On automatic homeomorphicity for topological monoids

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An intriguing question

How much information about an ω-categorical structure can be recovered from its endomorphism monoid?

In particular, when does the endomorphism monoid of an ω-categorical structure have automatic homeomorphicity?

Automatic homeomorphicity

Given:

- $\mathbf{M} \leq \mathbf{T}_A$ – a closed transformation monoid and
- $\mathcal{K}$ – a set of closed transformation monoids on A.

$\mathbf{M}$ has **automatic homeomorphicity with respect to $\mathcal{K}$** if every monoid-isomorphism from $\mathbf{M}$ to an element of $\mathcal{K}$ is a homeomorphism.

$\mathbf{M}$ has **automatic homeomorphicity** if it has automatic homeomorphicity with respect to the set of all closed submonoids of $\mathbf{T}_A$.

Examples

Monoids with automatic homeomorphicity

- Monoid of injective selfmappings of A
- Full transformation monoid on A
- Monoid of selfembeddings of the Rado graph

Monoids with automatic homeomorphicity with respect to $\mathcal{K}$

- Monoid of selfembeddings of countable universal homogeneous tournament

Strong gate coverings

Given:

- A – a countably infinite set,
- $\mathcal{M} \subseteq T_A$ – a transformation monoid,
- G – the set of units in $\mathcal{M}$ and
- $\overline{G}$ – the closure of G in $\mathcal{M}$.

A **strong gate covering** of $\mathcal{M}$ is an open covering $\mathcal{U}$ of $\mathcal{M}$ with elements $f_U \in U$, for every $U \in \mathcal{U}$, such that

- for all $U \in \mathcal{U}$ and
- for all Cauchy-sequences $(g_n)_{n\in\mathbb{N}}$ in U

there exist Cauchy-sequences $(\kappa_n)_{n\in\mathbb{N}}$ und $(\iota_n)_{n\in\mathbb{N}}$ in $\overline{G}$ such that

$$\forall n \in \mathbb{N}: \quad g_n = \kappa_n \circ f_U \circ \iota_n.$$

Automatic homeomorphicity and strong gate coverings

Proposition

Let $\mathbf{A}$ and $\mathbf{B}$ be two countable relational structures such that

- $\mathrm{End}(\mathbf{A})$ has a strong gate covering.

Let $h : \mathrm{End}(\mathbf{A}) \to \mathrm{End}(\mathbf{B})$ be a monoid-isomorphism such that

- h is open and
- $h\restriction_{\mathrm{Aut}(\mathbf{A})}$ is continuous.

Then h is continuous. In particular, h is a homeomorphism.

Can we recognize when monoid-isomorphisms are open?

Proposition

Let $\mathbf{A}$ and $\mathbf{B}$ be two relational structures such that

- $\mathrm{End}(\mathbf{A})$ contains all constant functions and
- $\mathrm{Aut}(\mathbf{B})$ is oligomorphic.

Then every monoid-isomorphism $h : \mathrm{End}(\mathbf{A}) \to \mathrm{End}(\mathbf{B})$ is open.

When are group-isomorphisms homeomorphisms?

Small index property

Let A be a countable set and let $\mathbf{G} \leq \mathbf{S}_A$ be a closed subgroup.
Then $\mathbf{G}$ has the **small index property** if every subgroup of index less than $2^{\aleph_0}$ in $\mathbf{G}$ is open.

Small index property and automatic homeomorphicity

If $\mathbf{G} \leq \mathbf{S}_A$ has a small index property, than it has automatic homeomorphicity.

Weak $\forall\exists$ interpretations and automatic homeomorphicity

If an oligomorphic closed subgroup of $\mathbf{S}_A$ has a weak $\forall\exists$ interpretation, then it has automatic homeomorphicity with respect to the closed oligomorphic subgroups of $\mathbf{S}_A$.

Examples

Rationals

$\mathrm{Aut}(\mathbb{Q}, \leq)$ has the small index property.

Generic poset

$\mathrm{Aut}(\mathbb{P}, \leq)$ has a weak $\forall\exists$ interpretation.
It is not known whether it has the small index property.

How to obtain strong gate coverings?

Use universal homogeneous endomorphisms!

Let $\mathbf{U}$ be a countable structure.

