

On Syntactic Structures

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An Outline

- I. **Algebraic Theory of Regular Languages**
Varieties of regular languages, syntactic monoid, Eilenberg correspondence and its generalizations.
- II. **Varieties of Automata**
Minimal DFA, closure operators, Eilenberg type correspondence.
- III. **Automata Enriched with an Algebraic Structure**
Ordered automata, meet-automata, DL-automata, Eilenberg type correspondence.
- IV. **Syntactic Structures**

I. Algebraic Theory of Regular Languages

Examples

- Goal of the study: effective characterizations of certain natural classes of regular languages.
- Typical result: a language belongs to a given class iff its syntactic monoid belongs to a certain class of monoids.

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- Typical result: a language belongs to a given class iff its syntactic monoid belongs to a certain class of monoids.

Theorem (Schützenberger – 1966)

A regular language L is star-free if and only if its syntactic monoid is aperiodic.

Examples

Concrete Examples. For $A = \{a, b\}$, $L = (ab)^*$, we have
 $A^* = 0'$, $L = \lambda + ((bA^*)' \cap (A^*a)' \cap (A^*aaA^*)' \cap (A^*bbA^*)')$.
 $K = (ab + ba)^* = \dots$

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Theorem (Simon — 1972)

A regular language L is piecewise testable if and only if the syntactic monoid of L is \mathcal{J} -trivial.

- General framework – Eilenberg correspondence.

Varieties of Languages

Definition

A **variety of languages** \mathcal{V} associates to every finite alphabet A a class $\mathcal{V}(A)$ of regular languages over A in such a way that

- $\mathcal{V}(A)$ is closed under finite unions, finite intersections and complements (in particular $\emptyset, A^* \in \mathcal{V}(A)$),
- $\mathcal{V}(A)$ is closed under quotients, i.e.
 $L \in \mathcal{V}(A)$, $u, v \in A^*$ implies
 $u^{-1}Lv^{-1} = \{w \in A^* \mid uwv \in L\} \in \mathcal{V}(A)$,
- \mathcal{V} is closed under preimages in morphisms, i.e.
 $f : B^* \rightarrow A^*$, $L \in \mathcal{V}(A)$ implies
 $f^{-1}(L) = \{v \in B^* \mid f(v) \in L\} \in \mathcal{V}(B)$.

Pseudovarieties of Monoids

Definition

A **pseudovariety** of finite monoids is a class of finite monoids closed under submonoids, morphic images and products of finite families.

- For a regular language $L \subseteq A^*$ we define a relation \sim_L on A^* by the rule $u \sim_L v$ iff u and v have the same contexts in L .

Formally: $u \sim_L v \iff \{(p, q) \mid puq \in L\} = \{(p, q) \mid pvq \in L\}$.

- \sim_L is the **syntactic congruence** of L and $A^*/\sim_L = M_L$ is the **syntactic monoid** of L .

The Eilenberg Correspondence

- For each pseudovariety of monoids \mathbf{V} , we denote $\alpha(\mathbf{V})$ the variety of regular languages given by

$$(\alpha(\mathbf{V}))(A) = \{L \subseteq A^* \mid M_L \in \mathbf{V}\}.$$

- For each variety of regular languages \mathcal{L} , we denote by $\beta(\mathcal{L})$ the pseudovariety of monoids generated by syntactic monoids M_L , where $L \in \mathcal{L}(A)$ for some alphabet A .

Theorem (Eilenberg – 1976)

The mappings α and β are mutually inverse isomorphisms between the lattice of all pseudovarieties of finite monoids and the lattice of all varieties of regular languages.

A Formal Definition of a DFA

Definition

A **deterministic finite automaton** over the alphabet A is a five-tuple $\mathcal{A} = (Q, A, \cdot, i, F)$, where

- Q is a nonempty set of states,
- $\cdot : Q \times A \rightarrow Q$ is a **complete** transition function, which can be extended to a mapping $\cdot : Q \times A^* \rightarrow Q$ by $q \cdot \lambda = q$, $q \cdot (ua) = (q \cdot u) \cdot a$,
- $i \in Q$ is the initial state,
- $F \subseteq Q$ is the set of final states.

The automaton \mathcal{A} accepts a word $u \in A^*$ iff $i \cdot u \in F$.

