Finitely generated varieties with edge term

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Question

When are the subvarieties of a finitely generated variety finitely generated?

1. Background

Algebras and identities

A variety is a class of algebras of fixed type defined by identities.

1. $\mathcal{F}$ ... operation symbols with arities (**type**)
   E.g., $\mathcal{F} := \{\cdot, ^{-1}, 1\}$ with $\mathrm{ar}(\cdot) = 2$, $\mathrm{ar}(^{-1}) = 1$, $\mathrm{ar}(1) = 0$

2. A ... set
   $f^{\mathbf{A}}\colon A^{\mathrm{ar}(f)} \to A$ for $f \in \mathcal{F}$ ... operation
   $\mathbf{A} := (A, \{f^{\mathbf{A}} \mid f \in \mathcal{F}\})$ ... **algebra** of type $\mathcal{F}$
   E.g., $S_3 := (\{\text{permutations on}\{1,2,3\}\}, \circ, ^{-1}, ())$

3. $t(x_1, \dots, x_k)$ ... **term** over $\mathcal{F}$
   $t^{\mathbf{A}} : A^k \to A$ ... induced function on $\mathbf{A}$
   E.g., $t(x_1, x_2) = x_1 \cdot x_2^{-1}$ induces $t^{S_3}(a_1, a_2) := a_1 \circ a_2^{-1}$

4. $\mathbf{A}$ satisfies the **identity** $s \approx t$ (short $\mathbf{A} \models s \approx t$) iff $s^{\mathbf{A}} = t^{\mathbf{A}}$.
   E.g., $S_3 \models x \cdot x^{-1} \approx 1$, $S_3 \not\models x \cdot y \approx y \cdot x$

Varieties

Definition

A class V of algebras of type $\mathcal{F}$ is a **variety** if $\exists$ set of identities Φ:

$$V = \{\mathbf{A} \mid \mathbf{A} \text{ has type } \mathcal{F} \text{ and } \mathbf{A} \models \Phi\}.$$

Example

1. $\Phi := \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ x \cdot 1 \approx x,\ x \cdot x^{-1} \approx 1\}$
   defines the variety of groups.
2. Fields do not form a variety.

Theorem (Birkhoff, 1935)

V is a variety iff V is closed under

- $\mathbb{H}$ homomorphic images,
- $\mathbb{S}$ subalgebras,
- $\mathbb{P}$ direct products.

Finite generation

For an algebra $\mathbf{A}$, $\mathbb{V}(\mathbf{A})$ is the smallest variety that contains $\mathbf{A}$.

Corollary (Birkhoff, 1935)

$\mathbb{V}(\mathbf{A}) = \mathbb{HSP}(\mathbf{A})$.

Definition

A variety V is **finitely generated** if $\exists$ a finite algebra $\mathbf{A}$ with $V = \mathbb{V}(\mathbf{A})$.
(Equivalently, V is generated by finitely many finite algebras.)

Example

1. The variety of groups is not finitely generated. Why?
2. $\mathbb{V}(S_3) =$ groups satisfying $x^6 \approx 1,\ x^2y^2 \approx y^2x^2,\ [x, y]^3 \approx 1$

Finite basis

Definition

A variety V is **finitely based** if $\exists$ a finite set of equations Φ with $V = \{\mathbf{A} \mid \mathbf{A} \models \Phi\}$.

Note

1. Variety of groups is finitely based.
2. $\mathbb{V}(S_3)$ is finitely based.
3. In general, being finitely generated and finitely based are independent.

How to recognize finitely generated subvarieties

V is **locally finite** if every finitely generated algebra in V is finite.
E.g. Finitely generated varieties are locally finite.

Theorem (cf. Oates MacDonald, Vaughan-Lee, 1978)

Let V be a locally finite variety, $\mathcal{W}$ the set of subvarieties.

1. Then every $W \leq V$ is finitely generated iff $(\mathcal{W}, \subseteq)$ satisfies the (ACC).
2. Every $W \leq V$ is finitely based relative to V (ie, $\exists$ finite Φ with $W = \{\mathbf{A} \in V \mid A \models \Phi\}$ iff $(\mathcal{W}, \subseteq)$ satisfies the (DCC).
3. Then $\mathcal{W}$ is finite iff $(\mathcal{W}, \subseteq)$ satisfies (DCC) and (ACC).

Note

Item 2. also holds if V is not locally finite.

Finitely many subvarieties

Theorem (cf. Jónsson's Lemma, 1967)

Let $\mathbf{A}$ finite, $\mathbb{V}(\mathbf{A})$ congruence distributive.
Then all subvarieties of $\mathbb{V}(\mathbf{A})$ are finitely generated.

Proof.

