Uncountable critical points for congruence distributive varieties

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Congruence lattices

Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

Or, at least,

for given classes $\mathcal{K}$, $\mathcal{L}$ determine if Con $\mathcal{K}$ = Con $\mathcal{L}$
and, if Con $\mathcal{K} \nsubseteq$ Con $\mathcal{L}$, determine

$$\mathrm{Crit}(\mathcal{K}, \mathcal{L}) = \min\{\mathrm{card}(L_c) \mid L \in \mathrm{Con}\,\mathcal{K} \setminus \mathrm{Con}\,\mathcal{L}\}$$

(L_c = compact elements of L)

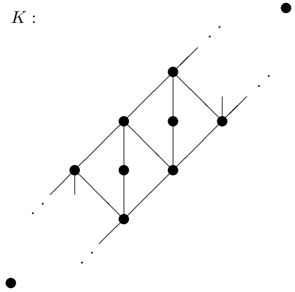
Some critical points

We are interested in the case when $\mathcal{K}$ and $\mathcal{L}$ are (congruence-distributive) varieties. For instance,
$\mathrm{Crit}(\mathbf{N}_5, \mathbf{M}_3) = 5$,
$\mathrm{Crit}(\mathbf{M}_3, \mathbf{N}_5) = \mathrm{Crit}(\mathbf{M}_3, \mathbf{D}) = \aleph_0$,
$\mathrm{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$,
$\mathrm{Crit}(\mathbf{Maj}, \mathbf{Lat}) = \aleph_2$.
($\mathbf{N}_5$, $\mathbf{M}_3$, $\mathbf{M}_4$ are well-known lattice varieties, $\mathbf{Lat}$ = all lattices, $\mathbf{Maj}$ = all majority algebras.)
P. Gillibert: under some reasonable finiteness conditions, the critical point between two varieties cannot be larger than $\aleph_2$.

Critical points $\aleph_1$

First such example has been discovered by P. Gillibert. We present three more examples.

Let $\mathbf{K}$ be the variety generated by the bounded lattice



Critical points $\aleph_1$

Theorem

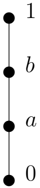
(1) $\mathrm{Crit}(\mathbf{N_5}, \mathbf{K}) = \aleph_1$;

(2) $\mathrm{Crit}(\mathbf{K}, \mathbf{N_5}) = \aleph_0$.

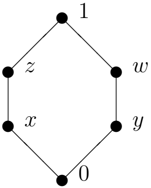
C_4 and N_6

Let $\mathbf{C}_4^*$ and $\mathbf{N}_6^*$ be the varieties generated by the bounded lattices C_4 and N_6 with an additional unary operation:
on C_4 ... $f(0) = 0$, $f(a) = b$, $f(b) = a$, $f(1) = 0$;
on N_6 ... 180° rotation $(f(x) = w \ldots)$.

C_4 :



N_6 :



Critical points $\aleph_1$

Theorem

(1) $\mathrm{Crit}(\mathbf{N}_6^*, \mathbf{N}_5) = \aleph_1$;

(2) $\mathrm{Crit}(\mathbf{N}_5, \mathbf{N}_6^*) = \aleph_0$.

(3) $\mathrm{Crit}(\mathbf{N}_6^*, \mathbf{C}_4^*) = \aleph_1$;

(4) $\mathrm{Crit}(\mathbf{C}_4^*, \mathbf{N}_6^*) = \infty$.

Question

What is the mechanism behind these examples?

For any homomorphism of algebras $f: A \to B$ we define

$$\mathrm{Con}_c\, f: \mathrm{Con}_c\, A \to \mathrm{Con}_c\, B$$

by
$\alpha \mapsto$ congruence generated by $\{(f(x), f(y)) \mid (x, y) \in \alpha\}$.

Fact. $\mathrm{Con}_c\, f$ preserves $\vee$ and 0, not necessarily $\wedge$.

For every commutative diagram $\mathcal{A}$ of algebras we have a commutative diagram $\mathrm{Con}\,\mathcal{A}$ of $(\vee, 0)$-semilattices.