The automaton \mathcal{A} recognizes the language

$$L_{\mathcal{A}} = \{u \in A^* \mid i \cdot u \in F\}.$$

A Relationship between DFAs and Monoids

If we have a DFA $\mathcal{A} = (Q, A, \cdot, i, F)$, then:

- Each word $u \in A^*$ performs the transformation $\tau_u : Q \rightarrow Q$ where $\tau_u(q) = q \cdot u$ for each $q \in Q$.
- The **transition monoid** of \mathcal{A} is $(\{\tau_u \mid u \in A^*\}, \circ)$.
- The transition monoid of the minimal automaton of L is isomorphic to the syntactic monoid M_L of L .

Motivations for a Notion of a Variety of Automata

- Why monoids instead of automata?
 - An equational description of pseudovarieties of monoids by pseudoidentities.
 - Other algebraic constructions, e.g. products (semidirect, wreath, Mal'cev).

Motivations for a Notion of a Variety of Automata

- Why monoids instead of automata?
 - An equational description of pseudovarieties of monoids by pseudoidentities.
 - Other algebraic constructions, e.g. products (semidirect, wreath, Mal'cev).
- Why are we still interested in automata characterizations?
 - Usually, a regular language is given by an automaton. And computation of the syntactic monoid need not to be effective (can be exponentially larger).
 - Sometimes a “graph condition” on automata can be easier to test than an equational condition on monoids.

Generalizations of the Eilenberg Correspondence

Since not all natural classes of regular languages are varieties, one of the recent directions of the research in algebraic theory of regular languages is devoted to generalizations of the Eilenberg correspondence.

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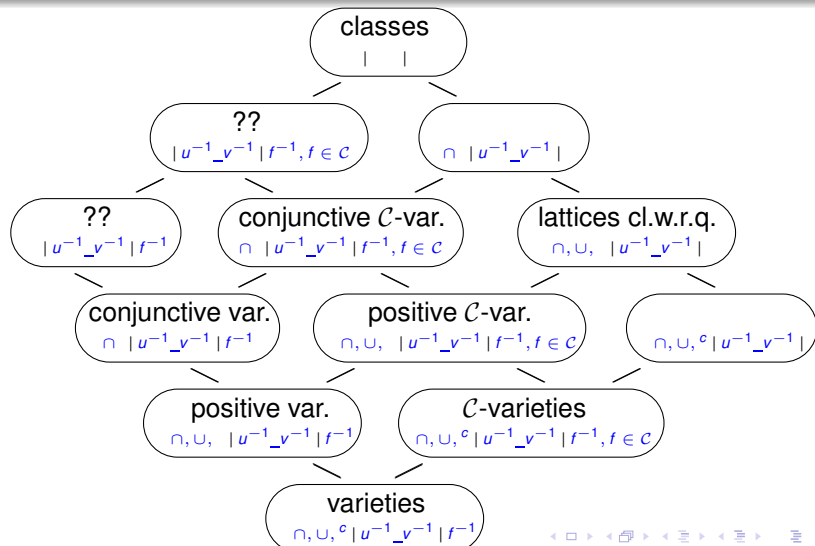
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(Syntactic monoid is implicitly ordered.)

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- Pin (1995): Positive varieties of regular languages — closure under complementation is not required. Algebraic counterparts are pseudovarieties of finite ordered monoids.
(Syntactic monoid is implicitly ordered.)
- Polák (1999): Conjunctive (and disjunctive) varieties.
- Straubing (2002): \mathcal{C} -varieties of languages.
- Ésik, Larsen (2003): literal varieties of languages.
- Gehrke, Grigorieff, Pin (2008): Lattices of regular languages.

Variants of Varieties of Regular Languages



II. Varieties of Automata

The Construction of a Minimal DFA by Brzozowski

- For a language $L \subseteq A^*$ and $u \in A^*$, we define a **left quotient** $u^{-1}L = \{ w \in A^* \mid uw \in L \}$.

Definition

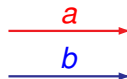
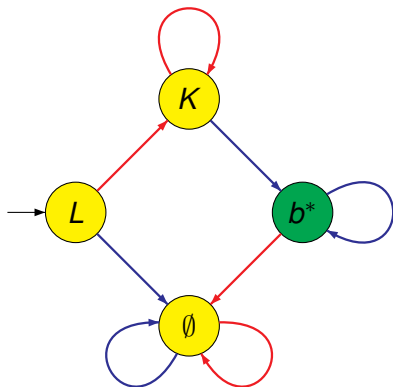
The **canonical deterministic automaton** of L is

$\mathcal{D}_L = (D_L, A, \cdot, L, F)$, where

- $D_L = \{ u^{-1}L \mid u \in A^* \}$,
- $q \cdot a = a^{-1}q$, for each $q \in D_L$, $a \in A$,
- $q \in F$ iff $\lambda \in q$.