Every $W \leq \mathbb{V}(\mathbf{A})$ is determined by its subdirectly irreducible members.
All SI algebras of $\mathbb{V}(\mathbf{A})$ are in $\mathbb{H}\mathbb{S}(\mathbf{A})$, hence finite in size and number. $\square$

Theorem (Oates, Powell, 1964)

For a finite group $\mathbf{A}$, all subvarieties of $\mathbb{V}(\mathbf{A})$ are finitely generated.

Proof.
$W \leq \mathbb{V}(\mathbf{A})$ is determined by its critical members.
$\mathbf{B}$ is **critical** if finite and $\mathbf{B} \notin \mathbb{V}(\{\mathbf{C} \in \mathbb{HS}(\mathbf{B}) \mid |C| < |B|\})$
$\mathbb{V}(\mathbf{A})$ contains only finitely many critical groups. $\square$

Theorem (Kruse, 1973)

For a finite ring $\mathbf{A}$, all subvarieties of $\mathbb{V}(\mathbf{A})$ are finitely generated.

In all three cases $\mathbb{V}(\mathbf{A})$ has only finitely many subvarieties.

Not finitely generated

Example

For the following **A** not all subvarieties of $\mathbb{V}(\mathbf{A})$ are finitely generated:

1. (Oates MacDonald, Vaughan-Lee, 1978) the expansion of the Murskii groupoid $(\{0,1,2\},\cdot,0)$

$\cdot$	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

2. (Lee, 2006) the 5-element Brandt semigroup $\mathbf{B}_2 :=$

$$(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \cdot)$$

Few subpowers

Definition

A, finite, has **few subpowers** if $\exists$ polynomial p $\forall n \in \mathbb{N}$:
$|\mathbb{S}(\mathbf{A}^n)| \leq |A|^{p(n)}$

Example

1. The set $\mathbf{A} := (A, \emptyset)$ has **many subpowers**, $2^{|A|^n}$.
2. Groups, loops, lattices,... have few subpowers.

Theorem (Berman, Idziak, Markovic, McKenzie, Valeriote, Willard, 2010)

Every finite **A** has few subpowers or $\exists c$ $\forall n \in \mathbb{N}$: $|\mathbb{S}(\mathbf{A}^n)| \geq 2^{c2^n}$.

Edge terms

Theorem (BIMMVW, 2010, Kearnes, Szendrei 2012)

A has few subpowers iff it has an edge (cube, parallelogram) term.

Definition

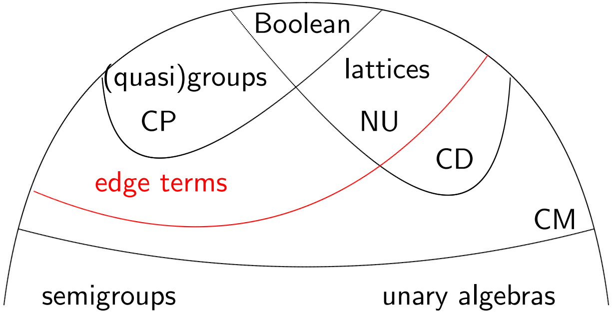
For $k \geq 2$, a $(k+1)$-ary term t on **A** is a k-**edge term** if

$$\mathbf{A} \models t \begin{pmatrix} x & x & y & \dots & \dots & y \\ x & y & x & \ddots & & \vdots \\ y & y & y & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & y \\ y & y & y & \dots & y & x \end{pmatrix} \approx \begin{pmatrix} y \\ \vdots \\ \vdots \\ \vdots \\ y \end{pmatrix}.$$

$k-$near unanimity

Mal'cev

Where is your favorite algebra?



?

2. Result

Theorem (Aichinger, M, arXiv 2014)

Every subvariety of a finitely generated variety with edge term is finitely generated.

Corollary (Aichinger, M)

A finitely generated variety with edge term has at most countably many subvarieties.

Note

Generalizes known results on groups, rings, algebras with NU-term. New for loops.

3. Proof

Outline of the proof for our main theorem

1. Represent identities as functions.
2. These functions form generalized clones (**clonoids**).
3. Nice generating sets (**forks**) and order theory yield chain condition for clonoids.
4. Obtain (ACC) for subvarieties.

Encoding identities as functions

A finite
$\mathrm{Clo}(\mathbf{A})$... set of term operations on **A** (**clone**)

Note

1. $\mathrm{Clo}_k(\mathbf{A})$ forms the free algebra on $x_1, \dots, x_k$ in $\mathbb{V}(\mathbf{A})$.
   Modulo identities of $\mathbb{V}(\mathbf{A})$, terms are functions on A.
2. Represent identities valid for $\mathbf{B} \in \mathbb{V}(\mathbf{A})$ by pairs of functions on A (ie., functions into A^2).

