TECHNISCHE UNIVERSITÄT WIEN
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Centraliser clones on finite domains[1]

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2nd July 2014
Prague, Czech Republic

[1]Supported by the Austrian Science Fund (FWF) under grant I836-N23

Outline

Why centraliser clones?

CSP, again!

$\underset{\sim}{\mathbf{T}} = \langle T; Q\rangle$ finite rel structure, of finite signature, template.
$\mathrm{CSP}(\underset{\sim}{\mathbf{T}})$ is the decision problem $\exists$ hom. $h\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{T}}$? for finite $\underset{\sim}{\mathbf{A}}$ of the same signature.

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$\mathrm{CSP}\,(\underset{\sim}{\mathbf{T}})$ is always in $\mathtt{NP}$, some CSP's are in $\mathtt{P}$.

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CSP dichotomy conjecture

Feder/Vardi: $|T| = 2 \Rightarrow$ either $\mathrm{CSP}\,(\underset{\sim}{\mathbf{T}})$ in P or NP-complete.
Conjecture: This extends to all finite domains.

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Connection to centraliser clones

Feder/Madelaine/Stewart | Broniek: It suffices to establish the dichotomy conjecture for $\underset{\sim}{\mathbf{T}}$, where $\operatorname{Pol}\underset{\sim}{\mathbf{T}}$ is a centraliser clone.

Outline

Operations and their graphs

Finitary operations

- For $k \in \mathbb{N}_+$ a func $f\colon A^k \longrightarrow A$ is a k-ary operation on A
- $\mathrm{O}_A^{(k)} := A^{A^k}$ set of k-ary operations on A
- $\mathrm{O}_A := \bigcup_{k\in\mathbb{N}_+} \mathrm{O}_A^{(k)}$ set of all finitary operations on A

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Graphs

For $f \in \mathrm{O}_A^{(n)}, F \subseteq \mathrm{O}_A$

$$f^\bullet := \left\{ (x_1, \ldots, x_n, x_{n+1}) \in A^{n+1} \;\middle|\; x_{n+1} = f(x_1, \ldots, x_n) \right\}$$
$$= \left\{ (x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \;\middle|\; (x_1, \ldots, x_n) \in A^n \right\}$$
$$F^\bullet := \left\{ f^\bullet \;\middle|\; f \in F \right\}$$

Basis of the Galois connection

Commutation

For $n, m \in \mathbb{N}_+$, $f \in O_A^{(n)}$, $g \in O_A^{(m)}$,

$$f \text{ commutes with } g \;:\Longleftrightarrow\; f \perp g \;:\Longleftrightarrow\; f \rhd g^\bullet$$

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Lemma

For $n, m \in \mathbb{N}_+$, $f \in \mathrm{O}_A^{(n)}$, $g \in \mathrm{O}_A^{(m)}$, we have

$$f \perp g \iff f\colon \langle A; g\rangle^n \longrightarrow \langle A; g\rangle \text{ is a homomorphism.}$$

A more concrete characterisation of commutation

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$$f \perp g \iff \forall\, X = (x_{i,j})_{\substack{1 \le i \le m \\ 1 \le j \le n}} \in A^{m \times n}:$$

$$\begin{matrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{matrix}$$

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$$\begin{array}{ccc}
g & & g \\
\frown & & \frown \\
x_{1,1} & \cdots & x_{1,n} \\
\vdots & \ddots & \vdots \\
x_{m,1} & \cdots & x_{m,n} \\
\smile & & \smile \\
= & & = \\
g\,(x\,(\cdot,1)) & \cdots & g\,(x\,(\cdot,n))
\end{array}$$