Lifting of semilattice morphisms

Let

- $\varphi : \ S \to T$ be a homomorphism of $(\vee, 0)$-semilattices;
- $f : \ A \to B$ be a homomorphisms of algebras.

We say that f *lifts* φ, if there are isomorphisms $\psi_1 : \ S \to \operatorname{Con}_c A$, $\psi_2 : \ T \to \operatorname{Con}_c B$ such that

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ \psi_1 \Big\downarrow & & \psi_2 \Big\downarrow \\ \operatorname{Con}_c A & \xrightarrow{\operatorname{Con}_c f} & \operatorname{Con}_c B \end{array}$$

commutes.

Diagrams indexed by posets 1

Let

- $(P, \leq)$ be a poset;
- $\mathcal{K}$ be a category of algebras

Definition. A $(P, \leq)$-indexed diagram in $\mathcal{K}$ is a functor

$$\mathcal{A}: \ (P, \leq) \to \mathcal{K}.$$

Diagrams indexed by posets 2

That means:

- an algebra $\mathcal{A}(j) \in \mathcal{K}$ for every $j \in P$;
- a homomorphisms $\mathcal{A}(j,k): \ \mathcal{A}(j) \to \mathcal{A}(k)$ for every $j \leq k$;

such that

- $\mathcal{A}(j,j) = \mathrm{id}(\mathcal{A}(j))$ for every $j \in P$;
- $\mathcal{A}(j,k) \circ \mathcal{A}(i,j) = \mathcal{A}(i,k)$ for every $i \leq j \leq k$.

Lifting of diagrams

Let P be a poset and let

- $\mathcal{D}: \ P \to \mathcal{S}$ be a diagram of $(\vee, 0)$-semilattices;
- $\mathcal{A}: \ P \to \mathcal{K}$ be a diagram of algebras;

We say that $\mathcal{A}$ *lifts* $\mathcal{D}$, if there are isomorphisms $\psi_j: \ \mathcal{D}(j) \to \operatorname{Con}_c \mathcal{A}(j)$ such that

$$\begin{tikzcd} \mathcal{D}(j) \arrow[r, "\mathcal{D}(j{,}k)"] \arrow[d, "\psi_j"'] & \mathcal{D}(k) \arrow[d, "\psi_k"'] \\ \operatorname{Con}_c \mathcal{A}(j) \arrow[r, "\operatorname{Con}_c \mathcal{A}(j{,}k)"] & \operatorname{Con}_c \mathcal{A}(k) \end{tikzcd}$$

commutes for every $j \leq k$.

Let $\mathcal{K}$, $\mathcal{L}$ be locally finite congruence distributive varieties.

Theorem

TFAE

- $\mathrm{Con}\,\mathcal{K} \nsubseteq \mathrm{Con}\,\mathcal{L}$;
- *there exists a diagram of finite* $(\vee, 0)$*-semilattices indexed by* $\{0,1\}^n$ *(for some* n*) liftable in* $\mathcal{K}$ *but not in* $\mathcal{L}$

Theorem

(2) implies (1), where

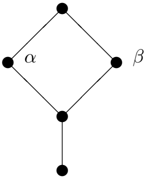
- $\mathrm{Crit}(\mathcal{K}, \mathcal{L}) \leq \aleph_n$;
- *there exists a diagram of finite* $(\vee, 0)$*-semilattices indexed by a product of* $n+1$ *finite chains liftable in* $\mathcal{K}$ *but not in* $\mathcal{L}$

If $n = 0$ *then also (1)*$\Longrightarrow$ *(2).*

Especially, if there exists a diagram of finite $(\vee, 0)$-semilattices indexed by a square liftable in $\mathcal{K}$ but not in $\mathcal{L}$, then $\mathrm{Crit}(\mathcal{K}, \mathcal{L}) \leq \aleph_1$.

