Weighted Clones and Valued CSPs

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EPSRC
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Motivation

$$\text{CSPs} \quad \Leftrightarrow \quad \text{Clones}$$

[*Jeavons et al. JACM*'97, *Jeavons TCS*'98, *Bulatov et al. SICOMP*'05]

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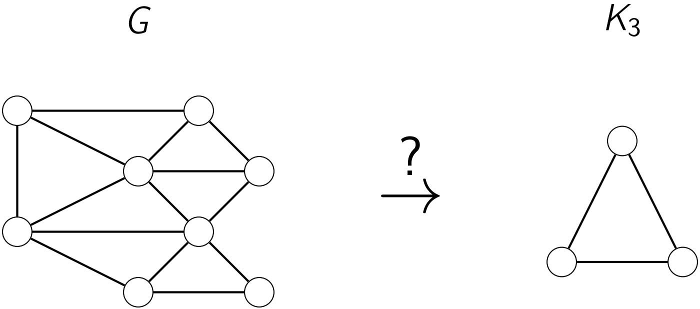
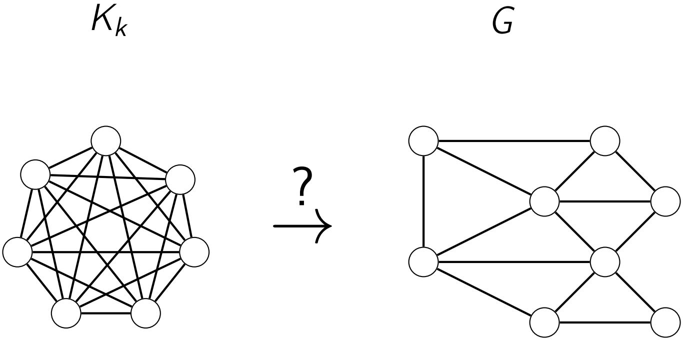
3-Colouring



k-Clique



Constraint Satisfaction Problems

$$\mathsf{CSP}(\mathcal{A}, \mathcal{B})$$

Given two classes $\mathcal{A}$ and $\mathcal{B}$ of relational structures and $\mathbf{A} \in \mathcal{A}$, $\mathbf{B} \in \mathcal{B}$, decide whether $\mathbf{A} \to \mathbf{B}$.

Non-Uniform CSP$(\mathcal{A},\mathcal{B})$: $\mathcal{A} = \mathsf{all}, \mathcal{B} = \{\Gamma\}$

CSP(Γ) $=$ CSP$(-, \{\Gamma\})$, finite relational structure Γ

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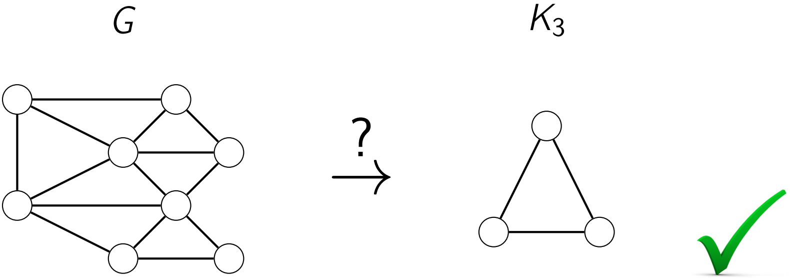

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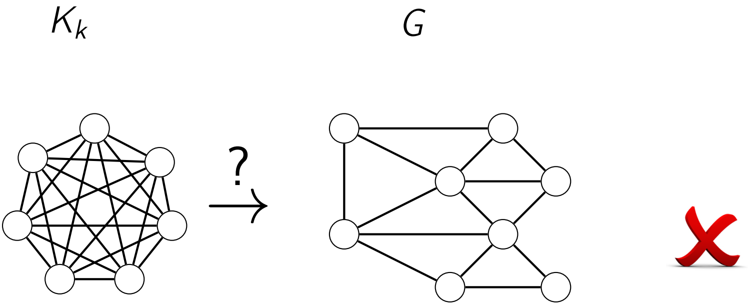

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Complexity of CSP(Γ)

1. CSP(Γ') $\equiv_p$ CSP(Γ)
   where Γ' = RelClone(Γ) is the closure of Γ under $\exists, \wedge, =$
2. relational clones RelClone(Γ)
   $\approx$ clones Pol(Γ)
3. properties of Pol(Γ)

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3. properties of Pol(Γ)

Example: Pol(Γ) contains a Mal'cev operation $\Rightarrow$ CSP(Γ) tractable

[*Bulatov & Dalmau SICOMP'*06]

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This Talk

Valued CSPs $\Leftrightarrow$ Weighted Clones

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Let Γ be a finite set of functions $\phi : D^m \to \overline{\mathbb{Q}}$. An instance of VCSP($\Gamma$) is an optimisation problem of the form

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where $x_{i,j} \in \{x_1,\ldots,x_n\}$ and all functions $\phi_i \in \Gamma$.

