

# On automatic homeomorphicity for topological monoids

Maja Pech

Institute of Algebra  
TU Dresden  
Germany

Department of Mathematics and Informatics  
University of Novi Sad  
Serbia

02.07.2014

*joint work with Christian Pech*

# An intriguing question

How much information about an  $\omega$ -categorical structure can be recovered from its endomorphism monoid?

In particular, when does the endomorphism monoid of an  $\omega$ -categorical structure have automatic homeomorphicity?

# Automatic homeomorphicity

Given:

- $\mathbf{M} \leq \mathbf{T}_A$  – a closed transformation monoid and
- $\mathcal{K}$  – a set of closed transformation monoids on  $A$ .

$\mathbf{M}$  has **automatic homeomorphicity with respect to  $\mathcal{K}$**  if every monoid-isomorphism from  $\mathbf{M}$  to an element of  $\mathcal{K}$  is a homeomorphism.

$\mathbf{M}$  has **automatic homeomorphicity** if it has automatic homeomorphicity with respect to the set of all closed submonoids of  $\mathbf{T}_A$ .

# Examples

## Monoids with automatic homeomorphicity

- Monoid of injective selfmappings of  $A$
- Full transformation monoid on  $A$
- Monoid of selfembeddings of the Rado graph

## Monoids with automatic homeomorphicity with respect to $\mathcal{K}$

- Monoid of selfembeddings of countable universal homogeneous tournament

# Strong gate coverings

Given:

- $A$  – a countably infinite set,
- $\mathcal{M} \subseteq T_A$  – a transformation monoid,
- $G$  – the set of units in  $\mathcal{M}$  and
- $\overline{G}$  – the closure of  $G$  in  $\mathcal{M}$ .

A **strong gate covering** of  $\mathcal{M}$  is an open covering  $\mathcal{U}$  of  $\mathcal{M}$  with elements  $f_U \in U$ , for every  $U \in \mathcal{U}$ , such that

- for all  $U \in \mathcal{U}$  and
- for all Cauchy-sequences  $(g_n)_{n \in \mathbb{N}}$  in  $U$

there exist Cauchy-sequences  $(\kappa_n)_{n \in \mathbb{N}}$  und  $(\iota_n)_{n \in \mathbb{N}}$  in  $\overline{G}$  such that

$$\forall n \in \mathbb{N} : \quad g_n = \kappa_n \circ f_U \circ \iota_n.$$

# Automatic homeomorphicity and strong gate coverings

## Proposition

Let  $\mathbf{A}$  and  $\mathbf{B}$  be two countable relational structures such that

- $\text{End}(\mathbf{A})$  has a strong gate covering.

Let  $h : \text{End}(\mathbf{A}) \rightarrow \text{End}(\mathbf{B})$  be a monoid-isomorphism such that

- $h$  is open and
- $h|_{\text{Aut}(\mathbf{A})}$  is continuous.

Then  $h$  is continuous. In particular,  $h$  is a homeomorphism.

# Can we recognize when monoid-isomorphisms are open?

## Proposition

Let  $\mathbf{A}$  and  $\mathbf{B}$  be two relational structures such that

- $\text{End}(\mathbf{A})$  contains all constant functions and
- $\text{Aut}(\mathbf{B})$  is oligomorphic.

Then every monoid-isomorphism  $h : \text{End}(\mathbf{A}) \rightarrow \text{End}(\mathbf{B})$  is open.

# When are group-isomorphisms homeomorphisms?

## Small index property

Let  $A$  be a countable set and let  $\mathbf{G} \leq \mathbf{S}_A$  be a closed subgroup. Then  $\mathbf{G}$  has the **small index property** if every subgroup of index less than  $2^{\aleph_0}$  in  $\mathbf{G}$  is open.

## Small index property and automatic homeomorphicity

If  $\mathbf{G} \leq \mathbf{S}_A$  has a small index property, then it has automatic homeomorphicity.

## Weak $\forall\exists$ interpretations and automatic homeomorphicity

If an oligomorphic closed subgroup of  $\mathbf{S}_A$  has a weak  $\forall\exists$  interpretation, then it has automatic homeomorphicity with respect to the closed oligomorphic subgroups of  $\mathbf{S}_A$ .



# Examples

## Rationals

$\text{Aut}(\mathbb{Q}, \leq)$  has the small index property.

## Generic poset

$\text{Aut}(\mathbb{P}, \leq)$  has a weak  $\forall\exists$  interpretation.

