On optimal Mal'cev conditions for congruence meet-semidistributivity

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Mal'cev conditions

Definitions

- Mal'cev condition are syntactical conditions describing various semantical properties of all algebras in a a variety and/or their congruence lattices.
- If a property can be described by a fixed number of term operations of fixed arities satisfying a fixed number of linear equations we call it a *strong* Mal'cev property (and a corresponding system would be a *strong Mal'cev condition*).
- A usual Mal'cev property (also known as *weak*) is equivalent to satisfying one strong Mal'cev condition (for some $n \in \omega$) from a given countable sequence of strong Mal'cev conditions (for every $n \in \omega$).

...and some more definitions...

Definition

A locally finite variety is congruence meet–semidistributive iff its congruence lattices satisfy the following implication:
$x \wedge z = y \wedge z \Rightarrow (x \vee y) \wedge z = x \wedge z.$

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An n–ary term t, for $n > 1$ is a weak near–unanimity term for an algebra $\mathbf{A}$ if it is idempotent and the identities $t(x, x, \dots, x, y) \approx t(x, x, \dots, y, x) \approx \cdots \approx t(x, y, \dots, x, x) \approx t(y, x, \dots, x, x)$ hold in $\mathbf{A}$.

Strong Mal'cev conditions for congruence meet–semidistributivity

Theorem (Kozik)

A locally finite variety $\mathcal{V}$ satisfies congruence $SD(\wedge)$ iff is has 3–ary and 4–ary weak near–unanimity terms, v and w respectively, that satisfy the identity $v(y,x,x) \approx w(y,x,x,x)$.

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Corollary (Maroti, Janko)

A locally finite variety $\mathcal{V}$ satisfies congruence $SD(\wedge)$ iff is has a 3–ary weak near–unanimity term s and 3–ary terms r and t satisfying

$$r(x,x,y) \approx r(x,y,x) \approx t(y,x,x) \approx t(x,y,x) \approx s(x,x,y)$$
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A question: ARE GIVEN CONDITIONS OPTIMAL or the same can be done by two at most 3–ary idempotent terms?

Examples of algebras generating congruence meet–semidistributive varieties

Example (1)

Let **A** be a finite algebra with at least two elements and a single idempotent basic operation $f(x_1, x_2, x_3)$, which is a majority term:

$$f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \approx x.$$

In case no arguments are equal we can define f like this: $f(a, b, c) = a$, for all a, b, c in **A** and $a \neq b$, $b \neq c$, $c \neq a$.

This algebra generates a congruence meet–semidistributive variety (Kozik's theorem stated earlier).

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Example (2)

Let $\mathbf{B} = \langle \{0\ ,\ 1\}\ ,\ \wedge \rangle$ be the semilattice with two elements. For this algebra holds the same and for the same reason as above.

The method used – examples given and the following theorem

This theorem implicitly follows from the results proved by Hobby and McKenzie in *The structure of finite algebras* and the Fundamental theorem of Abelian algebras in commutator theory (when applied to idempotent algebras):

Theorem

Let $\mathbf{A}$ *be a finite idempotent algebra and* $\mathcal{V}$ *the variety generated by* $\mathbf{A}$*. Then* $\mathcal{V}$ *satisfies congruence* $SD(\wedge)$ *iff it does not contain an algebra that is term equivalent to either a set or a full idempotent reduct of a module over some finite ring.*

A system of identities implying and possibly describing congruence $SD(\wedge)$ of a locally finite variety

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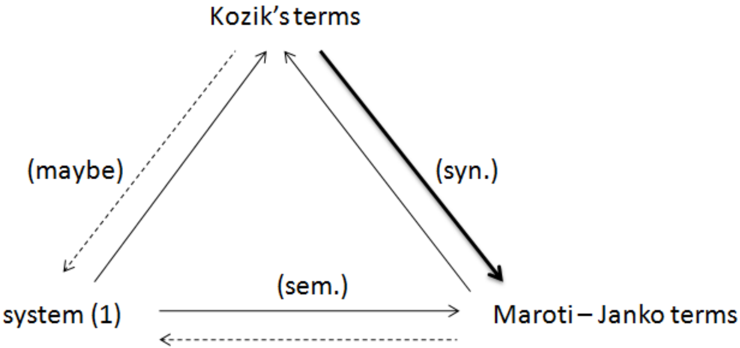
$$\left\{ \begin{array}{c} p(x,x,y) \approx p(x,y,y) \\ p(x,y,x) \approx q(x,x,y) \approx q(x,y,x) \approx q(y,x,x) \end{array} \right. \tag{1}$$

so this is how things look like...



Is there a counterexample for the system (**1**)?

We examined algebras of polymorphisms of small digraphs (up to size 5).

Definitions

An n–ary polymorphism of a digraph $G = (V, E)$ is a mapping $f : V^n \to V$ which preserves edges, that is for any $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \in E$ the pair $(f(a_1, \ldots a_n), f(b_1, \ldots b_n))$ is also in E.
An algebra of polymorphisms for a given digraph $G = (V, E)$ is an algebra with the universe V whose basic operations are polymorphisms of this digraph.

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This search actually leans on the results obtained by L. Barto and D.Stanovsky (*Polymorphisms of small digraphs*).

Algebras of polymorphisms

For algebras of polymorphisms of 2–element and 3–element digraphs holds the following: if they generate a congruence $SD(\wedge)$ variety they have a majority and/or a 2sml polymorphism (Barto/Stanovsky). In either of the cases they satisfy the system $(\mathbf{1})$ – this is easily seen from the examples given before.

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So we examined algebras of polymorphisms of 4–element and 5–element digraphs using both our own algorithms written for this purpose and Paradox model–builder.

What do we have?

The result

For all algebras of polymorphisms of digraphs up to size five holds the following: they generate a congruence $SD(\wedge)$ variety iff they satisfy the system $(\mathbf{1})$.

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The following fact is a part of a theorem proven by Bulin, Delić, Jackson and Niven (2011):

Fact

For every finite relational structure A there exists a digraph $\mathbf{D_A}$ such that for all Mal'cev conditions Σ holds $\mathbf{D_A} \models \Sigma \Rightarrow A \models \Sigma$.

The conclusion

So, if the system (1) describes congruence $SD(\wedge)$ on algebras of polymorphisms of all digraphs it means the same holds for algebras of polymorphisms of all finite relational structures.

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We do know it holds for digraphs up to size five.

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So, if the system (1) describes congruence $SD(\wedge)$ on algebras of polymorphisms of all digraphs it means the same holds for algebras of polymorphisms of all finite relational structures.

Whether it does – we do not know.

We do know it holds for digraphs up to size five.

Conjecture

The system $(\mathbf{1})$ *(given here once again) is the optimal Mal'cev condition for congruence meet–semidistributivity of a locally finite variety.*

$$\begin{cases} p(x,x,y) \approx p(x,y,y) \\ p(x,y,x) \approx q(x,x,y) \approx q(x,y,x) \approx q(y,x,x) \end{cases}$$