On the cardinality of the C-clone lattice.

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Conference on Algebras and Clones (ALC fest).

Presenting joint work with...

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Outline

Clausal Relations

GALOIS connection Pol – CInv

Lattice of C-clones

Clausal Relations

Let $p, q \in \mathbb{N}_+ := \{1, 2, \ldots\}$, $D := \{0, 1, \ldots, n-1\}$ and $(D; \leq)$ chain.

Clausal Relations

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Definition

For given parameters $\mathbf{a} = (a_1, \ldots, a_p) \in D^p$ and $\mathbf{b} = (b_1, \ldots, b_q) \in D^q$, the *clausal relation* $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ *of arity* $p + q$ is the set of all tuples $(x_1, \ldots, x_p, y_1, \ldots, y_q) \in D^{p+q}$ satisfying

$$(x_1 \geq a_1) \vee \cdots \vee (x_p \geq a_p) \vee (y_1 \leq b_1) \vee \cdots \vee (y_q \leq b_q).$$

In this expression $\leq$ denotes the canonical linear order on D and $\geq$ its dual.

Let $D = \{0, 1, 2\}$, then

$$\begin{aligned} \mathrm{R}_1^2 &= \{(x_1, y_1) \in D^2 \mid x_1 \geq 2 \vee y_1 \leq 1\} \\ &= \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 \end{pmatrix} = D^2 \setminus \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}. \end{aligned}$$

Fact

- *If* $(\exists i \in \{1, \ldots, p\} : a_i = 0) \implies \mathrm{R}^{\mathbf{a}}_{\mathbf{b}} = D^{p+q}$.
- *If* $(\exists j \in \{1, \ldots, q\} : b_j = n - 1) \implies \mathrm{R}^{\mathbf{a}}_{\mathbf{b}} = D^{p+q}$.

$$CR_D := \bigcup_{(p,q)\in\mathbb{N}_+^2} \{\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in D^p, \mathbf{b} \in D^q\}$$

the *set of all finitary clausal relations* on D.

If $|D| \leq \aleph_0$, then $|CR_D| \leq \aleph_0$.

$$\begin{aligned} \varphi : D^p \times D^q &\longrightarrow \mathrm{R}_q^p := \{\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in D^p, \mathbf{b} \in D^q\} \\ (\mathbf{a}, \mathbf{b}) &\mapsto \quad \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \end{aligned}$$

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$$\Rightarrow |\mathrm{R}^p_q| \leq |D^p \times D^q| = |D^{p+q}| \leq \aleph_0$$

$$|CR_D| \leq \aleph_0.$$

Lemma

$$
\begin{aligned}
CR_D \cap \operatorname{diag}(D) &= \{ D^{p+q} \mid p, q \in \mathbb{N}_+ \} . \\
CR_D^* &:= CR_D \setminus \operatorname{diag}(D) \\
&= \{ \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in (D \setminus \{0\})^p, \mathbf{b} \in (D \setminus \{n-1\})^q ; p, q \in \mathbb{N}_+ \} .
\end{aligned}
$$

Non-trivial clausal relations (Essential predicates)

$$
(0, \ldots, 0, n-1, \ldots, n-1) \notin \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \in CR_D^* \qquad \text{but}
$$

$$
(a_1, 0, \ldots, 0, n-1, \ldots, n-1), \ldots, (0, \ldots, 0, n-1, \ldots, n-1, b_q) \in \mathrm{R}_{\mathbf{b}}^{\mathbf{a}}
$$

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GALOIS connection Pol – CInv

Lattice of C-clones

For $n \in \mathbb{N}_+$ called arity,

$$f : D^n \longrightarrow D$$

$$(x_1, \ldots, x_n) \longmapsto f(x_1, \ldots, x_n)$$

n-ary operation on D

$$\mathrm{O}_D^{(n)} := D^{D^n}$$

set of n-ary operations on D

$$\mathrm{O}_D := \bigcup_{k \in \mathbb{N}_+} \mathrm{O}_D^{(k)}$$

set of all finitary operations on D.