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$$\begin{matrix}
\overbrace{\underbrace{\begin{matrix} x_{1,1} \\ \vdots \\ x_{m,1} \end{matrix}}_{=\; g(x(\cdot,1))}}^{g} & \begin{matrix} \cdots \\ \ddots \\ \cdots \end{matrix} & \overbrace{\underbrace{\begin{matrix} x_{1,n} \\ \vdots \\ x_{m,n} \end{matrix}}_{=\; g(x(\cdot,n))}}^{g} \\
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$$
\begin{array}{ccccccc}
 & g & & g & & & \\
f(& x_{1,1} & \cdots & x_{1,n} &) & = & f\left(x\left(1,\cdot\right)\right) \\
 & \vdots & \ddots & \vdots & & & \vdots \\
f(& x_{m,1} & \cdots & x_{m,n} &) & = & f\left(x\left(m,\cdot\right)\right) \\
 & = & & = & & & \\
f(& g\left(x\left(\cdot,1\right)\right) & \cdots & g\left(x\left(\cdot,n\right)\right) &) & = & y \\
 & \in g^{\bullet} & & \in g^{\bullet} & & &
\end{array}
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 & \smile & & \smile & & & & \smile\\
 & = & & = & & & & =
\end{array}$$

$$f\left(g\left(x\left(\cdot,1\right)\right)\cdots g\left(x\left(\cdot,n\right)\right)\right) \color{red}{=} g\left(f\left(x\left(1,\cdot\right)\right),\ldots,f\left(x\left(m,\cdot\right)\right)\right)$$

$$\in g^{\bullet} \qquad\qquad \in g^{\bullet}$$

Commutativity is symmetric

Corollary

For $f, g \in \mathrm{O}_A$ we have $f \perp g \iff g \perp f$.

Galois correspondence induced by commutation

Definition

For $F \subseteq \mathrm{O}_A$ we put

$$F^* := \{ g \in \mathrm{O}_A \mid \forall f \in F \colon g \perp f \} \qquad \text{(centraliser of } F\text{)}$$

$$F^{**} \qquad \text{(bicentraliser of } F\text{, bicentrical closure of } F\text{)}$$

$$\mathcal{C}_A := \{ F^* \mid F \subseteq \mathrm{O}_A \} \qquad \text{(lattice of centraliser clones)}$$

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Alternatively

For $F \subseteq \mathrm{O}_A$ we have

$$F^* = \mathrm{Pol}_A F^{\bullet} \qquad \text{(centralisers are clones!)}$$

$$= \{f \in \mathrm{O}_A \mid f^{\bullet} \in \mathrm{Inv}_A F\}$$

Comparison with clone closure

Observation

Centralisers
$=$ usual $\mathrm{Pol} - \mathrm{Inv}$ where relations are confined to graphs

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For $F \subseteq \mathrm{O}_A$ we have $F^* = \langle F\rangle^*_{\mathrm{O}_A}$.

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Lemma

For $F \subseteq \mathrm{O}_A$ we have $F^* = \langle F \rangle_{\mathrm{O}_A}^*$.

Proof.

$g \in F^* \iff g^\bullet \in \mathrm{Inv}_A F = \mathrm{Inv}_A \langle F \rangle_{\mathrm{O}_A} \iff g \in \langle F \rangle_{\mathrm{O}_A}^*$. □

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For $F \subseteq \mathrm{O}_A$ it is $F^{**} = \langle F\rangle^{*\,*}_{\mathrm{O}_A}$, i.e. $\langle F\rangle_{\mathrm{O}_A} \subseteq F^{**}$.

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For $F \subseteq \mathrm{O}_A$ *we have* $F^* = \langle F \rangle_{\mathrm{O}_A}^*$.

Corollary

For $F \subseteq \mathrm{O}_A$ *it is* $F^{**} = {\langle F \rangle_{\mathrm{O}_A}^*}^*$, *i.e.* $\langle F \rangle_{\mathrm{O}_A} \subseteq F^{**}$.

Proof.