- Each state $q = u^{-1}L$ is formed by all words transforming the state q into a final state.

An Example of a Canonical Automaton



$$L = a^+ b^+$$

$$K = a^{-1}L = a^* b^+$$

$$b^{-1}K = b^*$$

Preimages in Morphisms, Varieties of Automata

- Let $f : B^* \rightarrow A^*$ be a morphism, We say that (P, B, \circ) is an f -subautomaton of (Q, A, \cdot) if $P \subseteq Q$ and $q \circ b = q \cdot f(b)$ for every $q \in P, b \in B$.

Definition

A **variety of automata** \mathbb{V} associates to every finite alphabet A a class $\mathbb{V}(A)$ of automata (without initial and final states) over alphabet A in such a way that

- $\mathbb{V}(A) \neq \emptyset$ is closed under disjoint unions, finite direct products and morphic images,
- \mathbb{V} is closed under f -subautomata.

An Eilenberg Type Correspondence

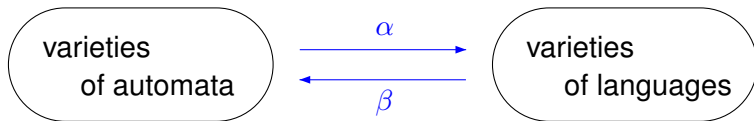
- For each variety of automata \mathbb{V} we denote by $\alpha(\mathbb{V})$ the variety of regular languages given by

$$(\alpha(\mathbb{V}))(A) = \{L \subseteq A^* \mid \exists \mathcal{A} = (Q, A, \cdot, i, F) :$$

$$L = L_{\mathcal{A}} \wedge (Q, A, \cdot) \in \mathbb{V}(A)\}.$$

- For each variety of regular languages \mathcal{L} we denote by $\beta(\mathcal{L})$ the variety of automata generated by all DFAs \mathcal{D}_L , where $L \in \mathcal{L}(A)$ for some alphabet A .

An Eilenberg Type Correspondence II



Theorem (Ésik and Ito, Chaubard, Pin and Straubing)

The mappings α and β are mutually inverse isomorphisms between the lattice of all varieties of automata and the lattice of all varieties of regular languages.

- A version for \mathcal{C} -varieties is obvious: we consider f -subautomata (etc.) just for $f \in \mathcal{C}$.
- Ésik and Ito were working with literal varieties (morphisms map letters to letters, i.e. $f(B) \subseteq A$) and used disjoint union.
- Chaubard, Pin and Straubing called the automata \mathcal{C} -actions and used trivial automata.

An Examples – Acyclic Automata

- One of the conditions in Simon's characterization of piecewise testable languages is that a minimal DFA is acyclic.
- A content $c(u)$ of a word $u \in A^*$ is the set of all letters occurring in u .
- We say that (Q, A, \cdot) is a **acyclic** if for each $u \in A^*$ and $q \in Q$ we have

$$q \cdot u = q \implies (\forall a \in c(u) : q \cdot a = q).$$

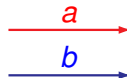
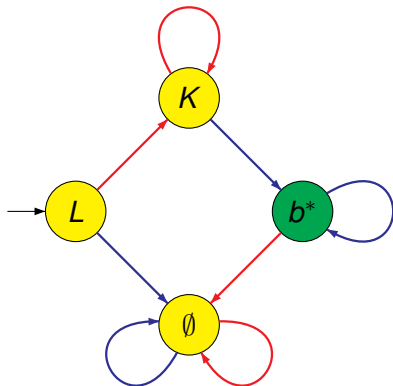
- The class of all acyclic automata is a variety.
- The corresponding variety of languages (well-known): (disjoint) unions of the languages of the form

$$A_0^* a_1 A_1^* a_2 A_2^* \dots A_{n-1}^* a_n A_n^*, \quad \text{where } a_i \notin A_{i-1} \subseteq A.$$

An Example – Piecewise Testable Languages

- In DLT'13 we gave an alternative condition for automata recognizing piecewise testable languages.
- We call an acyclic automaton (Q, A, \cdot) **locally confluent**, if for each state $q \in Q$ and every pair of letters $a, b \in A$, there is a word $w \in \{a, b\}^*$ such that $(q \cdot a) \cdot w = (q \cdot b) \cdot w$.
- A stronger condition: an acyclic automaton (Q, A, \cdot) is **confluent**, if for each state $q \in Q$ and every pair of words $u, v \in \{a, b\}^*$, there is a word $w \in \{a, b\}^*$ such that $(q \cdot u) \cdot w = (q \cdot v) \cdot w$.
- Each acyclic automaton is confluent iff it is locally confluent.
- The class of all acyclic confluent automata is a variety which corresponds to the variety of piecewise testable languages.