For $m \in \mathbb{N}_+$ subsets $\varrho \subseteq D^m$ are

m-ary relations on D

$$\mathrm{R}_D^{(m)} := \mathcal{P}(D^m)$$

set of m-ary relations on D

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GALOIS correspondence connecting

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$$f \circ \left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{pmatrix} \right)$$

$$\begin{array}{ccc} \Cap & \qquad & \Cap \\ \varrho & & \varrho \end{array}$$

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$$f \circ \left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{pmatrix} \right) = \begin{pmatrix} f(a_{11} & \dots & a_{1k}) \\ & \ddots & \\ f(a_{m1} & \dots & a_{mk}) \end{pmatrix}$$

$$\begin{array}{ccc} \Cap & \Cap & \Cap \\ \varrho & \varrho & \varrho \end{array}$$

Polymorphisms and Invariant relations

Definition

$F \subseteq \mathrm{O}_D$, $Q \subseteq \mathrm{R}_D$

$$\begin{aligned} \mathrm{Pol}_D\, Q &:= \{f \in \mathrm{O}_D \mid \forall \varrho \in Q : f \rhd \varrho\} \\ \mathrm{Inv}_D\, F &:= \{\varrho \in \mathrm{R}_D \mid \forall f \in F : f \rhd \varrho\} \end{aligned}$$

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The mappings

$$\mathrm{Pol}\, Q \leftarrow\!\shortmid Q$$
$$F \mapsto \mathrm{Inv}\, F$$

define a GALOIS *connection* Pol – Inv induced by $\rhd$.

On finite D, clones(of operations) are

Theorem (Bodnarčuk, Kalužnin, Kotov, Romov 69)

For D finite, F is a clone $\implies F = \mathrm{Pol}_D\, Q$ *for* $Q = \mathrm{Inv}_D\, F$.

Every clone can be described by relations.

Idea

To confine the allowed relations to be clausal relations.

$$\mathit{C}\mathrm{Inv}\, F := \mathrm{Inv}_D\, F \cap \mathit{C}\mathrm{R}_D$$

Definition

$F \subseteq \mathrm{O}_D$ is C-clone $:\iff$ $F = \mathrm{Pol}_D\, Q$, where Q is a set of clausal relations.

Lemma

$a, b \in D, f \in \mathrm{O}_D^{(1)}$. If $im(f) \subseteq \{0, \ldots, b\}$ or $im(f) \subseteq \{a, \ldots, n-1\}$, then

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Example

$$D = \{0, 1, 2, 3\}, \quad f(x) := \begin{cases} 2 & \text{if } x = 1 \\ 3 & \textit{otherwise.} \end{cases} \quad \rhd \ \mathrm{R}_1^2.$$

The set of all C-clones forms a lattice w.r.t. $\subseteq$.

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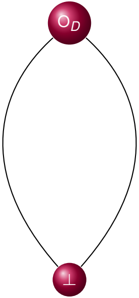

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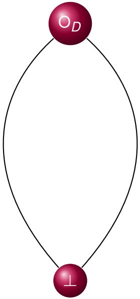
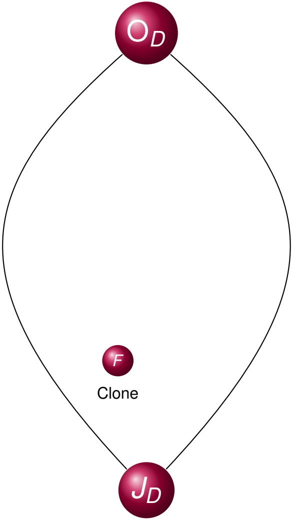


$$\left\langle \left\{ c_0^{(k)}, c_{n-1}^{(k)} \right\} \right\rangle_{O_D}$$

Where do the C-clones live?



