The structure of polynomial operations associated with smooth digraphs

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Simply smooth digraphs

Digraph G

- smooth: every vertex has both ingoing and and outgoing edges
- simply smooth: smooth, connected, and there is a closed path that has one more forward edges than backward edges

Fact:
connected reflexive digraphs $\subseteq$ simply smooth digraphs

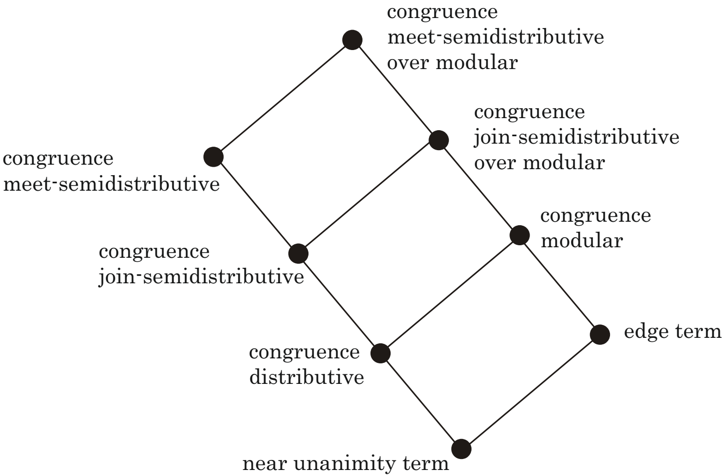
Algebras associated with digraphs

Algebra A associated with a digraph G:

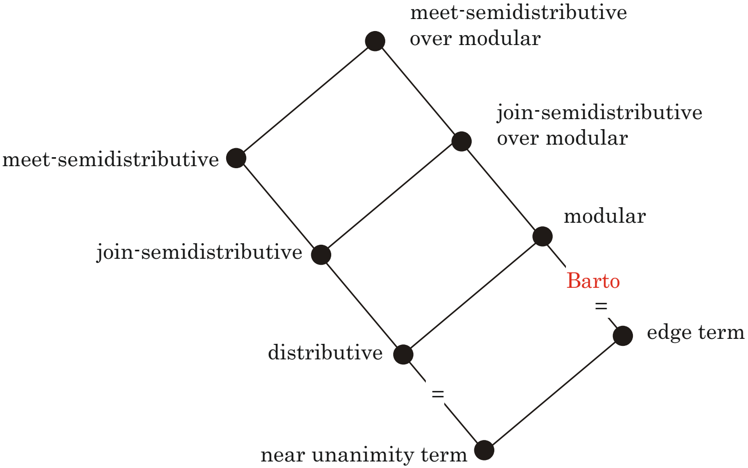
- underlying set: vertex set of G
- operations: the polymorphisms of G

Goal: investigate the relationship between digraph properties and varietal properties of varieties generated by algebras associated with digraphs

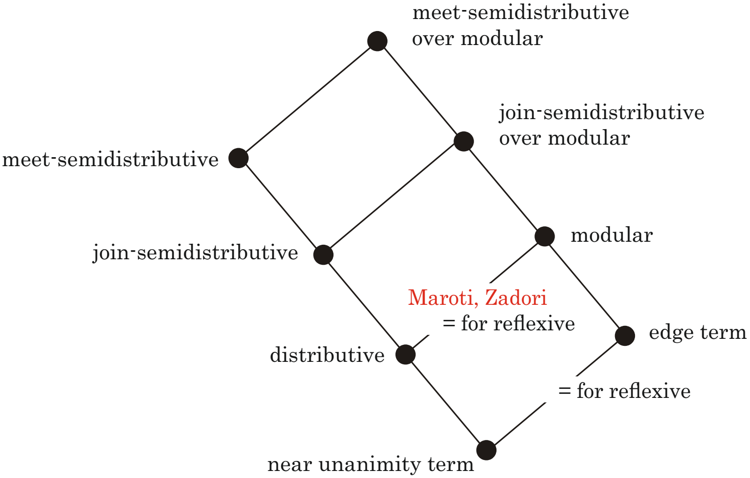
Some classes of locally finite varieties



