The structure of medial quandles

Přemysl Jedlička[1], Agata Pilitowska[2], David Stanovský[3], Anna Zamojska-Dzienio[2]

[1]Department of Mathematics, Faculty of Engineering, Czech University of Life Sciences,
[2]Faculty of Mathematics and Information Science, Warsaw University of Technology,
[3]Department of Algebra, Faculty of Mathematics and Physics, Charles University

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Quandles

A binary algebra $(Q, \cdot)$ is called a **quandle** if it is:

- **idempotent**: $xx = x$ for each $x \in Q$,
- **left distributive**: $x(yz) = (xy)(xz)$ for every $x, y, z \in Q$,
- a **left quasigroup**: the equation $xu = y$ has a unique solution $u \in Q$ for every $x, y \in Q$.

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A quandle Q is called **medial** if

$$(xy)(uv) = (xu)(yv)$$

for every $x, y, u, v \in Q$.

Affine quandles

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Example

Let A be an abelian group, k its endomorphism, and define an operation on the set A by

$$a * b = (1 - k)(a) + k(b).$$

The resulting algebra $\mathrm{Aff}(A, k) = (A, *)$ is called **affine** over the group A. It is idempotent and medial. If k is an automorphism then it is a medial quandle, called **affine quandle** over A.

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What is the role of affine quandles in the class of medial quandles?

Two important groups

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The (left) **multiplication group** of a quandle Q is the permutation group generated by left translations, i.e.,

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Both groups are normal subgroups of $\mathrm{Aut}(Q)$.

Two important groups

Proposition

Let Q be a quandle. Then

1. $\mathrm{Dis}(Q) = \{L_{a_1}^{k_1} \dots L_{a_n}^{k_n} : \ a_1, \dots, a_n \in Q$ *and* $\sum_{i=1}^{n} k_i = 0\}$;
2. *the natural actions of* $\mathrm{LMlt}(Q)$ *and* $\mathrm{Dis}(Q)$ *on* Q *have the same orbits;*
3. Q *is medial if and only if* $\mathrm{Dis}(Q)$ *is abelian.*

Orbit decomposition

Orbits of Q are the orbits of transitivity of the groups $\mathrm{LMlt}(Q)$ and $\mathrm{Dis}(Q)$ denoted

$$Qe = \{\alpha(e) \mid \alpha \in \mathrm{LMlt}(Q)\} = \{\alpha(e) \mid \alpha \in \mathrm{Dis}(Q)\},$$

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Let $\alpha(e), \beta(e) \in Qe$ with $\alpha, \beta \in \mathrm{Dis}(Q)$ and put

$$\alpha(e) + \beta(e) = \alpha\beta(e) \qquad \text{and} \qquad -\alpha(e) = \alpha^{-1}(e).$$

Then $\mathrm{Orb}_Q(e) = (Qe, +, -, e)$ is an abelian group, called the **orbit group** for Qe.

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Every orbit of a medial quandle is an affine quandle:

$$Qe = \mathrm{Aff}(\mathrm{Orb}_Q(e), L_e).$$

Construction

An **affine mesh** over a non-empty set I is a triple

$$\mathcal{A} = ((A_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I}, (c_{i,j})_{i,j\in I})$$

where A_i are abelian groups, $\varphi_{i,j} : A_i \to A_j$ homomorphisms, and $c_{i,j} \in A_j$ constants, satisfying the following conditions for every $i, j, j', k \in I$:

(M1) $1 - \varphi_{i,i}$ is an automorphism of A_i;

(M2) $c_{i,i} = 0$;

(M3) $\varphi_{j,k}\varphi_{i,j} = \varphi_{j',k}\varphi_{i,j'}$, i.e., the following diagram commutes:

$$\begin{tikzcd} A_i \arrow[r, "\varphi_{i,j}"] \arrow[d, "\varphi_{i,j'}"] & A_j \arrow[d, "\varphi_{j,k}"] \\ A_{j'} \arrow[r, "\varphi_{j',k}"] & A_k \end{tikzcd}$$

(M4) $\varphi_{j,k}(c_{i,j}) = \varphi_{k,k}(c_{i,k} - c_{j,k})$.

Construction

The **sum of an affine mesh** $((A_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I}, (c_{i,j})_{i,j\in I})$ over a set I is a binary algebra defined on the disjoint union of the sets A_i, with operation

$$a * b = c_{i,j} + \varphi_{i,j}(a) + (1 - \varphi_{j,j})(b)$$

for every $a \in A_i$ and $b \in A_j$.

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Lemma

The sum of an affine mesh is a medial quandle.

Representation theorem

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An affine mesh $((A_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I}, (c_{i,j})_{i,j\in I})$ over a set I is called **indecomposable** if

$$A_j = \left\langle \bigcup_{i\in I} (c_{i,j} + \mathrm{Im}(\varphi_{i,j})) \right\rangle,$$

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Theorem

A binary algebra is a medial quandle if and only if it is the sum of an indecomposable affine mesh. The orbits of the quandle coincide with the groups of the mesh.

Homologous meshes

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Two affine meshes $\mathcal{A} = ((A_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I}, (c_{i,j})_{i,j\in I})$ and $\mathcal{A}' = ((A'_i)_{i\in I}, (\varphi'_{i,j})_{i,j\in I}, (c'_{i,j})_{i,j\in I})$, over the same index set I, are **homologous**, if there is a permutation π of the set I, group isomorphisms $\psi_i : A_i \mapsto A'_{\pi i}$, and constants $d_i \in A'_{\pi i}$, such that, for every $i, j \in I$,

(H1) $\psi_j\varphi_{i,j} = \varphi'_{\pi i,\pi j}\psi_i$, i.e., the following diagram commutes:

$$\begin{tikzcd} A_i \arrow[r, "\varphi_{i,j}"] \arrow[d, "\psi_i"] & A_j \arrow[d, "\psi_j"] \\ A'_{\pi i} \arrow[r, "\varphi'_{\pi i,\pi j}"] & A'_{\pi j} \end{tikzcd}$$

(H2) $\psi_j(c_{i,j}) = c'_{\pi i,\pi j} + \varphi'_{\pi i,\pi j}(d_i) - \varphi'_{\pi j,\pi j}(d_j)$.

Isomorphism theorem

Theorem

Let $\mathcal{A} = ((A_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I}, (c_{i,j})_{i,j\in I})$ *and* $\mathcal{A}' = ((A'_i)_{i\in I}, (\varphi'_{i,j})_{i,j\in I}, (c'_{i,j})_{i,j\in I})$ *be two indecomposable affine meshes, over the same index set* I. *Then the sums of* $\mathcal{A}$ *and* $\mathcal{A}'$ *are isomorphic quandles if and only if the meshes* $\mathcal{A}$, $\mathcal{A}'$ *are homologous.*