For $F \subseteq O_D$

$$\operatorname{Inv} F \supseteq CR_D \cap \operatorname{Inv} F = C\operatorname{Inv} F$$

$$\Rightarrow \langle F \rangle_{O_D} = \operatorname{Pol} \operatorname{Inv} F \subseteq \operatorname{Pol} C\operatorname{Inv} F := \langle F \rangle_C$$

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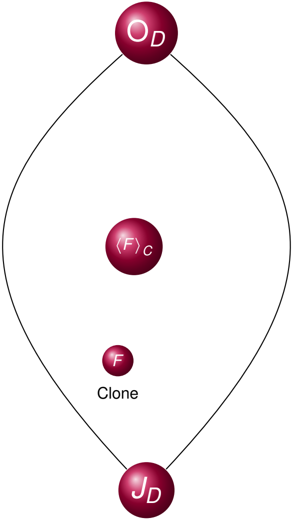

For $F \subseteq \mathrm{O}_D$

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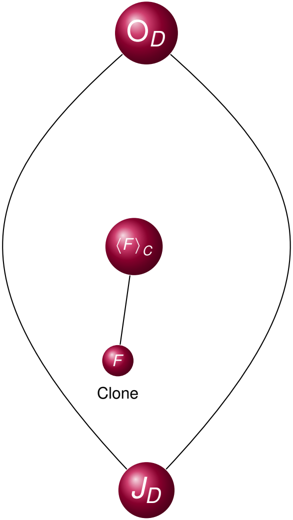

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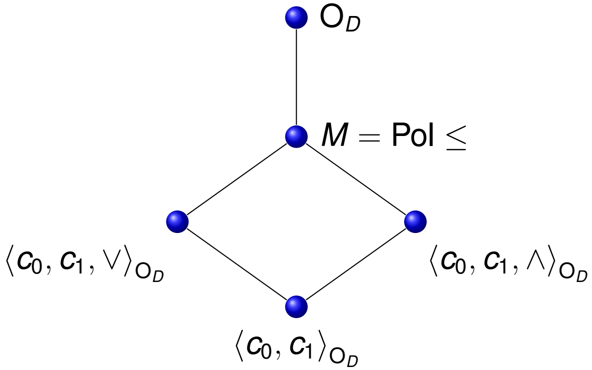
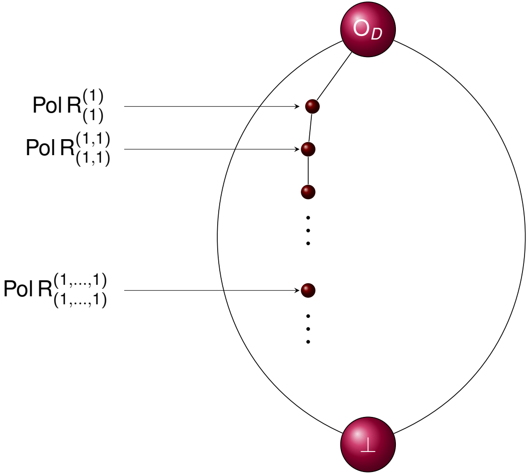
How many C-clones do exist for an arbitrary finite set D?

- For $D = \{0, 1\}$, there are five different C-clones.



Lattice of C-clones for $D = \{0, \ldots, n-1\}, n \geq 3$

- Contains countably infinite descending chains.



Let $\varrho_m := \mathrm{R}^{(\overbrace{1,\ldots,1}^{m \text{ times}})}_{(\underbrace{1,\ldots,1}_{m \text{ times}})}$

$$(x_1 \geq 1) \vee \ldots \vee (x_m \geq 1) \vee (y_1 \leq 1) \vee \ldots \vee (y_m \leq 1)$$

- $\mathrm{Pol}\,(\varrho_m) \subsetneq \mathrm{Pol}\,(\varrho_{m-1})$

$$f(x_1,\ldots,x_{2m}) := \begin{cases} 0 & \text{if there is only one 1 among } x_1,\ldots,x_m \\ & \text{and 0 in the other entries,} \\ n-1 & \text{if there is only one 1 among } x_{m+1},\ldots,x_{2m} \\ & \text{and } n-1 \text{ in the other entries,} \\ 1 & \text{otherwise.} \end{cases}$$

$$\text{If } \sigma_m := \mathrm{R}^{(2,\ldots,\overset{m}{2})}_{(2,\ldots,2)}, \text{ then } \mathrm{Pol}\,(\sigma_m) \subsetneq \mathrm{Pol}\,(\sigma_{m-1}).$$

