Structural completeness for discriminator varieties

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Deductive systems

$Sent$ - set of propositional sentences
Ax - axioms ($\subseteq Sent$)
$+$ inference rules: $\dfrac{\Delta}{\varphi}$, $\Delta \subseteq_{fin} Sent$, $\varphi \in Sent$
(only structural rules - closed on substitutions)

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deductive system (Ax, R)

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deductive system (Ax, R)

φ is a theorem of (Ax, R) provided

there is a proof of φ from Ax with the use of R

Shortening proofs

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Can we do better?

Yes, we can by adding admissible rules

Deductive systems, Structural Completeness

Associate with (Ax, R) a consequence relation

$$\vdash \subseteq \mathcal{P}(Sent) \times Sent$$

A rule $\quad \dfrac{\Delta}{\varphi} \quad$ is derivable if

$$\Delta \vdash \varphi$$

is admissible if for each substitution σ

$$(\forall \psi \in \Delta) \ \vdash \sigma(\psi) \quad \text{implies} \quad \vdash \sigma(\varphi)$$

Structural completeness

Examples of admissible non-derivable rules

$$\text{Harrop rule } \frac{\neg p \to q \vee r}{(\neg p \to q) \vee (\neg p \to r)} \quad \text{in INT}$$

$$\frac{\Diamond x \wedge \Diamond \neg x}{\bot} \quad \text{in S5}$$

Structural completeness

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A deductive system is structurally complete (SC) if every its admissible rule is derivable

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A deductive system is structurally complete (SC) if every its admissible rule is derivable

Examples for SC

classical logic, Gödel-Dummett logic, $\mathrm{INT}^{\to}$, Medvedev logic, S4.3Grz

Almost structural completeness

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Almost structural completeness

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A rule $\dfrac{\Delta}{\varphi}$ is passive if

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A deductive system is almost structurally complete (ASC) if its admissible non-derivable rules are passive

Almost structural completeness

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A rule $\dfrac{\Delta}{\varphi}$ is passive if

$$(\forall \sigma \in \mathit{Subst})(\exists \delta \in \Delta) \nvdash \sigma(\delta)$$

A deductive system is almost structurally complete (ASC) if its admissible non-derivable rules are passive

Examples for ASC\SC

S5, S4.3, $Ł_n$

SC vs ASC problem

How common is to be ASC$\setminus$SC

Maybe it is a negligible issue.

Problem

Determine which ASC deductive systems are SC.

Quasivarieties

Quasi-identities look like

$$(\forall \bar{x})\ s_1(\bar{x}) \approx t_1(\bar{t}) \wedge \cdots \wedge s_n(\bar{x}) \approx t_n(\bar{x}) \to s(\bar{x}) \approx t(\bar{x})$$

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Correspondence for algebraizable deductive systems

consequence relation $\vdash$	$\leftrightsquigarrow$	quasivariety $\mathcal{Q}$
deductive system (Ax, R)	$\leftrightsquigarrow$	axiomatization of $\mathcal{Q}$
logical connectives	$\leftrightsquigarrow$	basic operations
theorems	$\leftrightsquigarrow$	identities true in $\mathcal{Q}$
derivable rules	$\leftrightsquigarrow$	quasi-identities true in $\mathcal{Q}$
admissible rules	$\leftrightsquigarrow$	quasi-identities true in $\mathbf{F}$

SC and ASC algebraically

F - $\mathcal{Q}$-algebra over $\aleph_0$ generators
Q(**F**) - quasivariety generated by **F**

A quasivariety $\mathcal{Q}$ is SC if $\mathcal{Q} = \mathrm{Q}(\mathbf{F})$, i.e., every quasi-identity valid in **F** is valid in $\mathcal{Q}$ too.

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$\mathcal{Q}$ is ASC if for every quasi-identity q valid in **F** either q is valid in $\mathcal{Q}$ or its premises are not satisfiable in **F**,
i.e., every non-passive quasi-identity valid in **F** is also valid in $\mathcal{Q}$.

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i.e., every non-passive quasi-identity valid in $\mathbf{F}$ is also valid in $\mathcal{Q}$.

Theorem (W. Dzik, M.S.)

The following conditions are equivalent

- $\mathcal{Q}$ is ASC
- For every $\mathbf{A} \in \mathcal{Q}$, $(\exists h \colon \mathbf{A} \to \mathbf{F})$ yields $\mathbf{A} \in \mathrm{Q}(\mathbf{F})$

SC vs ASC problem algebraically

Problem

Determine which ASC quasivarieties are SC.

Partial solutions to SC vs ASC problem

Theorem (W. Dzik, M.S.)

$\mathcal{Q}$ is ASC iff it is SC provided

- $\mathbf{F}$ has an idempotent element (groups, lattices)
- every nontrivial algebra from $\mathcal{Q}$ admits a homomorphism into $\mathbf{F}$ (Heyting algebras, McKinsey algebras)

Partial solutions to SC vs ASC problem

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Theorem (W. Dzik, M.S.)

Let $\mathcal{V}$ be an ASC variety of closure algebras. The following conditions are equivalent

- $\mathcal{V}$ is SC
- $\mathcal{V}$ is a variety of McKinsey algebras
- There is no 4-element simple algebra in $\mathcal{V}$

General solution

Theorem (M.S.)

Let $\mathcal{Q}$ be an ASC quasivariety. The following conditions are equivalent

- $\mathcal{Q}$ is SC
- Every nontrivial $\mathcal{Q}$-finitely presented algebra admits a homomorphism into $\mathbf{F}$

Solution for semisimple quasivarieties

Theorem (M.S.)

Let $\mathcal{Q}$ be an ASC semisimple quasivariety. The following conditions are equivalent

- $\mathcal{Q}$ is SC
- $\mathcal{Q}$ is minimal or $\mathbf{F}$ has an elementary extension with an idempotent element

Discrimintor varieties

Theorem

Let $\mathcal{V}$ be a discriminator variety. Then

- $\mathcal{V}$ is semisimple
- $\mathcal{V}$ is ASC (S. Burris, W. Dzik)

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- $\mathcal{V}$ is semisimple
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Corollary

Let $\mathcal{V}$ be a discriminator variety. The following conditions are equivalent

- $\mathcal{V}$ is SC
- $\mathcal{V}$ is a minimal quasivariety or $\mathbf{F}$ has an elementary extension with an idempotent element

Corollary

Let $\mathcal{V}$ be an SC discriminator variety. Assume that there are two distinct constants in $\mathbf{F}$ (like $\mathbf{0}$ and $\mathbf{1}$). Then $\mathcal{V}$ must be minimal.

Discrimintor varieties in logic

Corollary

Let $\mathcal{V}$ be an SC discriminator variety. Assume that there are two distinct constants in **F** (like $\mathbf{0}$ and $\mathbf{1}$). Then $\mathcal{V}$ must be minimal.

Example

All (continuum many) subvarieties of $\mathcal{CA}_2$ are ASC. There are only countable many SC subvarieties of $\mathcal{CA}_2$

Minimal varieties

Problem

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Last Week Theorem (M. Campercholi, D. Vaggione)

Every minimal discriminator variety is minimal as a quasivariety.

Question

Problem

How many there are $\mathrm{ASC} \setminus \mathrm{SC}$ varieties of closure algebras?

The end

This is all Thank you!