Entropicity and generalized entropic property in idempotent n-semigroups

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Definition

Two operations $f\colon A^n \to A$ and $g\colon A^m \to A$ *commute*, if

$$\begin{aligned} & g(f(a_{11}, \dots, a_{1n}), \dots, f(a_{m1}, \dots, a_{mn})) \\ = {} & f(g(a_{11}, \dots, a_{m1}), \dots, g(a_{1n}, \dots, a_{mn})), \end{aligned}$$

for all $a_{11}, \dots, a_{nm} \in A$.

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A variety $\mathcal{V}$ is entropic, if every algebra in $\mathcal{V}$ is entropic.

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A groupoid $(G,\cdot)$ is entropic if it satisfies the identity

$$(xy)\cdot(zu) \approx (xz)\cdot(yu).$$

Endomorphism closure property

Definition

An algebra (A, F) has the *endomorphism closure property* (ECP), if the set $End(A)$ of all endomorphisms of (A, F) is closed under induced fundamental operations of (A, F), i.e. for any $f \in F$, and all $\varphi_1, \ldots, \varphi_n \in End(A)$,

$$f(\varphi_1, \ldots, \varphi_n) \in End(A).$$

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A variety $\mathcal{V}$ has the endomorphism closure property, if all algebras from $\mathcal{V}$ have EPC.

Theorem [T. Evans; L. Klukovits]

A variety $\mathcal{V}$ has ECP iff it is entropic.

Generalized entropic property

Definition

An algebra (A, F) has the *generalized entropic property* if, for every n-ary $f \in F$ and every m-ary $g \in F$, there exist m-ary terms $t_1, \dots, t_n$ of (A, F) such that (A, F) satisfies the identity

$$g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1m}, \dots, x_{nm}))$$
$$\approx f(t_1(x_{11}, \dots, x_{1m}), \dots, t_n(x_{n1}, \dots, x_{nm})).$$

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A variety $\mathcal{V}$ satisfies the generalized entropic property, if every algebra in $\mathcal{V}$ satisfies the gep.

A groupoid $(G, \cdot)$ has the generalized entropic property if there are two binary terms t and s such that $(G, \cdot)$ satisfies the identity

$$(xy) \cdot (zu) \approx t(x, z) \cdot s(y, u)$$

Subalgebra closure property

Definition

An algebra (A, F) has the *subalgebra closure property* (SCP), if the set $Sub(A)$ of all (non-empty) subalgebras of (A, F) is closed under complex fundamental operations of (A, F), i.e. for any $f \in F$, and all $A_1, \ldots, A_n \in Sub(A)$,

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A variety $\mathcal{V}$ has the subalgebra closure property, if all algebras from $\mathcal{V}$ have SPC.

Theorem [T. Evans; K. Adaricheva, A.P., D. Stanovský]

A variety $\mathcal{V}$ has SCP iff it has the generalized entropic property.

Relationship between entropicity and GEP

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Conjecture

Every idempotent algebra (A, f) with only one at least binary operation, with the generalized entropic property is entropic.

Definition

An algebra (A, f) with one n-ary operation f is called an *n-semigroup*, if the operation $f\colon A^n \to A$ is associative: the following associative laws hold

$$f(f(a_1, \ldots, a_n), a_{n+1}, \ldots, a_{2n-1}) = \cdots =$$

$$f(a_1, \ldots, a_r, f(a_{r+1}, \ldots, a_{r+n}), a_{r+n+1}, \ldots, a_{2n-1}) = \cdots =$$

$$f(a_1, \ldots, a_{n-1}, f(a_n, \ldots, a_{2n-1})),$$

for all $a_1, \ldots, a_{2n-1} \in A$.

Non-idempotent n-semigroups

Theorem [E. Lehtonen, A.P.]

For every integer $n \geq 2$, there exists non-idempotent n-semigroup that has the generalized entropic property but is not entropic.

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$n \geq 2$, $W = \{w \in \{1, \dots, n^2\}^* : 1 \leq |w| < n^2\}$, $A = W \cup \{\top, \bot\}$,
$\mathbf{c} = (1, 2, \dots, n^2)$, $\sigma \in S_{n^2}$
$f \colon A^n \to A$:

$$f(w_1, \dots, w_n) = \begin{cases} w_1 \cdots w_n, & \text{if } w_1, \dots, w_n \in W \text{ and } |w_1 \cdots w_n| < n^2, \\ \top, & \text{if } w_1, \dots, w_n \in W,\ |w_1 \cdots w_n| = n^2, \text{ and} \\ & \quad w_1 \cdots w_n = \mathbf{c}\sigma^{\mathbf{k}} \text{ for some } k \in \mathbb{N}, \\ \bot, & \text{otherwise.} \end{cases}$$

Theorem [K. Adaricheva, A.P., D. Stanovský]

If an idempotent semigroup (a band) satisfies the gep then it is entropic.

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Theorem [E. Lehtonen, A.P.]

Let $2 \leq n \in \mathbb{N}$ and $(S; \cdot)$ be a semigroup which satisfies the identity

$$x^n \approx x.$$

Then $(S; \cdot)$ has the generalized entropic property if and only if it is entropic.

n-semigroups derived from semigroups

$2 \leq n \in \mathbb{N}$

n-semigroup (S, f) is *derived from a semigroup* $(S, \cdot)$, if for any $x_1, \ldots, x_n \in S$,

$$f(x_1, \ldots, x_n) = x_1 \cdot \ldots \cdot x_n.$$

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Theorem [E. Lehtonen, A.P.]

Let (S, f) be an idempotent n-semigroup derived from a semigroup $(S, \cdot)$. Then the following statements are equivalent:

- (S, f) is entropic
- (S, f) has the generalized entropic property
- $(S, \cdot)$ is entropic
- $(S, \cdot)$ has the generalized entropic property

Mal'cev n-semigroups

$n \geq 3$, n-ary semigroup (S, f) that satisfies the identities

$$f(x, y, \dots, y) \approx x,$$
$$f(y, \dots, y, x) \approx x$$

is called Mal'cev n-semigroup with Mal'cev term

$$p_f(x, y, z) := f(x, y, \dots, y, z)$$

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Theorem [E. Lehtonen, A.P.]

For $n \geq 3$, a Mal'cev n-semigroup has the generalized entropic property if and only if it is entropic.

Semiabelian n-semigroups

An n-ary semigroup (S, f) is called *semiabelian*, if

$$f(a_1, \ldots, a_i, \ldots, a_n) = f(a_n, \ldots, a_i, \ldots, a_1)$$

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Theorem [W. Dörnte]

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Theorem [W. Dörnte]

A semiabelian n-semigroup is entropic.

Theorem [E. Lehtonen, A.P.]

An n-ary Mal'cev semigroup is entropic iff it is semiabelian.

Theorem

The free ternary Mal'cev semigroup on two generators is isomorphic to the algebra $(2\mathbb{Z}+1; f)$, where $2\mathbb{Z}+1$ denotes the set of odd integers and $f(x, y, z) = x - y + z$.

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Corollary

The free ternary Mal'cev semigroup on two generators is entropic.

Thank you for your attention

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It is time for a dinner!!