An Example of a Piecewise Testable Language

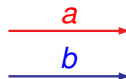
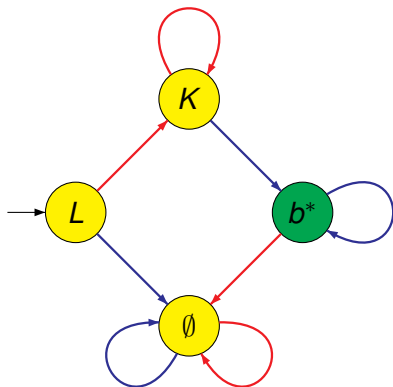


$$L = a^+ b^+$$

$$K = a^{-1}L = a^* b^+$$

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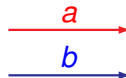
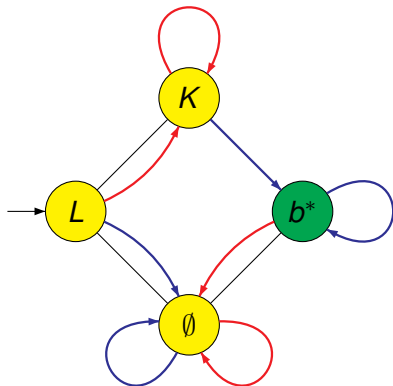
$$L = A^* a A^* b A^* \cap (A^* b A^* a A^*)^c$$

III. Automata Enriched with an Algebraic Structure

A Natural Ordering of the Canonical Automaton

- For a language $L \subseteq A^*$, we have defined a the canonical deterministic automaton: $\mathcal{D}_L = (D_L, A, \cdot, L, F)$, where
 - $D_L = \{ u^{-1}L \mid u \in A^* \}$,
 - $q \cdot a = a^{-1}q$, for each $q \in D_L$, $a \in A$,
 - $q \in F$ iff $\lambda \in q$.
- Therefore states are ordered by inclusion, which means that each minimal automaton is implicitly equipped with a partial order.
- The action by each letter a is an isotone mapping: for all states p, q such that $p \subseteq q$ we have
$$p \cdot a = a^{-1}p \subseteq a^{-1}q = q \cdot a.$$
- The final states form an upward closed subset w.r.t. \subseteq .

An Example of an Ordered Automaton

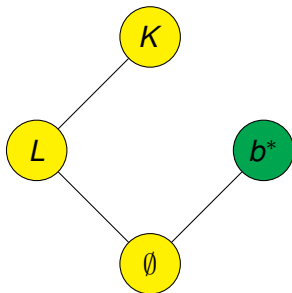


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An Example of an Ordered Automaton



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An Ordered Automaton

Definition

An **ordered automaton** over the alphabet A is a six-tuple $\mathcal{A} = (Q, A, \cdot, \leq, i, F)$, where

- $\mathcal{A} = (Q, A, \cdot, i, F)$ is a usual DFA;
- \leq is a partial order;
- an action by every letter is an isotone mapping from the partial ordered set (Q, \leq) to itself;
- F is an upward closed set, i.e. $p \leq q, p \in F \implies q \in F$.

A Transition Monoid of an Ordered Automaton

If we have an ordered automaton (Q, A, \cdot, \leq) , then:

- we have defined $\tau_u : Q \rightarrow Q$ transformation by a word $u \in A^*$.
- These transformations can be ordered:

$$\tau_u \leq \tau_v \iff \forall p \in Q : p \cdot \tau_u \leq p \cdot \tau_v.$$

- An **ordered transition monoid**.
- In particular, the ordered transition monoid of the canonical ordered automaton of L is isomorphic to the syntactic ordered monoid M_L of L .

Algebraic Constructions on Ordered Automata

Definition

A **variety of ordered automata** \mathbb{V} associates to every finite alphabet A a class $\mathbb{V}(A)$ of ordered automata (without initial and final states) over alphabet A in such a way that

- $\mathbb{V}(A) \neq \emptyset$ is closed under disjoint union, finite direct products and morphic images,
- \mathbb{V} is closed under f -subautomata.

