FORBIDDEN SUBALGEBRA THEOREMS IN SEMIGROUP THEORY

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But here is what really matters in life:

Useful terminology

Given a (quasi)variety $\mathcal{V}$ and algebras $A, B_1, B_2, \ldots \in \mathcal{V}$, we will say that A *avoids* $B_1, B_2, \ldots$ if A does not contain an isomorphic copy of any B_i as a subalgebra.

(One can also talk about avoiding homomorphic images or divisors or . . . , but today we will just discuss subalgebras.)

Forbidden Subalgebra Theorems

Data:

- a (quasi)variety $\mathcal{V}$,
- a sub(quasi)variety $\mathcal{W}$,
- a "manageable" list $\mathbf{L}$ of $\mathcal{V}$-algebras which are not in $\mathcal{W}$.

Theorem

Let $A \in \mathcal{V}$. Then:

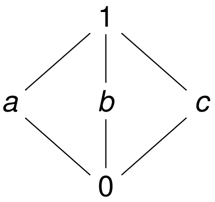
$$A \in \mathcal{W} \iff A \text{ avoids } \mathbf{L}.$$

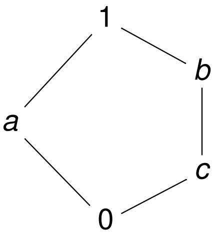
M_3-N_5 (Diamond-Pentagon) Theorem

The paradigm of such theorems....

Theorem

Let A be a lattice. Then A is distributive if and only if A avoids M_3 and N_5





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- We learn something about subquasivarieties avoiding subsets of **L** (*e.g.* modular lattices), especially if we can characterize them with quasi-identities.
- They are just very elegant!

How do we discover them?

How can we discover a forbidden subalgebra theorem in a setting where we think there might be one?

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Use software, such as MACE4 or GAP!

It's actually difficult to take an algebra and search it for an isomorphic copy of a particular algebra. For small algebras, the following approach is easier....

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- Find all non-$\mathcal{W}$ $\mathcal{V}$-algebras with no restrictions. If this list is longer, we have new non-$\mathcal{W}$ $\mathcal{V}$-algebras.

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- Concatenate the two lists and sort up to isomorphism. The algebras at the end of the list are new. Append them to **L**.
- Repeat until we learn something or run out of memory.

Green's Relations

For a semigroup S, let $S^1 = S$ if S is a monoid; otherwise, let $S^1 = S \cup \{1\}$, that is, S with an identity element adjoined.

Recall that *Green's relations* on a semigroup S are defined by

$$\begin{aligned} a \,\mathcal{L}\, b &\iff S^1 a = S^1 b\,, \\ a \,\mathcal{R}\, b &\iff aS^1 = bS^1\,, \\ a \,\mathcal{J}\, b &\iff S^1 a S^1 = S^1 b S^1\,, \end{aligned}$$

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}\,, \qquad \mathcal{H} = \mathcal{L} \cap \mathcal{R}\,.$$

We mostly need just $\mathcal{L}$ and $\mathcal{R}$ in this talk.

Regular semigroups

A semigroup S is *regular* if any of the following equivalent conditions holds:

- Each $a \in S$ has an *inverse* $b \in S$: $aba = a$, $bab = b$.
- Every $\mathcal{L}$-class contains an idempotent;
- Every $\mathcal{R}$-class contains an idempotent;

Examples: The full transformation semigroup on a set is regular. The semigroup of all linear transformations on a vector space is regular.

Inverse semigroups

A semigroup S is an *inverse semigroup* if any of the following equivalent conditions hold:

- Every element of S has a *unique* inverse;
- Each $\mathcal{L}$-class and each $\mathcal{R}$-class contain a *unique* idempotent;
- S is regular and $E(S)$ is a semilattice.

Example: The semigroup of partial bijections on a set is an inverse semigroup.

(The Wagner-Preston Theorem says that *every* inverse semigroup embeds in such an example.)