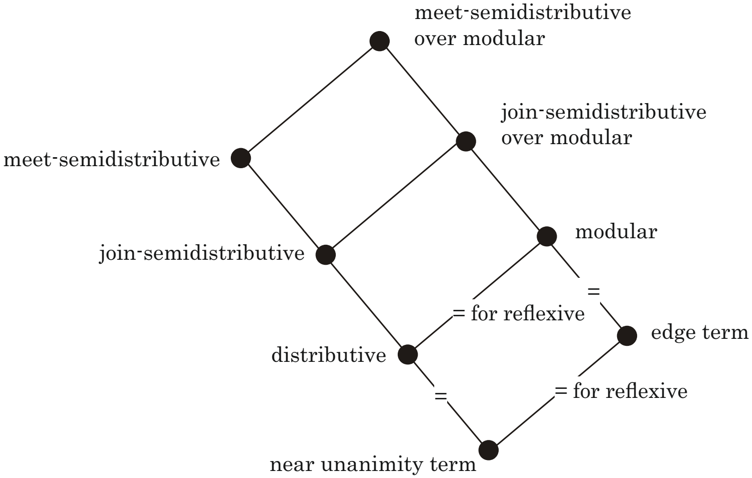
Classes of varieties generated by algebras associated with digraphs



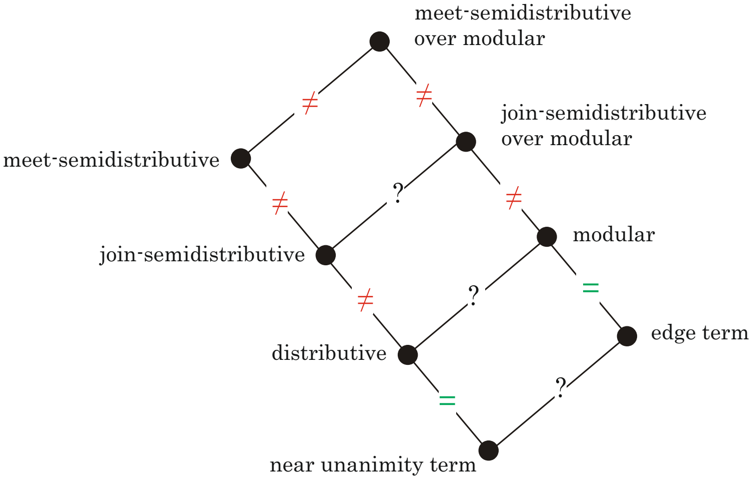
Classes of varieties generated by algebras associated with digraphs



Classes of varieties generated by algebras associated with digraphs



Some classes of varieties generated by algebras associated with simply smooth digraphs



Pol(A) as a subalgebra A^A

A is a finite algebra

Pol(A): the set of unary polynomial operations of A.

Pol(A) is a subalgebra of A^A.

The twin-congruence on Pol(A)

Twin relation on Pol(A): two polynomials f and g are twins, if there exist an $(n+1)$-ary term t and $a, b \in A^n$ such that $f(x) = t(x, a)$ and $g(x) = t(x, b)$.

The twin congruence τ on Pol(A): the transitive closure of the twin relation.

Some algebraic theorems

V is a locally finite variety

Theorem 1. If V is congruence join-semidistributive, then $\tau = 1$ for any finite algebra in V. If V is *idempotent*, the converse is also true.

Theorem 2. If $V(A)$ is congruence join-semidistributive over modular, then $\mathrm{Pol}(A)/\tau$ is a solvable algebra.

Pol(A) as a digraph

For a digraph G, G^G is a digraph

- vertices: all unary operations on the vertex set of G
- edges: $f \to g$ iff for all $a \to b$, $f(a) \to g(b)$

If A is an algebra associated with a digraph G, then Pol(A) induces a subdigraph of G^G.

Some properties of τ

Lemma: If G is a simply smooth digraph, then each of the τ-blocks induces a connected subdigraph of $\mathrm{Pol}(A)$.

Proposition: If G is a simply smooth digraph and the τ-block of the identity contains a non-permutation in $\mathrm{Pol}(A)$, then it contains a non-permutation r such that r is an endomorphism of G and $r^2 = r$.

The main result

Main Theorem. If G is a simply smooth digraph and $V(A)$ is congruence join-semidistributive over modular, then $\tau = 1$.

Proof: apply Theorems 1 and 2, the Proposition, some results of Hobby and McKenzie, a result of Kiss and one of Kazda.

Digraph properties of Pol(A)

Fact: If G is a simply smooth digraph, then Pol(A) has simply smooth connected components, but, in general, is not connected.

Corollary 1. If G is a simply smooth digraph and $V(A)$ is congruence join-semidistributive over modular, then Pol(A) is connected.

Maroti, Zadori (2012):
If G is a reflexive digraph and $V(A)$ is congruence modular, then $V(A)$ is congruence distributive.

A special case of the Loop Lemma

Barto, Kozik, Niven (2009):
The Loop Lemma. If G is a simply smooth digraph and $V(A)$ is congruence meet-semidistributive over modular, then G has a loop edge.

Corollary 2. If G is a simply smooth digraph and $V(A)$ is congruence join-semidistributive over modular, then G has a loop edge.