Definition

For $W \leq \mathbb{V}(\mathbf{A})$, define the "theory of W represented on **A**":

$$\mathrm{Th}_{\mathbf{A}}(W) := \{A^k \to A^2, x \mapsto (s^{\mathbf{A}}(x), t^{\mathbf{A}}(x)) \mid k \in \mathbb{N}, \\ s, t \text{ are } k-\text{variable terms}, W \models s \approx t\}$$

Abusing notation:

$$\mathrm{Th}_{\mathbf{A}}(W) = \{(s,t) \mid s,t \in \mathrm{Clo}(\mathbf{A}),\ \forall \mathbf{B} \in W\colon s^{\mathbf{B}} = t^{\mathbf{B}}\}$$

Example

1. $\mathrm{Th}_{\mathbf{A}}(\mathbb{V}(\mathbf{A})) = \{(t,t) \mid t \in \mathrm{Clo}(\mathbf{A})\}$
2. $W := \{\mathbf{G} \in \mathbb{V}(S_3) \mid \mathbf{G} \text{ abelian }\}$
   Then $(x_1, x_2) \mapsto (x_2^{-1} x_1 x_2, x_1)$ is in $\mathrm{Th}_{S_3}(W)$.
3. If $\mathbf{B} \leq \mathbf{A}$, then
   $\mathrm{Th}_{\mathbf{A}}(\mathbb{V}(\mathbf{B})) = \{(s,t) \mid s,t \in \mathrm{Clo}(\mathbf{A}), s|_B = t|_B\}$.

Question

What kind of structure is

$$\mathrm{Th}_{\mathbf{A}}(W) = \{(s,t) \mid s,t \in \mathrm{Clo}(\mathbf{A}),\ \forall \mathbf{B} \in W\colon s^{\mathbf{B}} = t^{\mathbf{B}}\}?$$

Answer

$C := \mathrm{Th}_{\mathbf{A}}(W)$ is not a clone since its functions map into A^2 not A.
But C has properties similar to clones:

1. closure under operations of $\mathbf{B} = \mathbf{A}^2$:
   $C_k \leq \mathbf{B}^{A^k}$
2. closure under identification and permutation of variables:
   $\forall (s,t) \in C_k\ \forall i_1,\dots,i_k \leq n$:
   $A^n \to A^2, (x_1,\dots,x_n) \mapsto (s(x_{i_1},\dots,x_{i_k}), t(x_{i_1},\dots,x_{i_k})),$ is in C_n

Clonoids

Definition

For a set A and an algebra $\mathbf{B}$, we call $C \subseteq \bigcup_{k\in\mathbb{N}} B^{A^k}$ a **clonoid** with **source** A and **target algebra B** if

1. $\forall k\colon C_k \leq \mathbf{B}^{A^k}$,
2. C is closed under identification and permutation of variables.

Example

1. $\mathrm{Clo}(\mathbf{A})$ is a clonoid with source A and target $\mathbf{A}$.
2. $\mathrm{Th}_{\mathbf{A}}(W)$ is a clonoid with source A and target $\mathbf{A}^2$.

Galois correspondences

For clones:

$$\begin{array}{rcl} \text{functions} & \leftrightarrow & \text{relations} \\ \mathrm{Clo}(\mathbf{A}) & \leftrightarrow & \bigcup_{n\in\mathbb{N}} \mathbb{S}(\mathbf{A}^n) \end{array}$$

Governed by: $f\colon A^k \to A$ preserves a relation $R \subseteq A^n$.

For clonoids:

$$\begin{array}{rcl} \text{identities} & \leftrightarrow & \text{algebras} \\ \mathrm{Th}_{\mathbf{A}}(W) & \leftrightarrow & W \leq \mathbb{V}(\mathbf{A}) \end{array}$$

Governed by: $s \approx t$ holds in $\mathbf{B}$

Recall

Theorem (Aichinger, M, McKenzie, arXiv 2009, JEMS 201?)

On a finite set, every clone with edge term is finitely related (= determined by a single relation).

Generalize to clonoids:

For $\mathbf{A}$ finite with edge term and $W \leq \mathbb{V}(\mathbf{A})$, $\mathrm{Th}_{\mathbf{A}}(W)$ is determined by a single algebra.

Representing functions into an algebra with edge term

Definition

Let A finite, linearly ordered, $C \subseteq B^{A^n}$.
The set of **forks of C at** $a \in A^n$ is

$$\varphi(C, a) := \{(f(a), g(a)) \in B^2 \mid f, g \in C,\ \forall x <_{lex} a : f(x) = g(x)\}.$$

Lemma (cf. Representation of subpowers, BIMMVW, 2010)

Let A finite, $\mathbf{B}$ with k-edge term, $C \leq \mathbf{B}^{A^n}$, $G \subseteq C$ such that

1. $\forall a \in A^n \colon \varphi(G, a) = \varphi(C, a)$,
2. $\forall T \subseteq A^n, |T| < k \colon G|_T = C|_T$.