It is not known whether it has the small index property.

# How to obtain strong gate coverings?

## Use universal homogeneous endomorphisms!

Let  $\mathbf{U}$  be a countable structure.

### Universal endomorphism

$u \in \text{End}(\mathbf{U})$  is **universal** if for every  $\mathbf{A} \in \overline{\text{Age}(\mathbf{U})}$  and every  $h : \mathbf{A} \rightarrow \mathbf{U}$  there exists an embedding  $\iota : \mathbf{A} \hookrightarrow \mathbf{U}$  with  $h = u \circ \iota$ .

### Homogeneous endomorphism

$u \in \text{End}(\mathbf{U})$  is **homogeneous** if for

- $\forall \mathbf{A} \in \text{Age}(\mathbf{U})$ ,
- $\forall h : \mathbf{A} \rightarrow \mathbf{U}$  and
- $\forall \iota_1, \iota_2 : \mathbf{A} \hookrightarrow \mathbf{U}$  with  $h = u \circ \iota_1 = u \circ \iota_2$ ,

there exists an  $f \in \text{Aut}(\mathbf{U})$  with

$$f \circ \iota_1 = \iota_2 \text{ and } u \circ f = u$$

# How to obtain strong gate coverings?

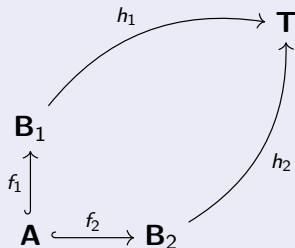
Use universal homogeneous endomorphisms! (cont.)

## Proposition

If  $\mathbf{U}$  is a countable infinite relational structure that has a universal homogeneous endomorphism, then  $\text{End}(\mathbf{U})$  has a strong gate covering.

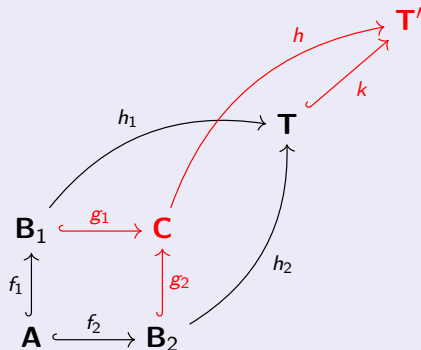
# Structures with a universal homogeneous endomorphism

## Amalgamated extension property (AEP)



# Structures with a universal homogeneous endomorphism

## Amalgamated extension property (AEP)



# Structures with a universal homogeneous endomorphism (cont.)

## Homoalmagamation property (HAP)

$$\begin{array}{ccc} & \mathbf{B} & \\ \uparrow g & & \\ \mathbf{A} & \xrightarrow{a} & \mathbf{T}_1. \end{array}$$

# Structures with a universal homogeneous endomorphism (cont.)

## Homoalmagamation property (HAP)

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{b} & \mathbf{T}_2 \\ \uparrow g & & \uparrow h \\ \mathbf{A} & \xrightarrow{a} & \mathbf{T}_1 \end{array}$$

# Structures with a universal homogeneous endomorphism (cont.)

## Homoalmagamation property (HAP)

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{b} & \mathbf{T}_2 \\ \uparrow g & & \uparrow h \\ \mathbf{A} & \xrightarrow{a} & \mathbf{T}_1 \end{array}$$

## Proposition

## Existence of UHE

Let  $\mathbf{U}$  be a countably infinite homogeneous structure. Then the following are equivalent:

- 1  $\mathbf{U}$  has a universal homogeneous endomorphism.
- 2  $\text{Age}(\mathbf{U})$  has the AEP and the HAP.



# Automatic homeomorphicity for $\text{End}(\mathbb{Q}, \leq)$ and $\text{End}(\mathbb{P}, \leq)$

## Theorem 1

Let  $\mathbf{B}$  be a countable  $\omega$ -categorical structure, and let

$$h : \text{End}(\mathbb{Q}, \leq) \rightarrow \text{End}(\mathbf{B})$$

be a monoid isomorphism. Then  $h$  is a homeomorphism.

## Theorem 2

Let  $\mathbf{B}$  be a countable  $\omega$ -categorical structure, and let

$$h : \text{End}(\mathbb{P}, \leq) \rightarrow \text{End}(\mathbf{B})$$

be a monoid isomorphism. Then  $h$  is a homeomorphism.