$$\mathrm{Pol}\,\mathrm{R}^{(1,\ldots,1)}_{(1,\ldots,1)} \nsubseteq \mathrm{Pol}\,\mathrm{R}^{(2,\ldots,2)}_{(2,\ldots,2)} \quad \text{and} \quad \mathrm{Pol}\,\mathrm{R}^{(2,\ldots,2)}_{(2,\ldots,2)} \nsubseteq \mathrm{Pol}\,\mathrm{R}^{(1,\ldots,1)}_{(1,\ldots,1)}$$

$$f\,(x_1,\ldots,x_{2m}) := \begin{cases} 1 & \text{if } \exists \text{ exactly one 2 among } x_1,\ldots,x_m \\ & \text{and 0 in the other entries} \\ 3 & \textit{otherwise.} \end{cases}$$

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$\mathrm{Pol}\,\mathrm{R}_{(1,\dots,1)}^{(1,\dots,1)} \nsubseteq \mathrm{Pol}\,\mathrm{R}_{(2,\dots,2)}^{(2,\dots,2)}$ and $\mathrm{Pol}\,\mathrm{R}_{(2,\dots,2)}^{(2,\dots,2)} \nsubseteq \mathrm{Pol}\,\mathrm{R}_{(1,\dots,1)}^{(1,\dots,1)}$

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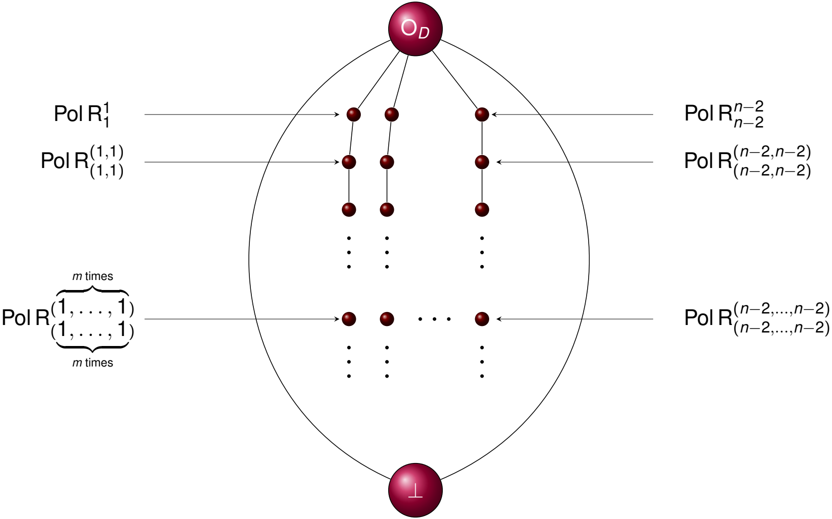
$$g\,(x_1,\dots,x_{2m}) := \begin{cases} 2 & \text{if } \exists \text{ exactly one 1 among } x_{m+1},\dots,x_{2m} \\ & \text{and 3 in the other entries} \\ 0 & \textit{otherwise.} \end{cases}$$


O_D
$\mathrm{Pol}\,\mathrm{R}_1^1$
$\mathrm{Pol}\,\mathrm{R}_{(1,1)}^{(1,1)}$
$\mathrm{Pol}\,\mathrm{R}_{\underbrace{(1,\ldots,1)}_{m \text{ times}}}^{\overbrace{(1,\ldots,1)}^{m \text{ times}}}$
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$\perp$

Are there countably many C-clones ?