Inverse semigroups form a variety

The “correct” way to study inverse semigroups is to think of them as algebras of type $\langle 2, 1\rangle$ where the unary operation is the inverse $x \mapsto x^{-1}$. Inverse semigroups are then axiomatized in many ways. B. Schein’s 4-base:

$$\begin{array}{cc} (xy)z = x(yz) & xx^{-1}x = x \\ (x^{-1})^{-1} = x & xx^{-1}y^{-1}y = y^{-1}yxx^{-1} \end{array}$$

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A 2-base! [J. Araújo, MK, R. Padmanabhan 2014]

$$x(x^{-1}x) = x$$
$$x(x^{-1}(y(y^{-1}((zu)^{-1}w^{-1})^{-1}))) = y(y^{-1}(x(x^{-1}((wz)u))))$$

Bicyclic semigroup

The *bicyclic semigroup* is an inverse semigroup defined by a monoid presentation or a semigroup presentation:

$$B = \langle a, b \mid ab = 1 \rangle = \langle a, b \mid aba = aab = a, abb = bab = b \rangle \,.$$

(B can be realized as the transformation semigroup on $\{0, 1, 2, \ldots\}$ generated by α, β where $n\alpha = n + 1$, $0\beta = 0$, $n\beta = n - 1$ if $n > 0$.)

From the semigroup presentation, we can write down a quasi-identity for avoiding B:

$$xyx = xxy = x \quad \& \quad xyy = yxy = y \quad \Longrightarrow \quad x = y \,.$$

Stability

A semigroup S is *stable* if for each $a, b \in S$,

$$Sa \subseteq Sab \implies Sa = Sab \quad \text{and} \quad aS \subseteq baS \implies aS = baS\,.$$

Every finite semigroup is stable.

Stability is considered to be one of the key properties of finite semigroups. For instance, in stable semigroups, $\mathcal{J} = \mathcal{D}$.

A classic

To most semigroup theorists, the following is the classic forbidden subalgebra theorem:

Theorem (essentially Koch & Wallace 1957)

Let S be a regular semigroup. The following are equivalent.

- *S is stable;*
- *S avoids the bicyclic semigroup.*

Uniqueness instead of existence

Let us forget regularity for a while, and focus on the structure of the set $E(S)$ of idempotents of a semigroup S.

For instance, whether the $\mathcal{L}$-classes and $\mathcal{R}$-classes have any idempotents or not, let us describe when there is *no more than one*.

Avoiding zero bands I

Restricted to $E(S)$, $\mathcal{L}$ and $\mathcal{R}$ have a nice form:

$$\begin{aligned} e\,\mathcal{L}\,f &\iff (ef = e \quad \& \quad fe = f)\,, \\ e\,\mathcal{R}\,f &\iff (ef = f \quad \& \quad fe = e)\,. \end{aligned}$$

Theorem

- *No $\mathcal{L}$-class has more than one idempotent $\iff$ S avoids the 2-element left zero semigroup.*
- *No $\mathcal{R}$-class has more than one idempotent $\iff$ S avoids the 2-element right zero semigroup.*

(Semigroup theorists do not really think of this as a forbidden subalgebra theorem because it is too elementary.)

Avoiding zero bands II

Observation: The class of all semigroups which avoid the 2-element bands forms a quasivariety axiomatized by associativity and

$$\begin{aligned} xx = x \quad \& \quad yy = y \quad \& \quad xy = x \quad \& \quad yx = y \implies x = y \\ xx = x \quad \& \quad yy = y \quad \& \quad xy = y \quad \& \quad yx = x \implies x = y \end{aligned}$$

Let's call this $\mathcal{V}$. (It doesn't have a special name in semigroup theory.)

When is $E(S)$ a semilattice?

A semigroup S is called *E-commutative* if $E(S)$ is a commutative subsemigroup, hence a semilattice. They form a quasivariety $\mathcal{E}$:

$$xx = x \quad \& \quad yy = y \implies xy = yx\,.$$

Clearly $\mathcal{E} \subseteq \mathcal{V}$.

Question: Can we characterize $\mathcal{E}$ in $\mathcal{V}$ via forbidden subsemigroups?

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Question: Can we characterize $\mathcal{E}$ in $\mathcal{V}$ via forbidden subsemigroups?

Answer: I wouldn't be here otherwise.