$F^* = \langle F \rangle_{\mathrm{O}_A}^* \implies F^{**} = {\langle F \rangle_{\mathrm{O}_A}^*}^* \supseteq \langle F \rangle_{\mathrm{O}_A}$. □

Outline

Finitely many centralisers

Theorem (R. Pöschel, unpublished)

$|A| = k$ *finite* $\implies \forall\, F \subseteq \mathrm{O}_A\ \exists\, Q_F \subseteq \mathrm{R}_A^{(4)}$ *4-ary rel's:*

$$F^* = \left(\langle F \rangle_{\mathrm{O}_A}^{(k)}\right)^* \cap \operatorname{Pol}_A Q_F.$$

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Corollary (S. Burris/R. D. Willard, 1987, different proof)

$|A| = k$ *finite* $\implies |\mathcal{C}_A|$ *finite.*

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$$\begin{array}{rcl} \mathrm{decomp}\colon \quad \mathcal{C}_A & \longrightarrow & \mathfrak{P}\left(\mathrm{O}_A^{(k)}\right) \times \mathfrak{P}\left(\mathfrak{P}\left(A^4\right)\right) \\ \Sigma & \longmapsto & \left(\Sigma^{*(k)}, Q_{\Sigma^*}\right) \end{array}$$ is inj., as

$$\left(\Sigma^{*(k)}\right)^* \cap \operatorname{Pol}_A Q_{\Sigma^*} = \left(\langle \Sigma^* \rangle_{\mathrm{O}_A}^{(k)}\right)^* \cap \operatorname{Pol}_A Q_{\Sigma^*} = \Sigma^{**} = \Sigma.$$

Special cases

$|A| = 2$, А. В. Кузнецов, 1977

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We don't know, but we would like to.

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- $\forall F \in \mathcal{C}_A \setminus \{\bot\}\ \exists A \in \mathcal{C}_A$ atom: $A \subseteq F$.
- $\forall F \in \mathcal{C}_A \setminus \{\top\}\ \exists C \in \mathcal{C}_A$ co-atom: $F \subseteq C$.

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- char of $\bigcap$-irreducibles? (not easy)
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Данильченко's method

Describe a generator for each $\bigcap$-irreducible $F \in \mathcal{C}_A$

$\top$ and $\bot$ of $\mathcal{C}_A$

Lemma

We have $\mathrm{O}_A^* = \mathrm{J}_A$.

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Proof.

W.l.o.g. $|A| \geq 2$.

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W.l.o.g. $|A| \geq 2$.

$f \in \mathrm{O}_A^* \implies f^\bullet \in \mathrm{Inv}_A \mathrm{O}_A = \mathrm{Diag}_A \implies \exists \varepsilon\colon f^\bullet = d_\varepsilon$

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$f \notin \mathrm{J}_A \implies \varepsilon = \Delta$, i.e. $f^\bullet = \nabla \lightning$, no graph. $\square$

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Corollary

For $F \subseteq \mathrm{O}_A$ *it is* $F^* = \mathrm{O}_A \iff F \subseteq \mathrm{J}_A$.

Roadmap

- Co-atoms are special $\bigcap$-irreducibles
- Co-atoms and atoms are related
- Atoms are related to minimal clones

Outline

1. A Galois theory based on commutation
2. Structure of the lattice of centralisers
3. Minimal/maximal centralisers

Parametrically pre-complete sets

Lemma

For $F \subseteq \mathrm{O}_A$ tfae:

$$F^{**} \subsetneq \mathrm{O}_A \qquad \forall G \supsetneq F: \qquad G^{**} = \mathrm{O}_A \tag{1}$$

$$\mathrm{J}_A \subsetneq F^{*} \qquad \forall G \supsetneq F: \qquad G^{*} = \mathrm{J}_A \tag{2}$$

$$\mathrm{J}_A \subsetneq F^{*} \qquad \forall g \in F^{*} \setminus \mathrm{J}_A: \qquad F = g^{*} \tag{3}$$