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possibly different m for different $\phi \in \Gamma$

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Complexity of VCSP(Γ) for all possible Γ!

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Monotone NAE 3-SAT

$$\min_{x_1,\ldots,x_n \in \{T,F\}} \sum_{(x_i,x_j,x_k)\in C} \phi_{\mathrm{nae}}(x_i,x_j,x_k)$$

x	y	z	$\phi_{\mathrm{nae}}(x,y,z)$
T	T	T	∞
F	F	F	∞
*	*	*	0

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where $x_{i,j} \in \{x_1,\ldots,x_n\}$ and all functions $\phi_i \in \Gamma$.

(s,t)-Min-Cut

$$\min_{x_1,\ldots,x_n \in \{a,b\}} \sum_{(x_i,x_j)\in E(G)} \phi_=(x_i,x_j)$$

$$\text{s.t.} \quad x_s = a, x_t = b$$

x	y	$\phi_=(x,y)$
a	a	0
a	b	1
b	a	1
b	b	0

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Max-Cut

$$\min_{x_1,\ldots,x_n \in \{a,b\}} \sum_{(x_i,x_j)\in E(G)} \phi_{\neq}(x_i,x_j)$$

x	y	$\phi_{\neq}(x,y)$
a	a	1
a	b	0
b	a	0
b	b	1

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Vertex Cover

$$\min_{x_1,\ldots,x_n \in \{0,1\}} \left(\sum_{(x_i,x_j)\in E(G)} \phi_{\mathrm{vc}}(x_i,x_j) + \sum_{x_i \in V(G)} \eta(x_i) \right)$$

x	$\eta(x)$
0	0
1	1

x	y	$\phi_{\mathrm{vc}}(x,y)$
0	0	∞
*	*	0

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- $\{0,\infty\}$: CSPs (feasibility)

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Approximation: strict CSPs=CSPs, CSPs=Max-CSPs, Finite-Valued CSPs=Generalised CSPs

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Outline

$$\begin{aligned} \text{operations} &\quad\rightarrow\quad \text{weightings} \\ \text{clones} &\quad\rightarrow\quad \text{weighted clones} \\ \text{polymorphisms} &\quad\rightarrow\quad \text{weighted polymorphisms} \end{aligned}$$

Operations and Clones

A k-ary operation on D is a mapping $f : D^k \to D$.

A clone C on D is a set of operations on D closed under superposition and containing all projections.

Weightings

A k-ary weighting of a clone C is a function $\omega : C^{(k)} \to \mathbb{Q}$ such that $\omega(f) < 0$ only if f is a projection and

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- $\omega(f) = \omega_1(f) + \omega_2(f)$ (addition)
- $\omega[g_1, \ldots, g_k](f') = \sum_{\substack{f \in C^{(k)} \\ f[g_1,\ldots,g_k]=f'}} \omega(f)$ (superposition)

Superposition Example

- D totally ordered

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- 4-ary weighting ω

$$\omega(f) = \begin{cases} -1 & \text{if } f \in \{\, \mathsf{e}_1^{(4)}, \mathsf{e}_2^{(4)}, \mathsf{e}_3^{(4)}, \mathsf{e}_4^{(4)} \} \\ +1 & \text{if } f \in \{\, \max_{12}^{(4)}, \min_{12}^{(4)}, \max_{34}^{(4)}, \min_{34}^{(4)} \} \\ 0 & \text{o/w} \end{cases}$$

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$$\begin{aligned} \omega &= (f_1, \ldots, f_{|C^{(k)}|}) \\ \omega[g_1, \ldots, g_k] &= (h_1, \ldots, h_{|C^{(\ell)}|}) \end{aligned}$$

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A k-ary weighting of a clone C is a function $\omega : C^{(k)} \to \mathbb{Q}$ such that $\omega(f) < 0$ only if f is a projection and

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Lemma: Improper superpositions can be eliminated!

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Structure of Weighted Clones!