The smallest counterexample

The smallest $\mathcal{V}$-semigroup which is not E-commutative:

M_0	a	b	c	d
a	a	c	c	c
b	d	b	c	d
c	c	c	c	c
d	d	c	c	c

This has a nice presentation in $\mathcal{V}$:

$$M_0 = \langle\, a, b \mid a^2 = a,\ b^2 = b,\ aba = ba \,\rangle\,.$$

So we *avoid* M_0 in $\mathcal{V}$ as follows:

$$xx = x \quad \& \quad yy = y \quad \& \quad xyx = yx \implies xy = yx\,.$$

An infinite example

At the other extreme, consider

$$M_\infty = \langle\, a, b \mid a^2 = a,\ b^2 = b \,\rangle\,.$$

Again, this is a $\mathcal{V}$-semigroup not in $\mathcal{E}$.

Here is a concrete copy of this (J. Fountain 1979). Let

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $M_\infty \cong \{2^n A, 2^n B, 2^n C, 2^n D \mid n \geq 0\}$.

The rest of the family

Order the elements of M_∞ like so:

$$a, b, ab, ba, aba, bab, abab, baba, \ldots.$$

Define a sequence of semigroups via presentations in $\mathcal{V}$:

$$
\begin{aligned}
M_0 &= \langle\, a, b \mid a^2 = a,\ b^2 = b,\ aba = ba \,\rangle\,, \\
M_1 &= \langle\, a, b \mid a^2 = a,\ b^2 = b,\ bab = aba \,\rangle\,, \\
M_2 &= \langle\, a, b \mid a^2 = a,\ b^2 = b,\ abab = bab \,\rangle\,, \\
M_3 &= \langle\, a, b \mid a^2 = a,\ b^2 = b,\ baba = abab \,\rangle\,, \\
&\ \ \vdots
\end{aligned}
$$

Again, these are $\mathcal{V}$-semigroups, but not in $\mathcal{E}$.

Lemma

For $0 \leq n < \infty, \qquad |M_n| = n + 4.$

Avoiding the family

Each presentation determines a quasi-identity to avoid that M_i. For example, to avoid M_3:

$$xx = x \quad \& \quad yy = y \quad \& \quad yxyx = xyxy \quad \Longrightarrow \quad xy = yx\,.$$

Let $\mathcal{M}_i$ denote the quasivariety of semigroups which avoid M_i.

Lemma

$\mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots$

Main result

Theorem (Araújo & K, June 2014)

Let S be a semigroup. The following are equivalent:

- *S is E-commutative;*
- *S avoids the 2-element zero semigroups and all M_i, $0 < i \leq \infty$.*

Idea of Proof

Suppose S is a semigroup which is not E-commutative. Assume S avoids all M_i for $0 \leq i \leq \infty$. In particular, $S \in \bigcap_{0 \leq i < \infty} \mathcal{M}_i$. Choose noncommuting idempotents a, b. Without loss, assume $S = \langle a, b \rangle$. Since $S \neq M_\infty$, there are coinciding distinct words in a, b. Find the shortest word where this occurs. Let $\mathcal{M}_i$ be the quasivariety defined by avoiding the M_i where this shortest word coincides with the preceding one. Use the quasi-identity and check all possible cases to show that $S = M_i$. (This is the messy part of the proof.) Contradiction.

Mathematics made difficult

Here is an amusing proof that

- Each $\mathcal{L}$-class and each $\mathcal{R}$-class contain a *unique* idempotent

implies

- S is regular and E-commutative.

Proof:

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implies

- S is regular and E-commutative.

Proof:

No M_i is regular.

(The standard textbook proof uses less machinery, of course.)

Generalized Green's relations

Generalized Green's $*$-relations:

$$a \,\mathcal{L}^*\, b \iff (\forall x, y \in S, ax = ay \iff bx = by)\,,$$
$$a \,\mathcal{R}^*\, b \iff (\forall x, y \in S, xa = ya \iff xb = yb)\,, \text{etc.}$$

Generalized Green's $\sim$-relations:

$$a \,\tilde{\mathcal{L}}\, b \iff (\forall e \in E(S), ae = a \iff be = b)\,,$$
$$a \,\tilde{\mathcal{R}}\, b \iff (\forall e \in E(S), ea = a \iff eb = b)\,, \text{etc.}$$

These are related by

$$\mathcal{L} \subseteq \mathcal{L}^* \subseteq \tilde{\mathcal{L}}\,.$$

and similarly for the $\mathcal{R}$'s.

Observation

What makes everything I am about to discuss work is the following:

For idempotents e, f,

$$e \,\tilde{\mathcal{L}}\, f \iff e \,\mathcal{L}^*\, f \iff e \,\mathcal{L}\, f\,.$$

So we can use everything we have discussed about the M_i's.

