Term-equivalent semigroups

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Indeed, $x - y = x + (n-1)y$ and $x + y = x - ((y - y) - y)$.

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Theorem (M. Behrisch, T. Waldhauser)

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- *The subsemigroups of $(A, \cdot)$ and $(A, +)$ are the same and term equivalent.*
- *The ideals of $(A, \cdot)$ and $(A, +)$ are the same.*
- *The idempotents of $(A, \cdot)$ and $(A, +)$ are the same.*
- *For every $a \in A$ that generates a finite semigroup $\langle a \rangle$, the index $m(a)$ and the period $n(a)$ of $\langle a \rangle$ are the same with respect to $\cdot$ and $+$.*

More observations

Lemma

Let $(S, \cdot)$ and $(S, +)$ be term equivalent semigroups. Assume that $+$ does not depend on both arguments. Then $xy = x + y$ for all $x, y \in S$ or $xy = y + x$ for all $x, y \in S$.

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Lemma

Let $(S, \cdot)$ and $(S, +)$ be term equivalent semigroups. Assume that for all $n \in \mathbb{N}$ there exists $a \in S$ such that $|\langle a \rangle| > n$ or a has index at least 3. Then $xy = x + y$ for all $x, y \in S$ or $xy = y + x$ for all $x, y \in S$.

By this lemma, two semigroups may be term equivalent without being identical or dual only if every element is contained in a group or has index 2. The order of every element is bounded from above by some fixed positive integer.

Nilpotent semigroups

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Proof. Let $s(x, y)$ be a term over $(S, +)$, let $t(x, y)$ be a term over $(S, \cdot)$ such that

$$xy = s(x, y) \text{ and } x + y = t(x, y) \text{ for all } x, y \in S.$$

Assume that lengths of s and t are at least 3. Then we obtain

$$S^2 \subseteq S + S \subseteq S^3 \subseteq S^2$$

Thus $S^2 = S^3$ and consequently $S^2 = S^n$ for every $n \geqslant 3$. Since $(S, \cdot)$ is nilpotent, it follows that $S^2 = 0$. Thus $(S, \cdot)$ and $(S, +)$ are identical 0-semigroups.

Commutative semigroups

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Corollary

Let $(S, \cdot), (S, +)$ *be term equivalent semigroups. Then for every* $k \in \mathbb{N}$ *and for every* $x \in S$ *we have* $x^k = kx$*.*

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If a semigroup $(S,\cdot)$ is term equivalent to a commutative semigroup $(S,+)$, then both are identical.

Corollary

Let $(S,\cdot),(S,+)$ be term equivalent semigroups. Then for every $k \in \mathbb{N}$ and for every $x \in S$ we have $x^k = kx$.

Proof. Monogenic subsemigroups are commutative.

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1. *Assume* $(S,\cdot)$ *has a zero* 0. *Then* 0 *is also a zero with respect to* $*$.
2. *Assume* $(S,\cdot)$ *has an identity* 1. *Then* 1 *is also an identity with respect to* $*$, *and* $(S,\cdot)$ *and* $(S,*)$ *are identical, dual, or both completely regular (i.e. are union of groups).*

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3. *Assume $(S, \cdot)$ is finitely generated. Then $(S, *)$ is finitely generated.*

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4. *Assume* $(S,\cdot)$ *is regular. Then* $(S,*)$ *is regular.*

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3. *Assume $(S,\cdot)$ is finitely generated. Then $(S,*)$ is finitely generated.*
4. *Assume $(S,\cdot)$ is regular. Then $(S,*)$ is regular.*
5. *Assume $(S,\cdot)$ is inverse. Then $(S,*)$ is inverse.*

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Let $(S, \cdot), (S, *)$ be term equivalent semigroups. Then we have:

1. *Assume $(S, \cdot)$ has a zero 0. Then 0 is also a zero with respect to $*$.*
2. *Assume $(S, \cdot)$ has an identity 1. Then 1 is also an identity with respect to $*$, and $(S, \cdot)$ and $(S, *)$ are identical, dual, or both completely regular (i.e. are union of groups).*
3. *Assume $(S, \cdot)$ is finitely generated. Then $(S, *)$ is finitely generated.*
4. *Assume $(S, \cdot)$ is regular. Then $(S, *)$ is regular.*
5. *Assume $(S, \cdot)$ is inverse. Then $(S, *)$ is inverse.*
6. *Assume $(S, \cdot)$ is a group. Then $(S, *)$ is a group.*

Term-equivalent semigroups that are not completely regular.

From the examples of term-equivalent groups we can obtain semigroups that are not regular and still term equivalent in a nontrivial way.

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Example

Let $(G, \cdot), (G, *)$ be finite term equivalent, non-isomorphic groups with identity 1. Let a be distinct from the elements of G, and extend the operation $\circ \in \{\cdot, *\}$ to $S := G \cup \{a\}$ as follows:

$$\begin{aligned} x \circ a &= a \circ x = x \quad \forall x \in G, \\ a \circ a &= 1. \end{aligned}$$

Then $(S, \cdot), (S, *)$ are term equivalent semigroups that are not regular, not isomorphic, and not dual to each other.

Completely regular semigroups

Problem

Let $(S, *), (S, \cdot)$ be term equivalent completely regular semigroups such that $*$ and $\cdot$ are identical on all subgroups. Are $(S, *)$ and $(S, \cdot)$ identical or dual to each other?

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Generally, the answer to this question is negative.

Example

Let G be a finite non-commutative group. Let $S = (G \times \mathbb{Z}_2)$ be an universe of semigroups $(S, \cdot)$ and $(S, *)$, where

$$(g, a) \cdot (h, b) := (gh, b), \qquad (g, a) * (h, b) := (gh, a).$$

Semigroups $(S, \cdot)$ and $(S, *)$ are term-equivalent, since

$$x \cdot y = y^{|G|} * x * y, \qquad x * y = x * y * x^{|G|}.$$

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Proposition

Let $(S, *), (S, \cdot)$ be completely regular and term equivalent inverse semigroups. Assume the operations are identical on all subgroups. Then $(S, *)$ and $(S, \cdot)$ are identical.

Thank you for your attention!

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