An Eilenberg Type Correspondence

Theorem (Pin)

There are mutually inverse isomorphisms between the lattice of all varieties of ordered automata and the lattice of all positive varieties of regular languages.

The Level 1/2

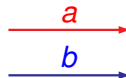
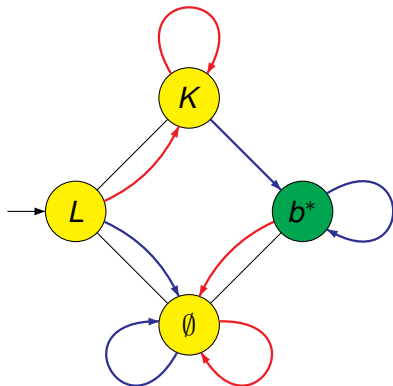
- Piecewise testable languages are Boolean combinations of languages of the form

$$A^* a_1 A^* a_2 A^* \dots A^* a_\ell A^*, \text{ where } a_1, \dots, a_\ell \in A, \ell \geq 0.$$

- Piecewise testable languages form level 1 in Straubing-Thérien hierarchy.
- Level 1/2 is formed just by finite unions of intersections of languages above.
- The corresponding variety of ordered automata is the class of all ordered automata where actions by letters are increasing mappings. I.e. ordered automata satisfying:

$$\forall q \in Q, a \in A : q \cdot a \geq q.$$

An Example of an Ordered Automaton outside 1/2



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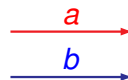
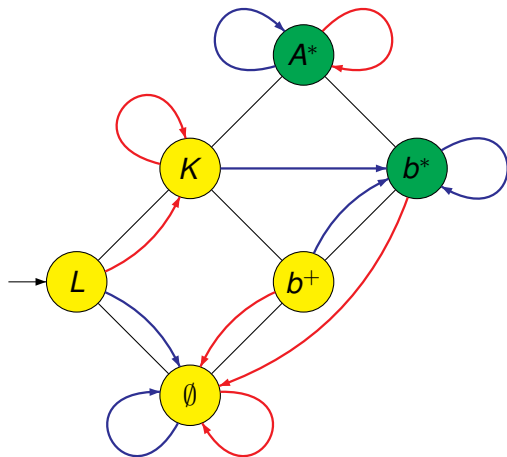
$$L \not\subseteq L \cdot b = \emptyset$$

III.2 Meet Automata

Intersections of Left Quotients

- For a language $L \subseteq A^*$ we extend the canonical automaton (D_L, A, \cdot) , where states are subsets of A^* .
- We can consider intersections of states:
 $U_L = \{\bigcap_{j \in I} K_j \mid I \text{ finite set}, K_j \in D_L\}$. If $I = \emptyset$ then we put $\bigcap_{j \in I} K_j = A^*$.
- The finite set U_L is equipped with the operation intersection \cap and we can define $(\bigcap_{j \in I} K_j) \cdot a = \bigcap_{j \in I} (K_j \cdot a)$.
- We have the automaton (U_L, A, \cdot) with semilattice operation \cap . Moreover, A^* is the largest element in the semilattice (U_L, \cap) and it is an absorbing state in (U_L, A, \cdot) .
- Naturally $F = \{K \mid \lambda \in K\}$ is closed w.r.t. \cap and F is upward closed, i.e. F is a filter.

An Example of a Meet Automaton



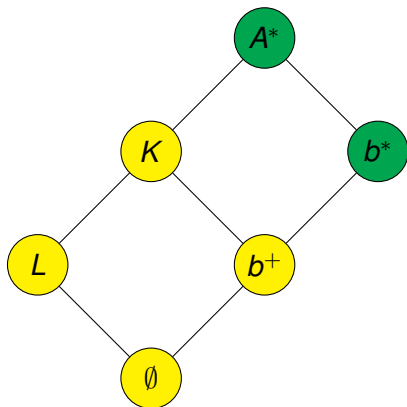
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$$K \cap b^* = b^+$$

$$A^* = \bigcap_{\emptyset}$$

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Meet Automata

Definition

A structure $(Q, A, \cdot, \wedge, \top)$ is a **meet automaton** if

- (Q, A, \cdot) is a DFA,
- (Q, \wedge) is a semilattice with the largest element \top ,
- actions by letters are endomorphisms of the semilattice (Q, \wedge) , i.e. $\forall p, q \in Q, a \in A : (p \wedge q) \cdot a = p \cdot a \wedge q \cdot a$
- \top is an absorbing state.