Parametrically pre-complete sets

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For $F \subseteq \mathrm{O}_A$ tfae:

$$
\begin{aligned}
F^{**} &\subsetneq \mathrm{O}_A & \forall G \supsetneq F:&& G^{**} &= \mathrm{O}_A && (1)\\
\mathrm{J}_A &\subsetneq F^{*} & \forall G \supsetneq F:&& G^{*} &= \mathrm{J}_A && (2)\\
\mathrm{J}_A &\subsetneq F^{*} & \forall g \in F^{*} \setminus \mathrm{J}_A:&& F &= g^{*} && (3)
\end{aligned}
$$

Corollary

$F \subseteq \mathrm{O}_A$ *param. pre-complete* $\implies F = f^{*}$ *for some* $f \in F^{*} \setminus \mathrm{J}_A$.

Minimal functions

Proposition

$F \subseteq \mathrm{O}_A$ *parametrically pre-complete*
$\implies \exists M$ *minimal clone*: $F = M^*$
$\implies \exists f \in \mathrm{O}_A^{(\leq \max(3,|A|))}$ *minimal function*: $F = f^*$.

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Proof.

- $F^* \supsetneq \mathrm{J}_A \implies F^* \supseteq M \supsetneq \mathrm{J}_A$
  for some minimal clone $M = \langle f \rangle_{\mathrm{O}_A}$ generated by one of Rosenberg's minimal functions.

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Proof.

- $F^* \supsetneq \mathrm{J}_A \implies F^* \supseteq M \supsetneq \mathrm{J}_A$ for some minimal clone $M = \langle f \rangle_{\mathrm{O}_A}$ generated by one of Rosenberg's minimal functions.
- Equivalently: $F \subseteq M^* = \langle f \rangle_{\mathrm{O}_A}^* = f^* \subsetneq \mathrm{O}_A$.

Minimal functions

Proposition

$F \subseteq \mathrm{O}_A$ *parametrically pre-complete*
$\implies \exists M$ *minimal clone*: $F = M^*$
$\implies \exists f \in \mathrm{O}_A^{(\leq \max(3,|A|))}$ *minimal function*: $F = f^*$.

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  for some minimal clone $M = \langle f \rangle_{\mathrm{O}_A}$ generated by one of Rosenberg's minimal functions.
- Equivalently: $F \subseteq M^* = \langle f \rangle_{\mathrm{O}_A}^* = f^* \subsetneq \mathrm{O}_A$.
- Since $M^* \supseteq F$ and $(M^*)^{**} = M^* \neq \mathrm{O}_A$,
  $\implies F = M^* = f^*$. $\square$

Unary case

Unary minimal functions from Rosenberg's theorem

$f \neq \mathrm{id}_A$ and ($f \circ f = f$ or $f^p = \mathrm{id}_A$ for some prime p)

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All of these unary functions generate indeed minimal clones.

Question

Which of these yield maximal centralisers?

Almost all

Theorem (W. Harnau, 1974)

f^* is a co-atom in $\mathcal{V}_A := \left\{ g^* \;\middle|\; g \in \mathrm{O}_A^{(1)} \right\}$ if and only if

1. $f \neq \mathrm{id}_A$ *and* $f \circ f = f$, *or*
2. $f^p = \mathrm{id}_A$ *for some prime* p *and*
   f *does not have exactly one fixed point*

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$$f \in \mathrm{O}_A^{(1)} \implies f^{**} \subseteq \left\langle \mathrm{O}_A^{(1)} \right\rangle_{\mathrm{O}_A}$$

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Proposition

$$f \in \mathrm{O}_A^{(1)} \implies f^{**} \subseteq \left\langle \mathrm{O}_A^{(1)} \right\rangle_{\mathrm{O}_A}$$

Corollary

All functions from Harnau's theorem yield maximal centralisers (minimal bicentralisers).

Future work

- consider other classes of minimal functions
- exploit characterisation of maximal clones
- better algorithms for checking commutation condition
- automatic computation of bicentrical closure up to a certain arity
- better upper bounds for the minimal arity needed to $**$-generate centraliser clones