Complexity of variety membership

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Prague ALCfest, June 2014

Var-mem

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Instance: a finite algebra **B** of the same signature as **A**
Question: is $\mathbf{B} \in \mathbb{HSP}(\mathbf{A})$?

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▷ *decomposition into subdirectly irreducibles can be performed in polynomial time* (*Demlová, 1982*)
▷ *Var-Mem is solvable in 2-*`EXPTIME` (*Bergman and Slutzki, 2000*).

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Connections with finite basis problem

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Var-Mem(**L**) is in NL

- (J and McNulty, 2011)

Eilenberg-Schützenberger

A is a finite algebra of finite signature

If $\mathbb{HSP}_{\mathrm{fin}}(\mathbf{A})$ is definable by a finite system of equations, is $\mathbb{HSP}(\mathbf{A})$ finitely axiomatisable?

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If $\mathbf{C}$ a counterexample

Var-Mem$(\mathbf{C}) \in \mathtt{FO}$, even though $\mathbb{HSP}(\mathbf{C})$ is not finitely axiomatisable?!?!?

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A result: "no"

There exists a finite algebra **A** for which $\mathbb{HSP}_{\mathrm{fin}}(\mathbf{A})$ is definable by a $\forall^*\exists^*\forall^*$ sentence, but not by a $\forall^*\exists^*$-sentence, nor by any finite system of pseudo-equations. In particular, $\mathbb{HSP}(\mathbf{A})$ is not first order definable.

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Łos-Tarski Theorem

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This fact is crucial in the classification of first order definable CSPs

$\mathbb{ISP}$ preservation?

Problem 1, Alechina and Gurevich 1997

If $K = \mathsf{Mod}_{\mathrm{fin}}(\phi)$ is closed under $\mathbb{SP}_{\mathrm{fin}}$, then is it true that ϕ is equivalent to a universal Horn sentence on finite structures?

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- $\mathbb{SP}(M_{3-3})$ is not finitely based (Belkin, 1978)

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No

- $\mathbb{SP}(M_{3-3})$ is not finitely based (Belkin, 1978)
- $\mathbb{S}_{\mathrm{c}}\mathbb{P}(M_{3-3})$ (where M_{3-3} has discrete topology) is first order definable amongst Boolean topological algebras (Clark, Davey, J, Pitkethly, 2008)

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S finite, with a semilattice term $\wedge$

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T $\in \mathbb{SP}_{\text{fin}}(\mathbf{S})$ if and only if $\wedge$ is a semilattice operation on **T** and for each $a \neq b$ in T there is a subalgebra **U** of **S** and (distinct) $t_u \in T$ for each $u \in U$

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$\mathbf{T} \in \mathbb{SP}_{\mathrm{fin}}(\mathbf{S})$ if and only if $\wedge$ is a semilattice operation on $\mathbf{T}$ and for each $a \neq b$ in T there is a subalgebra $\mathbf{U}$ of $\mathbf{S}$ and (distinct) $t_u \in T$ for each $u \in U$ such that the equivalence relation $x \equiv y$ defined by $t_u \leq x \Leftrightarrow t_u \leq y$ is a congruence relation of $\mathbf{T}$

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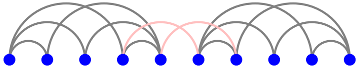
$\mathbb{HSP}_{\text{fin}}(\mathbf{C})$ finitely axiomatisable but $\mathbb{HSP}(\mathbf{C})$ not finitely axiomatisable

Example: $\mathbf{C} := M^{\flat}_{3-3}$



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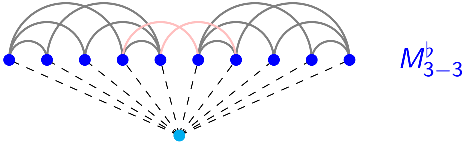

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$\triangleright$ Shows Var-Mem($\mathbf{C}$) $\in$ `FO` not equivalent to $\mathbf{C}$ being finitely based.

Related examples

Theorem

For every finite relational structure $\mathbf{R}$ of finite signature, there is a finite algebra $\mathbf{A}_{\mathbf{R}}$ such that CSP($\mathbf{R}$) is first order equivalent to both Var-Mem($\mathbf{A}_{\mathbf{R}}$) and $\mathrm{Var}_{\mathrm{si}}$-Mem($\mathbf{A}_{\mathbf{R}}$)

Related examples

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Corollary

Var-Mem($\mathbf{A_R}$) can be $\mathtt{L}$-complete, $\mathtt{NL}$-complete, $\mathtt{P}$-complete, $\mathtt{Mod}_p\mathtt{L}$-complete, etc (infinitely many tractable complexities)

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Corollary

Var-Mem($\mathbf{A_R}$) can be $\mathtt{L}$-complete, $\mathtt{NL}$-complete, $\mathtt{P}$-complete, $\mathtt{Mod}_p\mathtt{L}$-complete, etc (infinitely many tractable complexities)

Interesting problem for future

Is it possible that if Var-Mem(A) not in $\mathtt{P}$ then it is $\mathtt{NP}$-hard with respect to many-one reductions?

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Given finite algebra $\mathbf{A}$ it is undecidable whether or not Var-Mem($\mathbf{A}$) is in $\mathtt{K}$.

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Given finite algebra $\mathbf{A}$ it is undecidable whether or not Var-Mem($\mathbf{A}$) is in $\mathtt{K}$.

(*In contrast, it is* $\mathtt{NP}$*-complete to decide if* CSP($\mathbf{R}$) *is in* $\mathtt{FO}$; Larose, Loten, Tardif 2006)

Central technique

First order definable subdirect decomposition

There is a formula $\pi(x, y, u, v)$ such that $\{(x, y) \mid \mathbf{A} \models \pi(x, y, a, b)\}$ is a congruence $\theta_{a,b}$ maximal with respect to $(a, b) \notin \theta_{a,b}$

Consequence

$\mathbf{A}/\theta_{a,b}$ is first order definable from $\mathbf{A}$ for each $a \neq b$

Pointed semi-discriminator varieties

1. If $\mathbf{P}$ has a projection term $t(x, y) \approx x$, then

$$[\mathbb{HSP}(\mathbf{P}^\flat)]_{\mathrm{si}} = [\mathbb{SPP}_{\mathrm{u}}(\mathbf{P})]^\flat$$

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2. $\mathbb{V}(\mathbf{P}^\flat)$ has first order definable subdirect decomposition: for $a \not\leq b$

$$\pi(x,y,a,b) = \big(\neg t(x,a) \approx a \mathbin{\mathbb{\wedge}} \neg t(y,a) \approx a\big) \mathbin{\mathbb{\vee}} (t(x \wedge y, a) \approx a)$$

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$\Rightarrow$ Var-Mem($\mathbf{P}^\flat$) reduces to $\mathrm{Var}_{\mathrm{si}}$-Mem($\mathbf{P}^\flat$) by a sort of “first order Turing reduction”

3. If **P** comes from a relational structure, then a further trick enables a genuine first order reduction from Var-Mem($\mathbf{P}^\flat$) to $\mathrm{Var}_{\mathrm{si}}$-Mem($\mathbf{P}^\flat$)

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4. But first replace $\mathbf{P}$ by $\mathbf{P}^\sharp$

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A newish preservation theorem

If K is a uniformly locally finite $\mathbb{SP}_{\mathrm{fin}}$-class of structures contained in some locally finite and finitely axiomatisable $\mathbb{SP}$-class, then K is definable by a $\forall^*\exists^*$-sentence if and only if it is definable by a universal Horn sentence