This meet automaton recognizes a language L if there are $i, f \in Q$ such that $L = \{u \in A^* \mid i \cdot u \wedge f = f\}$.

Varieties of Meet Automata

Definition

A **variety of meet automata** \mathbb{V} associates to every finite alphabet A a class $\mathbb{V}(A)$ of meet automata over alphabet A in such a way that

- $\mathbb{V}(A) \neq \emptyset$ is closed under direct finite products and morphic images,
- \mathbb{V} is closed under f -subautomata.

An Eilenberg Type Correspondence

Theorem (Klíma, Polák)

There are mutually inverse isomorphisms between the lattice of all varieties of meet automata and the lattice of all conjunctive varieties of regular languages.

Varieties of Meet Automata – An Example

Example

For each alphabet A , a meet automata $(Q, A, \cdot, \wedge, \top)$ belongs to $\mathbb{S}(A)$ if $\forall q \in Q, a \in A : q \cdot a = q \cdot a \wedge q$ and

$$\forall q \in Q, a, b \in A : q \cdot ab = q \cdot a \wedge q \cdot b. \quad (*)$$

Then \mathbb{S} is a variety of meet automata and the corresponding conjunctive variety of languages \mathcal{S} satisfies $\mathcal{S}(A) = \{B^* \mid B \subseteq A\} \cup \{\emptyset\}$.

- \mathcal{S} is a conjunctive variety of languages.
- For all $B \subseteq A$, the canonical meet automaton of $L = B^*$ is $\mathcal{U}_L = (U_L, A, \cdot, \cap, A^*)$ and it has just three states: B^*, A^*, \emptyset . We see $B^* \cdot a \in \{B^*, \emptyset\}$ and \mathcal{U}_L satisfies $(*)$.

An Example

- Let $(Q, A, \cdot, \wedge, \top)$ be a meet automaton satisfying

$$\forall q \in Q, a, b \in A \cup \{\lambda\} : q \cdot ab = q \cdot a \wedge q \cdot b. \quad (*)$$

And we choose $i, f \in Q$.

- For $a, b, c \in A$ we have

$$(q \cdot a) \cdot bc = (q \cdot a) \cdot b \wedge (q \cdot a) \cdot c = q \cdot a \wedge q \cdot b \wedge q \cdot c.$$

- In general $q \cdot a_1 \dots a_n = q \cdot a_1 \wedge \dots \wedge q \cdot a_n$ and (for $q = i$) we have $a_1 \dots a_n \in L \iff a_1 \in L, \dots, a_n \in L$.
- We can denote $B = L \cap A$ and we have $L = B^*$.
- Note that an equational characterization is that syntactic semiring satisfies the equality $xy = x \wedge y$.

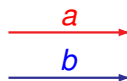
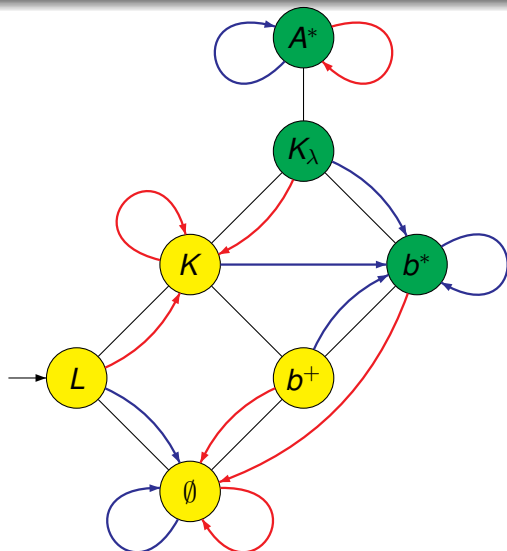
III.3 DL-automata

The Canonical DL-Automaton of a Language

- For a language $L \subseteq A^*$ we extend the canonical meet automaton $(U_L, A, \cdot, \wedge, A^*)$ by unions of states:
$$W_L = \{\bigcup_{j \in I} M_j \mid I \text{ finite set}, M_j \in U_L\}.$$

If $I = \emptyset$ then we put $\bigcup_{j \in I} M_j = \emptyset$.
- The finite set W_L is equipped with the operations intersection \cap and union \cup (due to distributive laws).
We can define $(\bigcup_{j \in I} M_j) \cdot a = \bigcup_{j \in I} (M_j \cdot a)$.
- We have the automaton (W_L, A, \cdot) and a distributive lattice (W_L, \cap, \cup) . Moreover, A^* is the largest element, \emptyset is the smallest element – both are absorbing states in (W_L, A, \cdot) .
- Naturally $F = \{M \mid \lambda \in M\}$ is closed w.r.t. \cap , upward closed, and $M_1 \cup M_2 \in F$ implies $M_1 \in F$ or $M_2 \in F$.
I.e. F is an ultrafilter. In other words the intersection of all elements in F (the minimum in F) is join-irreducible.

