Growth rates of solvable algebras

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Reason: The free algebras over $\mathbf{A}$ have polynomially bounded size.

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Motivating problem

What are the possible growth rates of finite algebras?

Pointed cube terms

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If Σ is a set of identities in a language $\mathcal{L}$, then an $\mathcal{L}$-term $F(x_1, \ldots, x_m)$ is a p-pointed, k-cube term for the variety axiomatized by Σ if there is a $k \times m$ matrix M consisting of variables and p distinct constant symbols, with every column of M containing a symbol different from x, such that

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Let $\mathbf{A}$ be an algebra with an m-ary, $p \geq 1$-pointed, k-cube term, with at least one constant symbol appearing in the cube identities.

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- There exist finite algebras with pointed cube terms whose growth rate is $\sim$ to a polynomial of any prescribed degree.
- The growth rate of any algebra with a pointed cube term arises as the growth rate of an algebra without a pointed cube term.
- If a basic Σ does not entail the existence of a pointed cube term, then Σ imposes no restriction on growth rates.
  "Basic" identity: at most one operation symbol on both sides.

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The proof uses a probabilistic argument of independent interest.

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Homomorphic images of abelian algebras are not always abelian.

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Homomorphic images of finite abelian algebras are right nilpotent.

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a closer relationship between various growth-restricting conditions.
Example: Both (5) and (4) imply that
the growth rate is non-exponential iff **A** has a Maltsev term
(in which case the growth rate is linear),

The hierarchy of abelianness properties

(1) **A** is solvable.

(2) **A** is (left) nilpotent.

(3) **A** is abelian.

(4) **A** is a subdirect product of simple abelian algebras.

(5) **A** generates an abelian variety.

We have (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (3).
No other implications hold (except the formal consequences).

We prove that stronger abelianness properties yield
a closer relationship between various growth-restricting conditions.
Example: Both (5) and (4) imply that
the growth rate is non-exponential iff **A** has a Maltsev term
(in which case the growth rate is linear), but (2) does not.

The hierarchy of growth-restricting conditions

(i) **A** has a Maltsev polynomial.

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Theorem (KKSz)

If $\mathbf{A}$ is a finite solvable algebra with a Maltsev polynomial, then $\mathbf{A}$ is a spread of its type $\mathbf{2}$ minimal sets.

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The proof uses the quasi-Hamiltonian property for the subalgebras of $\mathbf{A}^{|A|}$.

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An 8-element quasi-affine algebra shows that in the second statement the assumption that $\mathsf{V}(\mathbf{A})$ is abelian cannot be dropped.

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An 8-element quasi-affine algebra shows that in the second statement the assumption that $\mathrm{V}(\mathbf{A})$ is abelian cannot be dropped.

Another 8-element abelian algebra shows that in the first statement it is not sufficient to assume only that $\mathrm{H}(\mathbf{A})$ is abelian.

Semisimple algebras

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Example

There exist a 16-element algebra that is a direct product of two, 4-element affine (hence abelian, Maltsev) algebras, has a linear growth rate, but does not have a Maltsev polynomial.

Summary: arbitrary

(i) $\mathbf{A}$ has a Maltsev polynomial.

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(iv) $d_{\mathbf{A}}(n) \in O(n)$.

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All are equivalent if $\mathbf{A}$ is semisimple or if $\mathsf{V}(A)$ is abelian.

$$
\begin{array}{ccccccc}
 & & (i) & \Longrightarrow & (ii) & & \\
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For arbitrary finite algebras

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For finite, solvable algebras

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For finite, left nilpotent algebras

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Is there a finite algebra $\mathbf{A}$ such that $d_{\mathbf{A}}(n) \notin \Omega(n)$ and $d_{\mathbf{A}}(n) \notin O(\log(n))$? That is, whose growth rate is between logarithmic and linear?

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Does (ii)$\Rightarrow$(iii) hold for finite solvable algebras?

Which of the true implications (iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi) can be reversed for finite solvable algebras? In particular, is the growth rate of a finite solvable algebra always linear or exponential?

Literature

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Thank you for your attention.