A}$ of the type $\mathcal{F}$ and every a, b, c_1, $d_1, \ldots, c_k, d_k \in A$, if $(a, b) \in Cg^{\mathbf{A}}((c_1, d_1), \ldots, (c_k, d_k))$, then by Mal'cev's congruence generation theorem there exists such $\pi^k(x, y, z_1, w_1, \ldots, z_k, w_k)$, with $\mathbf{A} \models \pi^k(a, b, c_1, d_1, \ldots, c_k, d_k)$.

Remark 2

If Π_k denotes the set of all k-generated congruence formulas then we have:

$$''(x, y) \in Cg((z_1, w_1), \ldots (z_k, w_k))'' \leftrightarrow \bigvee_{\pi^k \in \Pi_k} \pi^k(x, y, z_1, w_1, \ldots, z_k, w_k).$$

Definitions

Definition

A variety $\mathcal{V}$ of the type $\mathcal{F}$ has *definable k-generated congruences* (k-DFGC), if $\bigvee_{\pi^k \in \Pi_k} \pi^k(x, y, z_1, w_1, \ldots, z_k, w_k)$ is equivalent in $\mathcal{V}$ to a finite disjunction

$$\pi^k_{i_1}(x, y, z_1, w_1, \ldots, z_k, w_k) \vee \cdots \vee \pi^k_{i_l}(x, y, z_1, w_1, \ldots, z_k, w_k).$$

We say that $\mathcal{V}$ has *strongly definable k-generated congruences* (k-SDFGC) if

$$\bigvee_{\pi^k \in \Pi_k} \pi^k(x, y, z_1, w_1, \ldots, z_k, w_k)$$

is equivalent in $\mathcal{V}$ to a single k-FGCF $\pi^k(x, y, z_1, w_1, \ldots, z_k, w_k)$.

Definitions

Remark 1

1-DFGC=DPC

Remark 2

k-DFGC implies l-DFGC for $1 \leq l \leq k$

Definitions

Definition

A variety $\mathcal{V}$ of type $\mathcal{F}$ has *definable finitely generated congruences* (shortly DFGC) if for every $k \geq 1$ $\mathcal{V}$ has definable k-generated congruences. $\mathcal{V}$ has *strongly definable finitely generated congruences* (shortly SDFGC) if for every $k \geq 1$ $\mathcal{V}$ has strongly definable k-generated congruences.

Example 1

The variety $\mathcal{R}_c$ of commutative rings with identity has SDFGC. Namely, given $k \geq 1$ we have:

$$(x, y) \in Cg((z_1, w_1), \ldots, (z_k, w_k)) \leftrightarrow \exists u_1 \ldots \exists u_k \left(x - y = \sum_{i=1}^{k} u_i(z_i - w_i)\right).$$

So, $\pi^k(x, y, z_1, w_1, \ldots, z_k, w_k) \equiv \exists u_1 \ldots \exists u_k \left(x - y \approx \sum_{i=1}^{k} u_i(z_i - w_i)\right)$.

Main Results

Theorem

Let $\mathcal{V}$ be a congruence permutable variety. Then $\mathcal{V}$ has (S)DPC if and only if $\mathcal{V}$ has (S)DFGC.

Remark 1

The previous Theorem remains true if congruence permutability is replaced by k-congruence permutability for $k \geq 3$.

Main Results

Theorem

Let $\mathcal{V}$ be a congruence permutable variety having SDPC. Then for every $\mathcal{K} \subseteq \mathcal{V}$ we have $HP_f(\mathcal{K}) \subseteq SP_f H(\mathcal{K})$.

Corollaries

Corollary

Let $\mathcal{V}$ be a congruence permutable variety having SDPC. Then for every $\mathcal{K} \subseteq \mathcal{V}$ we have:

1. $SHP_f(\mathcal{K}) = SP_f H(\mathcal{K})$, $SHP_f S(\mathcal{K}) = SP_f HS(\mathcal{K})$.
2. $Q(H(\mathcal{K})) = SHP(\mathcal{K})$.

Corollaries

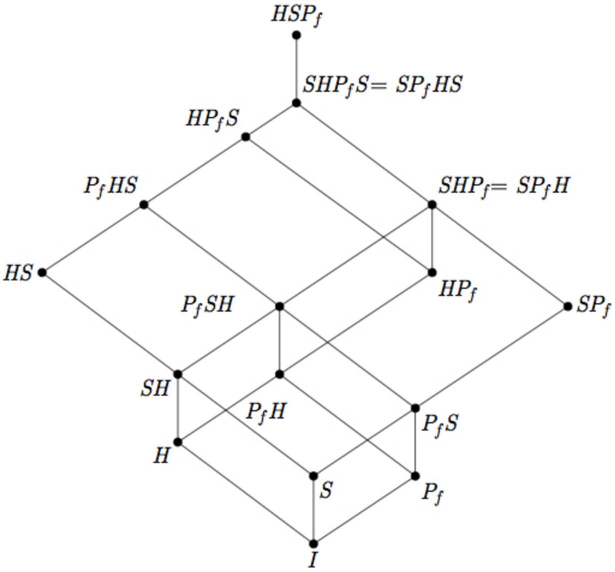
Corollary

Let $\mathcal{V}$ be a congruence permutable variety having SDPC and let $\mathcal{K} \subseteq \mathcal{V}$. Then $HSP(\mathcal{K}) = SHPS(\mathcal{K})$ if and only if for every quasi-identity $\left(\bigwedge_{i=1}^{n} p_i \approx q_i\right) \to p \approx q$ if $HS(\mathcal{K}) \vDash \left(\bigwedge_{i=1}^{n} p_i \approx q_i\right) \to p \approx q$ then $HSP(\mathcal{K}) \vDash \left(\bigwedge_{i=1}^{n} p_i \approx q_i\right) \to p \approx q$.

The structure of $\mathcal{M}_f[\mathcal{R}_c]$

Theorem

The standard monoid $\mathcal{M}_f[\mathcal{R}_c]$ of the variety of commutative rings with identity has 16 elements. The corresponding ordering is given by the following diagram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$$HSP_f(\mathcal{K}_1) \not\subseteq SHP_fS(\mathcal{K}_1), \tag{3}$$

$$HP_f(\mathcal{K}_2) \not\subseteq P_fHS(\mathcal{K}_2). \tag{4}$$

The structure of $\mathcal{M}_f[\mathcal{R}_c]$

Remark 1

The class $\mathcal{K} = \{\mathbf{Z}_8\}$ will simultaneously show both non-inclusions (3) and (4).

Remark 2

We want to show that $HP_f(\{\mathbf{Z}_8\}) \not\subseteq P_f HS(\{\mathbf{Z}_8\}) = P_f(\{\mathbf{Z}_8, \mathbf{Z}_4, \mathbf{Z}_2\})$.

The End