This Talk

Valued CSPs $\Leftrightarrow$ Weighted Clones

[*Cohen, Cooper, Creed, Jeavons,* ***Ž****. SICOMP'13*]

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Polymorphisms

Let $\phi : D^m \to \overline{\mathbb{Q}}$ and let $\mathrm{Feas}(\phi) = \{\mathbf{x} \in D^m \mid \phi(\mathbf{x}) \text{ is finite}\}$. An operation $f : D^k \to D$ is a polymorphism of ϕ if, for any $\mathbf{x_1}, \ldots, \mathbf{x_k} \in \mathrm{Feas}(\phi)$ we have $f(\mathbf{x_1}, \ldots, \mathbf{x_k}) \in \mathrm{Feas}(\phi)$.

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$$\omega_{sub}(\min) = \omega_{sub}(\max) = +1 \text{ and } \omega_{sub}(\mathrm{e}_1^{(2)}) = \omega_{sub}(\mathrm{e}_2^{(2)}) = -1$$

Weighted Polymorphisms

Let $\phi : D^m \to \overline{\mathbb{Q}}$ be a function and let $C \subseteq \mathrm{Pol}(\phi)$ be a clone of operations. A k-ary weighting $\omega : C^{(k)} \to \mathbb{Q}$ $(\sum_{f \in C^{(k)}} \omega(f) = 0, \omega(f) < 0$ only if f a projection$)$ is a k-ary weighted polymorphism of ϕ if for any $\mathbf{x_1}, \ldots, \mathbf{x_k} \in \mathrm{Feas}(\phi)$

$$\sum_{f \in C^{(k)}} \omega(f)\phi(f(\mathbf{x_1}, \ldots, \mathbf{x_k})) \;\leq\; 0\,.$$

Equivalently, a probability distribution μ over $C^{(k)} \subseteq \mathrm{Pol}^{(k)}(\phi)$

$$\mathbb{E}_{f \sim \mu}[\phi(f(\mathbf{x_1}, \ldots, \mathbf{x_k}))] \;\leq\; \mathrm{avg}\{\phi(\mathbf{x_1}), \ldots, \phi(\mathbf{x_k})\}.$$

$\phi : D^m \to \overline{\mathbb{Q}}$ is submodular if for all $\mathbf{x_1}, \mathbf{x_2} \in D^m$,

$$\phi(\min(\mathbf{x_1}, \mathbf{x_2})) + \phi(\max(\mathbf{x_1}, \mathbf{x_2})) - \phi(\mathbf{x_1}) - \phi(\mathbf{x_2}) \;\leq\; 0$$

$$\omega_{sub}(\min) = \omega_{sub}(\max) = +1 \text{ and } \omega_{sub}(e_1^{(2)}) = \omega_{sub}(e_2^{(2)}) = -1$$

$$\mu_{sub}(\min) = \mu_{sub}(\max) = \frac{1}{2}$$

This Talk

Valued CSPs $\Leftrightarrow$ Weighted Clones

[*Cohen, Cooper, Creed, Jeavons, **Ž**. SICOMP'13*]

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Galois Connection

wRelClone(Γ) closure of Γ under scaling, addition, min

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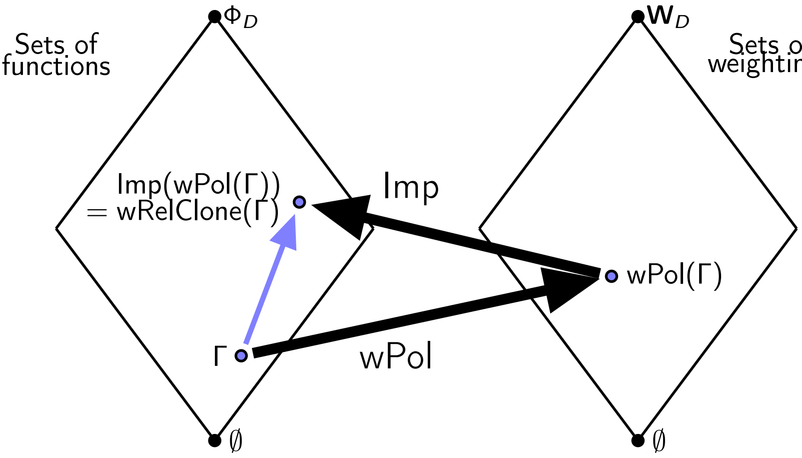
Galois Connection

$\mathrm{wRelClone}(\Gamma)$	closure of Γ under scaling, addition, min
$\mathrm{wClone}(W)$	smallest weighted clone containing W
$\mathrm{wPol}(\Gamma)$	weighted polymorphisms of Γ
$\mathrm{Imp}(W)$	functions ϕ with $\omega \in \mathrm{wPol}(\phi)$ for all $\omega \in W$