Claim to be proven:

$$(\forall\, \mathrm{Pol}_D\, Q;\, Q \subseteq C\mathrm{R}_D) \text{ there is a finite } Q_0 \subseteq C\mathrm{Inv}\,\mathrm{Pol}\, Q;$$
$$\mathrm{Pol}_D\, Q = \mathrm{Pol}_D\, Q_0.$$

Then following map is surjective

$$\begin{aligned} \psi : \mathcal{P}_{\text{fin}}(C\mathrm{R}_D) &\longrightarrow C\mathcal{L}_D \\ Q_0 &\mapsto \mathrm{Pol}\, Q_0. \end{aligned}$$

Hence,

$$|C\mathcal{L}_D| \leq |\mathcal{P}_{\text{fin}}(C\mathrm{R}_D)| \leq \aleph_0,$$

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$$|C\mathcal{L}_D| \leq |\mathcal{P}_{\mathrm{fin}}(C\mathrm{R}_D)| \leq \aleph_0, \text{ since } |C\mathrm{R}_D| \leq \aleph_0.$$

Are there uncountably many C-clones ?

Countable $\{f_i \mid i \in \mathbb{N}\}$, countable $\{\varrho_j \mid j \in \mathbb{N}\}$ of CR_D satisfying

$$\forall i, j \in \mathbb{N} \qquad f_i \rhd \varrho_j \iff i \neq j$$

For $I \subseteq \mathbb{N}$, we define $F_I := \operatorname{Pol} C\operatorname{Inv} \{f_i \mid i \in I\}$.

For $j \in \mathbb{N}$, we prove $f_j \in F_I \Rightarrow j \in I$.

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$$2^{\aleph_0} = |\mathcal{P}(\mathbb{N})| \leq |C\mathcal{L}_D| \leq |\mathcal{L}_D| = 2^{\aleph_0}$$

Inclusion of C-clones definable by one clausal relation

Let $R_a \in \mathbb{N}_+$, $a_{\wedge I} := \min\{a_\upsilon \mid \upsilon \in I \subseteq \{1,\dots,p\}\}$

$$\left\{I_1^a,\dots,I_{R_a}^a\right\} \subseteq \mathcal{P}(\{1,\dots,p\}) \setminus \{\emptyset\}$$

- $I_1^a,\dots,I_{R_a}^a$ are pairwise different
- $\bigcup_{t=1}^{R_a} I_t^a = \{1,\dots,p\}$
- $\exists k_1^a,\dots,k_{R_a}^a \in \{1,\dots,p\}$ pairwise different:

$$\forall t \in \{1,\dots,R_a\} : k_t^a \in I_t^a,\ a_{\wedge I_t^a} = a_{k_t^a}$$

$$\operatorname{Pol} \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \subseteq \operatorname{Pol} \mathrm{R}_{(b_{\vee I_1^b},\dots,b_{\vee I_{R_b}^b})}^{(a_{\wedge I_1^a},\dots,a_{\wedge I_{R_a}^a})}$$

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$\operatorname{Pol} \mathrm{R}_{(1,2,3,7,3)}^{(4,2,3,2)}$, $R_a = 2$

$I_1^a = \{1,2,3\}$, $I_2^a = \{2,4\}$

$k_1^a = 2, k_2^a = 4$

$a_2 = 2, a_4 = 2$

$\operatorname{Pol} \mathrm{R}_{(1,2,3,7,3)}^{(2,2)}$

$$\operatorname{Pol} \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \subseteq \operatorname{Pol} \mathrm{R}_{(b_{\vee I_1^b},\dots,b_{\vee I_{R_b}^b})}^{(a_{\wedge I_1^a},\dots,a_{\wedge I_{R_a}^a})}$$

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$$\forall s \in \{1,\ldots,R_b\} : k_s^b \in I_s^b,\ b_{\vee I_s^b} = b_{k_s^b}$$

$\mathrm{Pol}\,\mathrm{R}_{(1,2,3,7,3)}^{(4,2,3,2)}$, $R_b = 3$

$I_1^b = \{1,2,3\}$, $I_2^b = \{2,3,4\}$

$I_3^b = \{2,3,5\}$

$k_1^b = 3, k_2^b = 4, k_3^b = 5$

$b_3 = 3, b_4 = 7, b_5 = 3$

$\mathrm{Pol}\,\mathrm{R}_{(3,7,3)}^{(4,2,3,2)}$

Inclusion of C-clones definable by one clausal relation

Let $R_b \in \mathbb{N}_+$, $b_{\vee I} := \max\{b_\upsilon \mid \upsilon \in I \subseteq \{1,\ldots,q\}\}$

$$\left\{I_1^b,\ldots,I_{R_b}^b\right\} \subseteq \mathcal{P}(\{1,\ldots,q\}) \setminus \{\emptyset\}$$