Universal endomorphism

$u \in \mathrm{End}(\mathbf{U})$ is **universal** if for every $\mathbf{A} \in \overline{\mathrm{Age}(\mathbf{U})}$ and every $h : \mathbf{A} \to \mathbf{U}$ there exists an embedding $\iota : \mathbf{A} \hookrightarrow \mathbf{U}$ with $h = u \circ \iota$.

Homogeneous endomorphism

$u \in \mathrm{End}(\mathbf{U})$ is **homogeneous** if for

- $\forall \mathbf{A} \in \mathrm{Age}(\mathbf{U})$,
- $\forall h : \mathbf{A} \to \mathbf{U}$ and
- $\forall \iota_1, \iota_2 : \mathbf{A} \hookrightarrow \mathbf{U}$ with $h = u \circ \iota_1 = u \circ \iota_2$,

there exists an $f \in \mathrm{Aut}(\mathbf{U})$ with

$$f \circ \iota_1 = \iota_2 \text{ and } u \circ f = u$$

How to obtain strong gate coverings? Use universal homogeneous endomorphisms! (cont.)

Proposition

If $\mathbf{U}$ is a countable infinite relational structure that has a universal homogeneous endomorphism, then $\mathrm{End}(\mathbf{U})$ has a strong gate covering.

Structures with a universal homogeneous endomorphism

Amalgamated extension property (AEP)

$$
\begin{tikzcd}
 & & \mathbf{T} \\
\mathbf{B}_1 \arrow[urr, bend left, "h_1"] & & \\
\mathbf{A} \arrow[u, hook, "f_1"] \arrow[r, hook, "f_2"] & \mathbf{B}_2 \arrow[uur, bend right, "h_2"'] &
\end{tikzcd}
$$

Structures with a universal homogeneous endomorphism

Amalgamated extension property (AEP)

$$\begin{tikzcd}
 & & & \mathbf{T}' \\
 & & \mathbf{T} \arrow[ur, hook, "k"'] & \\
\mathbf{B}_1 \arrow[r, hook, "g_1"] \arrow[urr, bend left, "h_1"] & \mathbf{C} \arrow[uurr, bend left, "h"] & & \\
\mathbf{A} \arrow[u, hook, "f_1"] \arrow[r, hook, "f_2"] & \mathbf{B}_2 \arrow[u, hook, "g_2"'] \arrow[uu, bend right, "h_2"'] & &
\end{tikzcd}$$

Structures with a universal homogeneous endomorphism (cont.)

Homoalmagamation property (HAP)

$$\begin{tikzcd} \mathbf{B} & \\ \mathbf{A} \arrow[u, hook, "g"] \arrow[r, "a"] & \mathbf{T}_1. \end{tikzcd}$$

Structures with a universal homogeneous endomorphism (cont.)

Homoalmagamation property (HAP)

$$
\begin{tikzcd}
\mathbf{B} \arrow[r, "b"] & \mathbf{T}_2 \\
\mathbf{A} \arrow[u, hook, "g"'] \arrow[r, "a"] & \mathbf{T}_1. \arrow[u, hook, "h"]
\end{tikzcd}
$$

Structures with a universal homogeneous endomorphism (cont.)

Homoalmagamation property (HAP)

$$\begin{tikzcd} \mathbf{B} \arrow[r, "b"] & \mathbf{T}_2 \\ \mathbf{A} \arrow[u, hook, "g"] \arrow[r, "a"] & \mathbf{T}_1 \arrow[u, hook, "h"'] \end{tikzcd}.$$

Proposition — Existence of UHE

Let $\mathbf{U}$ be a countably infinite homogeneous structure. Then the following are equivalent:

1. $\mathbf{U}$ has a universal homogeneous endomorphism.
2. $\mathrm{Age}(\mathbf{U})$ has the AEP and the HAP.

Automatic homeomorphicity for $\mathrm{End}(\mathbb{Q}, \leq)$ and $\mathrm{End}(\mathbb{P}, \leq)$

Theorem 1

Let $\mathbf{B}$ be a countable ω-categorical structure, and let

$$h : \mathrm{End}(\mathbb{Q}, \leq) \to \mathrm{End}(\mathbf{B})$$

be a monoid isomorphism. Then h is a homeomorphism.

Theorem 2

Let $\mathbf{B}$ be a countable ω-categorical structure, and let

$$h : \mathrm{End}(\mathbb{P}, \leq) \to \mathrm{End}(\mathbf{B})$$

be a monoid isomorphism. Then h is a homeomorphism.