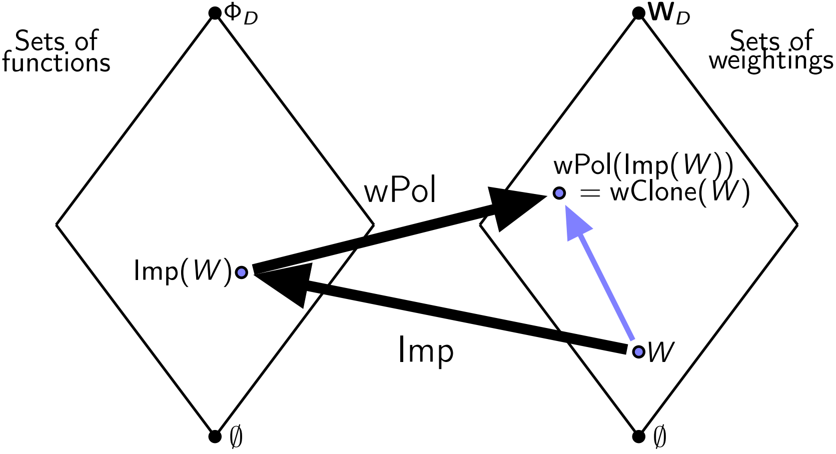
Theorem [*Cohen, Cooper, Creed, Jeavons, Ž. SICOMP'13*]

For any finite Γ, $\quad \mathrm{Imp}(\mathrm{wPol}(\Gamma)) = \mathrm{wRelClone}(\Gamma)$.
For any finite W, $\mathrm{wPol}(\mathrm{Imp}(W)) = \mathrm{wClone}(W)$.

Galois Connection in Picture 1



Galois Connection in Picture 2



Complexity of VCSP(Γ)

1. VCSP(Γ') $\equiv_p$ VCSP(Γ),
   Γ' = wRelClone(Γ) closure of Γ under $+$, scaling, min
2. weighted relational clones wRelClone(Γ)
   $\approx$ weighted clones wPol(Γ)
3. properties of wPol(Γ)

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Example: wPol(Γ) contains ω_{sub} $\Rightarrow$ VCSP(Γ) tractable

[*Iwata et al. JACM*'01, *Schrijver JCTB*'00]

Tractability

A weighted clone W with support clone C is tractable if VCSP($\mathrm{Imp}(W)$) belongs to PTIME.

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Easy to show:

- If $C = J_D$ (projections on D) then either $W = W_C$ or $W = W_C^0$, both are NP-hard.
- If $W = W_C^0$ for some C then W is NP-hard.

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Consequently, unless W is NP-hard W contains a nontrivial weighting ω, i..e, ω assigns a positive weight to a non-projection.

Idempotency

Wlog we can restrict to weighted clones W which are:

- surjective
  for every unary $\omega \in W$, $\omega(f) > 0 \Rightarrow f$ bijection [*Thapper & **Ž**. '14*]

Idempotency

Wlog we can restrict to weighted clones W which are:

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  for every unary $\omega \in W$, $\omega(f) > 0 \Rightarrow f$ bijection [*Thapper & **Ž**. '14*]
- idempotent
  for every $\omega \in W$, $\omega(f) > 0 \Rightarrow f$ idempotent [*Ochremiak '14*]
  (thus, the only $\omega(f) > 0$ in unary $\omega \in W$ are projections)

Necessary Conditions for Tractability

Theorem [*Creed & Ž. CP'11/SICOMP'13*]

Any weighted clone W containing a nontrivial weighting contains a weighting that assigns positive weight to either:

1. A set of unary operations that are not projections; or
2. A set of binary idempotent operations that are not projections; or
3. A set of ternary operations that are majority operations, minority operations, Pixley operations or semiprojections; or
4. A set of k-ary semiprojections (for some $k > 3$).

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Proof: follows the proof of Rosenberg's classification.

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(1) Pixley/semi out, (2) interplay of majorities and minorities.

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Proof: Gordan's Theorem (duality of linear programming).

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Easy: not possible/sufficient for tractability!

Sufficient Conditions for Tractability

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Easy: sufficient for tractability as $\mathrm{Imp}(W)$ contains only relations!

Sufficient Conditions for Tractability

Only near-unanimity operations/only edge operations!

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special case (2 maj & 1 minor) tractable [*Kolmogorov & **Ž**. JACM'13*]

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Holds true for any support clone $C \neq J_D$!