$\operatorname{Pol} \mathrm{R}^{(4,2,3,2)}_{(1,2,3,7,3)}$, $R_b = 3$

- $I_1^b,\ldots,I_{R_b}^b$ are pairwise different
- $\bigcup_{s=1}^{R_b} I_s^b = \{1,\ldots,q\}$
- $\exists k_1^b,\ldots,k_{R_b}^b \in \{1,\ldots,q\}$ pairwise different:

$$\forall s \in \{1,\ldots,R_b\} : k_s^b \in I_s^b,\ b_{\vee I_s^b} = b_{k_s^b}$$

$I_1^b = \{1,2,3\}$, $I_2^b = \{2,3,4\}$

$I_3^b = \{2,3,5\}$

$k_1^b = 3, k_2^b = 4, k_3^b = 5$

$b_3 = 3, b_4 = 7, b_5 = 3$

$\operatorname{Pol} \mathrm{R}^{(4,2,3,2)}_{(3,7,3)}$

$$\boxed{\operatorname{Pol} \mathrm{R}^{(4,2,3,2)}_{(1,2,3,7,3)} \subseteq \operatorname{Pol} \mathrm{R}^{(2,2)}_{(3,7,3)}}$$

Inclusions of C-clones on $D = \{0,1,2\}$, definible by one clausal relation.

Let $\mathrm{R}^{\mathbf{c}}_{\mathbf{d}} := \mathrm{R}^{2,\ldots,\overset{p_1}{2},1,\ldots,\overset{p_2}{1}}_{1,\ldots,\underset{q_1}{1},0,\ldots,\underset{q_2}{0}}$, $p_1, p_2, q_1, q_2 \geq 0$, $p_1 + p_2 > 0$ and $q_1 + q_2 > 0$.

Fine type

$$(p_1, p_2, q_1, q_2) := \mathrm{ft}(\mathrm{R}^{\mathbf{c}}_{\mathbf{d}})$$

When

$$\mathrm{ft}\left(\mathrm{R}^{\mathbf{c}}_{\mathbf{d}}\right) \leq \mathrm{ft}\left(\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}\right) \Rightarrow \mathrm{Pol}_D\,\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \subseteq \mathrm{Pol}_D\,\mathrm{R}^{\mathbf{c}}_{\mathbf{d}}?$$

Under the assumption:

whenever an entry of $\mathrm{ft}\left(\mathrm{R}^{\mathbf{c}}_{\mathbf{d}}\right) = 0$, then the corresponding entry of $\mathrm{ft}\left(\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}\right) = 0$.

$$(p_1, p_2, q_1, q_2) = (4, 1, 3, 2), (p'_1, p'_2, q'_1, q'_2) = (6, 4, 3, 4).$$

$$\mathrm{ft}\left(\mathrm{R}^{(2,2,2,2,1)}_{(1,1,1,0,0)}\right) \leq \mathrm{ft}\left(\mathrm{R}^{(2,2,2,2,2,2,1,1,1,1)}_{(1,1,1,0,0,0,0)}\right)$$

$$I^a_1 = \{1\}, I^a_2 = \{2\}, I^a_3 = \{3\}, I^a_4 = \{4, 5, 6\}, I^a_5 = \{7, 8, 9, 10\}$$

$$I^b_1 = \{1\}, I^b_2 = \{2\}, I^b_3 = \{3\}, I^b_4 = \{4\}, I^b_5 = \{5, 6, 7\}$$

$$\mathrm{Pol}\,\mathrm{R}^{(2,2,2,2,2,2,1,1,1,1)}_{(1,1,1,0,0,0,0)} \subseteq \mathrm{Pol}\,\mathrm{R}^{(2,2,2,2,1)}_{(1,1,1,0,0)}$$

Thank you for your attention :)