N6 versus N5

Both N_5 and N_6^* have the same congruence lattice, but N_6^* has an automorphism h (the vertical symmetry), such that $\mathrm{Con}_c\, h$ interchanges α and β:



N6 versus N5

Below: $\mathcal{D}$ is the diagram in $\mathbf{N}_6^*$, so that $\operatorname{Con}\mathcal{D}$ has a lifting in $\mathbf{N}_6^*$ but - no lifting in $\mathbf{N}_5$.

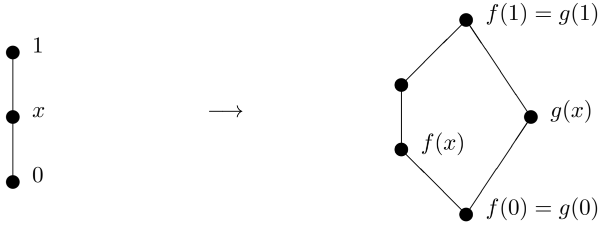
$$
\begin{tikzcd}
 & N_6^* \times N_6^* & \\
N_6^* \arrow[ur, "\mathrm{id}\times\mathrm{id}"] & & N_6^* \arrow[ul, "\mathrm{id}\times h"'] \\
 & \{0,1\} \arrow[ul, "\subseteq"'] \arrow[ur, "\subseteq"] &
\end{tikzcd}
$$

Observation

Every automorphism $f: A \to A$ induces an automorphism $\mathrm{Con}_c f: \mathrm{Con}_c A \to \mathrm{Con}_c A$. These induced automorphisms form a subgroup of the automorphism group of $\mathrm{Con}_c A$. And this subgroup has an influence on the class $\mathrm{Con}\,\mathbf{A}$, where A is the variety generated by A.

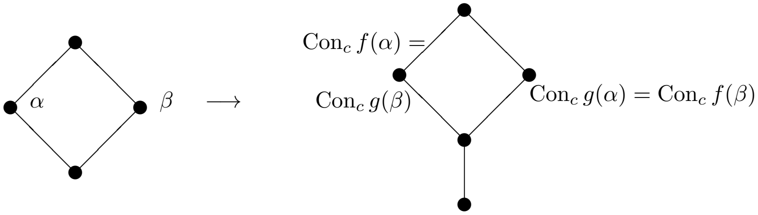
N5 versus K

The same idea as before, but more subtle. Not only automorphisms are important.
Consider the homomorphisms f, g in $\mathbf{N}_5$:



N5 versus K

The maps $\mathrm{Con}_c\, f$ and $\mathrm{Con}_c\, g$:



N5 versus K

If $\mathcal{D}$ is the diagram below, then $\operatorname{Con}\mathcal{D}$ has a lifting in $\mathbf{N}_5$ but not in $\mathbf{K}$.

$$\begin{tikzcd}
 & N_5 \times N_5 & \\
N_5 \arrow[ur, "f \times f"] & & N_5 \arrow[ul, "f \times g"'] \\
 & \{0,1\} \arrow[ul, "\subseteq"'] \arrow[ur, "\subseteq"] &
\end{tikzcd}$$

Gillibert's example

Different mechanism: a semilattice homomorphism $\varphi : S \to T$ with two liftings $f : A \to B_1$, $g : A \to B_2$ such that $\mathrm{Con}\, f$ and $\mathrm{Con}\, g$ have different kernels.

Possible general "theorem":

$\mathrm{Crit}(\mathbf{V}_1, \mathbf{V}_2) = \aleph_1$ occurs when all diagrams indexed by a finite chain liftable in $\mathbf{V}_1$ are also liftable in $\mathbf{V}_2$, but the liftings in $\mathbf{V}_2$ are "less symmetric".

Congruence intersection

A variety $\mathbf{V}$ has the *Compact intersection property* (CIP) if the intersection of two compact congruences on any $A \in \mathbf{V}$ is compact.

- If $\mathbf{V}$ has CIP, then Con $\mathbf{V}$ is tractable.
- Critical point $\aleph_1$ can still occur ($\mathbf{N}_6^*$ versus $\mathbf{C}_4^*$).
- Conjecture: $\aleph_2$ cannot occur.