York terminology

A semigroup is *right*...

- *abundant* if every $\mathcal{L}^*$-class contains an idempotent;
- *amiable* if every $\mathcal{L}^*$-class contains a *unique* idempotent;
- *adequate* if it is right abundant and E-commutative.

The left-handed versions are defined dually. Just say "abundant" for "both right and left abundant", *etc.*

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Replace $\mathcal{L}^*$ with $\tilde{\mathcal{L}}$, put the prefix "semi-" in front of everything above, get three more definitions.

Amiable and adequate as quasivarieties

The “correct” way of thinking about (semi-)amiable and (semi-)adequate semigroups is as an algebra of type $\langle 2, 1, 1\rangle$ where the unary operations $x \mapsto x^*$, $x \mapsto x^+$ assign to each element the corresponding unique idempotents in their $\mathcal{L}^*$- and $\mathcal{R}^*$-classes (resp. their $\tilde{\mathcal{L}}$- and $\tilde{\mathcal{R}}$-classes). They form a proper quasivariety.

Models

Example: Let T_X denote the full transformation semigroup on a set X, fix $Y \subseteq X$, and consider the semigroup $S = \{f \in T_X \mid f(Y) \subseteq Y\}$. Then S is abundant, but not generally regular (Sun & Wang 2014).

Example: Let T be an inverse semigroup and let S be any subsemigroup closed under the induced operations $x \mapsto x^* = x^{-1}x$ and $x \mapsto x^+ = xx^{-1}$. (*E.g.*, assign to each partial bijection the identity functions on its domain and range.) Then S is adequate (and more), but not an inverse semigroup in general.

Amiable and adequate

Every right adequate semigroup is right amiable. For the other way around...

- Left zero semigroups are not right amiable, so we avoid those anyway;
- We have to avoid right zero semigroups;
- What about our M_i's?

Checking the family

M_0	a	b	c	d
a	a	c	c	c
b	d	b	c	d
c	c	c	c	c
d	d	c	c	c

- M_0 is right amiable. The $\mathcal{L}^*$-classes are $\{a\}, \{b, d\}, \{c\}$.
- M_∞ is also right amiable.
- No other M_i is right amiable.

Results

Theorem (J. Araújo & K 2014)

Let S be a right amiable semigroup. The following are equivalent:

- *S is right adequate;*
- *S avoids the 2-element right zero semigroup, M_0 and M_∞.*

Corollary (J. Araújo, K & A. Malheiro 2013)

Let S be an amiable semigroup. The following are equivalent:

- *S is adequate;*
- *S avoids M_0 and M_∞.*

Semi-Results

It turns out that *all* the M_i's are semi-adequate.

Theorem (J. Araújo & K 2014)

Let S be a right semi-amiable semigroup. The following are equivalent:

- *S is right semi-adequate;*
- *S avoids the 2-element right zero semigroup and all M_i's, $0 \leq i \leq \infty$.*

Corollary (J. Araújo & K 2014)

Let S be a semi-amiable semigroup. The following are equivalent:

- *S is semi-adequate;*
- *S avoids all M_i's, $0 \leq i \leq \infty$.*

Clifford semigroups

An inverse semigroup is said to be a *Clifford* semigroup if its idempotents commute with everything (not just with each other).

Example: An inverse semigroup of partial bijections is Clifford if and only if each partial bijection is a permutation of its domain.

The smallest nonClifford inverse semigroup is the Brandt semigroup of order 5:

$\cdot$	0	e	f	a	b
0	0	0	0	0	0
e	0	e	0	0	b
f	0	0	f	a	0
a	0	a	0	0	f
b	0	b	e	0	0

Construction and characterization

For $n > 0$, let $X_n := \{-1, 0, 1, \ldots, n\}$. Define a partial bijection α by

$$(-1)\alpha = 0, \quad \alpha \restriction \{1, \ldots, n\} = (1, n, n-1, \ldots, 2)$$

(cycle notation). 0α is left undefined.

Let U_n be the inverse subsemigroup of the symmetric inverse semigroup on X_n generated by α.

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Theorem (J. Araújo & K 2014)

*Let S be a **finite** inverse semigroup. The following are equivalent:*

- *S is a Clifford semigroup;*
- *S avoids each U_n.*

(Probably no hope in the general case.)

References

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