Necessary Conditions for Finite-Valued Tractability

Theorem [*Thapper & Ž. '14*]

Any weighted clone W containing a nontrivial weighting contains a weighting that assigns positive weight to either:

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Easy: if the support clone $C = \mathcal{O}_D$ the only possible case

Complexity of Finite-Valued CSPs

Theorem [*Thapper & Ž. STOC'13*]

Let W be an idempotent weighted clone with the support clone $C = \mathcal{O}_D$ for some finite D.

1. Either W contains a binary weighting that assigns positive weight to commutative operations, in which case W is tractable;
2. or W is NP-hard.

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$C = \mathcal{O}_D$ means that functions in $\mathrm{Imp}(W)$ are $\mathbb{Q}$-valued

Necessary Conditions for Conservative Tractability

Theorem [*Thapper & **Ž**. '14*]

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4. A set of k-ary semiprojections (for some $k \geq 3$).

Easy: if all unary functions $\subseteq \mathrm{Imp}(W)$ the only possible cases

Complexity of Conservative Valued CSPs

Theorem *[Kolmogorov & Ž. JACM'13]*

Let W be weighted clone on D such that $\mathrm{Imp}(W)$ contains all unary functions (equivalently, all $\{0,1\}$-valued unary fns).

1. Either Γ admits a conservative binary multimorphism and a conservative ternary multimorphism and there is a family M of 2-element subsets of D, such that:
   - for every $\{a,b\} \in M$, $\omega|_{\{a,b\}}$ is a symmetric tournament pair and
   - for every $\{a,b\} \notin M$, $\omega'|_{\{a,b\}}$ is an MJN

   in which case W is tractable;
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2. or W is NP-hard.

multimorphism: $\omega(f) \in \mathbb{N}$ and $\omega(\mathrm{e}_i^{(k)}) = -1$
STP: binary mm with f dual of g and both conservative commutative
MJN: ternary mm with 2 majority and 1 minority operations

Weighted Clones on $|D| = 2$

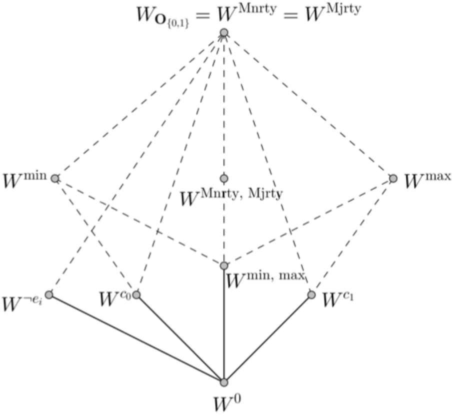
"Neccessary Theorem" gives 9 nontrivial weightings, which are atoms in the lattice of weighted clones.

8 are tractable, 1 is NP-hard.

[*Creed & **Ž**. CP'11/SICOMP'13*]

Weighted Clones on $|D| = 2$ with $C = \mathcal{O}_D$

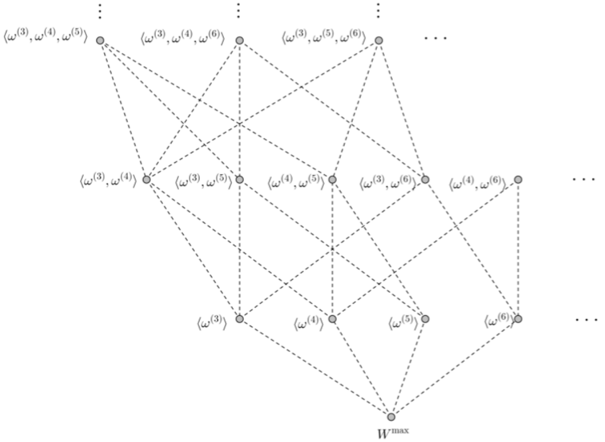
From the 9 before 4 are atomes, 2 generate everything.



[*Jeavons, Vaicenavičius,* ***Ž****. '14*]

Weighted Clones on $|D| = 2$ with $C = \mathcal{O}_D$, cont'd

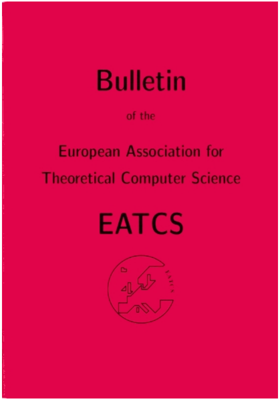
Uncountably many weighted clones above $W_{\max}$ (and $W_{\min}$).



[*Jeavons, Vaicenavičius, **Ž**. '14*]

Future Work, Open Problems, Where to Learn More?

The complexity of valued constraint satisfaction



[*Jeavons, Krokhin, Ž. '14*]

Thank You

http://zivny.cz/