An Example of a Canonical DL-Automaton

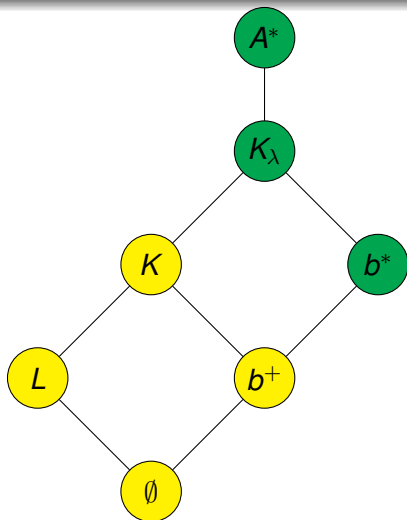


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$$K = a^{-1} L = a^* b^+ = L \cup b^+$$

$$K_\lambda = K \cup b^* = K + \lambda$$

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$$K_\lambda = K \cup b^* = K + \lambda$$

A DL-Automata – a Formal Definition

Definition (new)

A structure $(i, P, Q, A, \cdot, \wedge, \vee, \perp, \top)$ is a **DL-automaton** if

- $i \in P \subseteq Q$,
- (Q, A, \cdot) is a DFA,
- (Q, \wedge, \vee) is a distributive lattice with the minimum element \perp and the largest element \top ,
- actions by letters are endomorphisms of the lattice (Q, \wedge, \vee) ,
- \top and \perp are absorbing states,
- P is the set of all states reachable from i ,
- the lattice Q is generated by the set P .

Languages Recognized by a DL-Automaton

- A *DL*-automaton $(i, P, Q, A, \cdot, \wedge, \vee, \perp, \top)$ recognizes a language L if there are $j \in P, f \in Q$ such that f is a join-irreducible and $L = \{u \in A^* \mid j \cdot u \geq f\}$.

An Eilenberg Type Correspondence

Definition (new)

Let \mathcal{C} be a “Straubing” class of morphisms. A **weak \mathcal{C} -variety of languages** \mathcal{V} associates to every finite alphabet A a class $\mathcal{V}(A)$ of regular languages over A in such a way that

- $\mathcal{V}(A)$ is closed under quotients,
- \mathcal{V} is closed under preimages in morphisms from \mathcal{C} .

Theorem (new)

There are mutually inverse isomorphisms between the lattice of all \mathcal{C} -varieties of DL-automata and the lattice of all weak \mathcal{C} -varieties of regular languages.

An Eilenberg Type Correspondence – An Example

Example

Let \mathcal{V} be a class of languages such that

$$\mathcal{V}(A) = \{A^* a A^* \mid a \in A\} \cup \{A^*\}.$$

- $\mathcal{V}(A)$ is not closed under intersections nor unions, i.e. \mathcal{V} is not a conjunctive (nor disjunctive) variety of languages.
- Let $f : B^* \rightarrow A^*$, $a \in A$, $L = A^* a A^*$, then $f^{-1}(L) = B^* D B^*$ where $D = \{d \in B \mid f(d) \text{ contains } a\}$.

Therefore we should consider only f 's such that

$$\forall b, c \in B : b \neq c \implies c(f(b)) \cap c(f(c)) = \emptyset.$$

- \mathcal{V} is a weak \mathcal{C} -variety for such morphisms.

An Example

- Since $(A^*aA^*)^c = B^*$ for $B = A \setminus \{a\}$ we can take the dual condition characterizing the conjunctive variety \mathcal{S} given by $\mathcal{S}(A) = \{B^* \mid B \subseteq A\} \cup \{\emptyset\}$, namely

$$\forall q \in Q, a, b \in A \cup \{\lambda\} : q \cdot ab = q \cdot a \vee q \cdot b. \quad (*)$$

- In particular $q \cdot a \geq q$.
- Another property is the following

$$\forall q \in Q, a, b \in A : a \neq b \implies q \cdot a \wedge q \cdot b = q. \quad (**)$$

- If $L = A^*a_1A^* \cup \dots \cup A^*a_nA^*$ for a_1, \dots, a_n different letters, $n \geq 2$, then $L \cdot a_1 \cap L \cdot a_2 = A^* \neq L$, i.e. the canonical DL-automaton of L does not satisfy the condition $(**)$.
- The equational description of the property $(**)$ is $x \wedge y = 1$ with substitutions from the restricted class \mathcal{C} .