Then $\langle G \rangle = C$.

If G witnesses all forks and (k-1)-projections, it generates C.

Embedding order on words

Definition

$a, b \in A^+ \dots$ words over A
$a \leq_E b$ if b is obtained from a by inserting letters after their first occurence in a.

Example

hedgo $\leq_E$ hedgehog
an $\not\leq_E$ ant

Lemma (Aichinger, M, McKenzie, 2009, cf. Higman's Theorem, 1952)

Let A finite. Then $(A^+, \leq_E)$ is partially ordered with (DCC) and without infinite antichains (i.e., **well partially ordered**).

Transfering forks to distinct arities

Lemma (Transfering forks, cf. Aichinger, M, McKenzie 2009)

Let C be a clonoid with source A and target $\mathbf{B}$, let $a, b \in A^+$.
If $a \leq_E b$, then $\varphi(C_{|b|}, b) \subseteq \varphi(C_{|a|}, a)$.

Example

$a := (h, e, d, g, o)$, $b := (h, e, d, g, e, h, o, g)$
$f_1, f_2 \colon A^8 \to B$ witness a fork at b.

1. Define
$$g_i(x_1, x_2, x_3, x_4, x_5) := f_i(x_1, x_2, x_3, x_4, x_2, x_1, x_5, x_4)$$
2. Then $g_i(a) = f_i(b)$.
3. If $(x_1, \dots, x_5) <_{lex} a$, then $(x_1, x_2, x_3, x_4, x_2, x_1, x_5, x_4) <_{lex} b$ and $g_1(x_1, \dots, x_5) = g_2(x_1, \dots, x_5)$.
4. Hence $g_1, g_2 \in C_5$ witness the fork $(f_1(b), f_2(b))$ at a.

Sketch of the order argument

$C \dots$ clonoid with source A and target **B** with edge term

1. For $(u, v) \in B^2$,
$$\lambda_C(u, v) := \{a \in A^+ \mid (u, v) \text{ is } \textbf{not} \text{ a fork of } C \text{ at } a\}$$
is upward closed wrt. $\leq_E$ (Transfering-forks-Lemma).
2. If
$$C = C^1 \geq C^2 \geq \dots,$$
then
$$\lambda_{C^1}(u, v) \subseteq \lambda_{C^2}(u, v) \subseteq \dots$$
3. $S := \bigcup_{i \in \mathbb{N}} \lambda_{C^i}(u, v)$ is upward closed and has finitely many minimal elements M (Higman's Theorem).
4. Then $M \subseteq \lambda_{C^n}(u, v)$ and $S = \lambda_{C^n}(u, v)$ for some n.

5. Hence (u, v)-forks occur in the same places in all C_i $(i \geq n)$.
6. $\exists m \, \forall i \geq m \, \forall a \in A^+ : \; \varphi(C^i_{|a|}, a) = \varphi(C^m_{|a|}, a)$.
7. Thus
$$C = C^1 \geq C^2 \geq \dots,$$
is not infinitely descending (Representation Lemma).

We proved:

Theorem (Aichinger, M, 2014)

Let A finite, $\mathbf{B}$ finite algebra with edge term, $\mathcal{C}$ the set of clonoids with source A and target $\mathbf{B}$.
Then $(\mathcal{C}, \subseteq)$ satisfies the (DCC).

The result again

Corollary (Aichinger, M, 2014)

Let $\mathbf{A}$ be a finite algebra with edge term, $\mathcal{W}$ the set of subvarieties of $\mathbb{V}(\mathbf{A})$. Then

1. $(\mathcal{W}, \subseteq)$ satisfies the (ACC).
2. Every $W \in \mathcal{W}$ is finitely generated.

Proof.

For $W_1, W_2 \leq \mathbb{V}(\mathbf{A})$,

$$W_1 \subseteq W_2 \text{ iff } \mathrm{Th}_{\mathbf{A}}(W_1) \supseteq \mathrm{Th}_{\mathbf{A}}(W_2).$$

$\square$

Assuming (DCC)

Corollary (Aichinger, M, 2014)

Let $\mathbf{A}$ be a finite algebra with edge term, $\mathcal{W}$ the set of subvarieties of $\mathbb{V}(\mathbf{A})$. TFAE:

1. $(\mathcal{W}, \subseteq)$ satisfies the (DCC).
2. Every $\mathbf{B} \in \mathbb{V}(\mathbf{A})$ is finitely based relative to $\mathbb{V}(\mathbf{A})$.
3. $\mathcal{W}$ is finite.