IV. Syntactic Structures

Monoids. Let F be the free groupoid over an alphabet A with neutral element λ . We define inductively the actions of elements of F on 2^{A^*} :

$$L \circ \lambda = L, \quad L \circ a = a^{-1}L, \quad L \circ (u \cdot v) = (L \circ u) \circ v.$$

Put $u \rho v$ iff (for each $L \subseteq A^*$, we have $L \circ u = L \circ v$). Clearly, $M = F/\rho$ is a free monoid over A .

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Semirings. Let F' be the absolutely free over A with respect to the operational symbols \cdot, \wedge .

$$\dots, \quad L \circ (u \wedge v) = (L \circ u) \cap (L \circ v).$$

Put $u \rho' v$ iff (for each $L \subseteq A^*$, we have $L \circ u = L \circ v$). Clearly, $S = F'/\rho'$ is a free semiring S over A . Its elements can be represented as finite subsets of A^* , the operation is union.

Basic algebras II

DL-algebras. Let F'' be the absolutely free over A with respect to the operational symbols \cdot , \wedge , \vee .

$$\dots, L \circ (u \vee v) = (L \circ u) \cup (L \circ v).$$

Put $u \rho'' v$ iff (for each $L \subseteq A^*$, we have $L \circ u = L \circ v$). One can show that $D = F'' / \rho''$ is a free distributive lattice over A^* . The elements can be represented as

$$\{\{u_{1,1}, \dots, u_{1,\ell_1}\}, \dots, \{u_{k,1}, \dots, u_{k,\ell_k}\}\}, \quad u_{i,j} \in A^*$$

with incomparable inner sets. The interpretation is $(u_{1,1} \wedge \dots) \vee \dots \vee (u_{k,1} \wedge \dots)$.

Syntactic structures

Let $L \subseteq A^*$ be a regular language.

Monoids. For $u, v \in M = A^*$, put

$$u \sim v \text{ iff } \forall p, q \in A^* (puq \in L \iff pvq \in L).$$

Equivalently, $\forall p \in A^* p^{-1}L \circ u = p^{-1}L \circ v$. Then M/\sim is the syntactic monoid of L .

Syntactic structures

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Equivalently, $\forall p \in A^* p^{-1}L \circ u = p^{-1}L \circ v$. Then M/\sim is the syntactic monoid of L .

Semirings. For $U = \{u_1, \dots, u_k\}$, $V = \{v_1, \dots, v_\ell\} \in S$, put $U \sim' V$ iff

$$\forall p, q \in A^* (pu_1q \in L, \dots, pu_kq \in L \iff pv_1q \in L, \dots, pv_\ell q \in L).$$

Equivalently, $\forall p \in A^* p^{-1}L \circ U = p^{-1}L \circ V$. Then S/\sim' is the syntactic semiring of L .

DL-algebras. Let

$$\mathcal{U} = \{U_1, \dots, U_k\},$$

$$U_1 = \{u_{1,1}, \dots, u_{1,m_1}\}, \dots, U_k = \{u_{k,1}, \dots, u_{k,m_k}\},$$

$$\mathcal{V} = \{V_1, \dots, V_\ell\},$$

$$V_1 = \{v_{1,1}, \dots, v_{1,n_1}\}, \dots, V_\ell = \{v_{\ell,1}, \dots, v_{\ell,n_\ell}\}.$$

Put $\mathcal{U} \sim'' \mathcal{V}$ iff $\forall p, q \in A^*$

$$(pU_1q \in L \text{ or } \dots \text{ or } pU_kq \in L \iff pV_1q \in L \text{ or } \dots \text{ or } pV_\ell q \in L).$$

Equivalently, $\forall p \in A^* p^{-1}L \circ \mathcal{U} = p^{-1}L \circ \mathcal{V}$. Then D/\sim'' is the syntactic DL-algebra of L .

A Future Research

Future Research

- We plan to clarify all details in the Eilenberg type correspondence for DL-automata.
- We are trying to characterize classes of regular languages given by certain models of quantum automata.
- We would like to improve our knowledge of an equational logic of meet automata and of DL-automata.
- A further step: BA-automata
(automata equipped with a structure of a Boolean algebra).

Thank you.