Presentation for semigroups $x \approx x^k, k > 2$

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Outline

The aim of this talk is to look at a construction of some semigroups satisfying the identity $x \approx x^3$.

Theorem

Let Y be a semilattice. For each $\alpha \in Y$ let S_α be a semigroup such that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. Let $\alpha, \beta \in Y$ with $\beta \leq \alpha$, let $\{\phi_{\alpha,\beta} : S_\alpha \to S_\beta\}$ be a collection of homomorphisms such that

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Then $S = \bigcup\limits_{\alpha \in Y} S_\alpha$ is a semigroup where for $x \in S_\alpha, y \in S_\beta$ the product in S is given by

$$xy = (x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta}).$$

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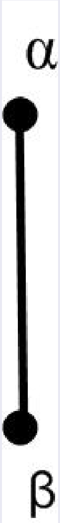
If replace a semigroup S_α by a group G_α then it will be called a Clifford semigroup

Example



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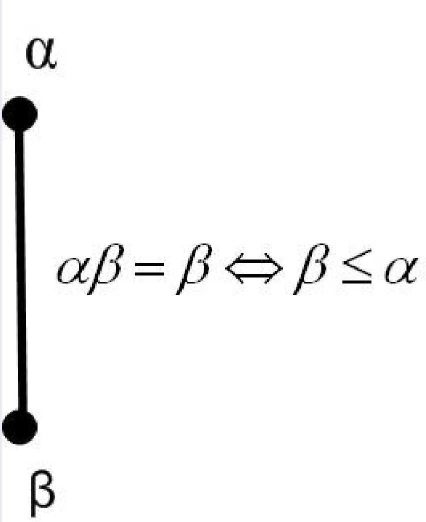

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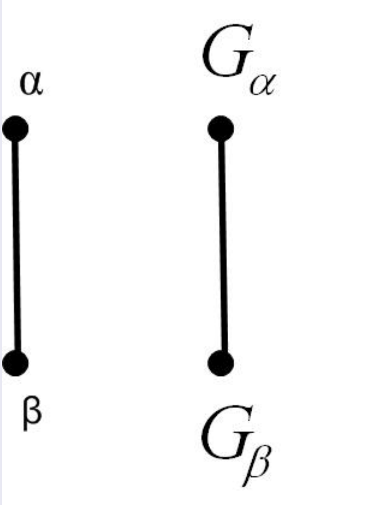

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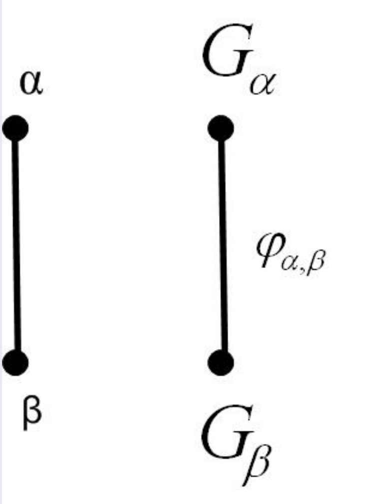

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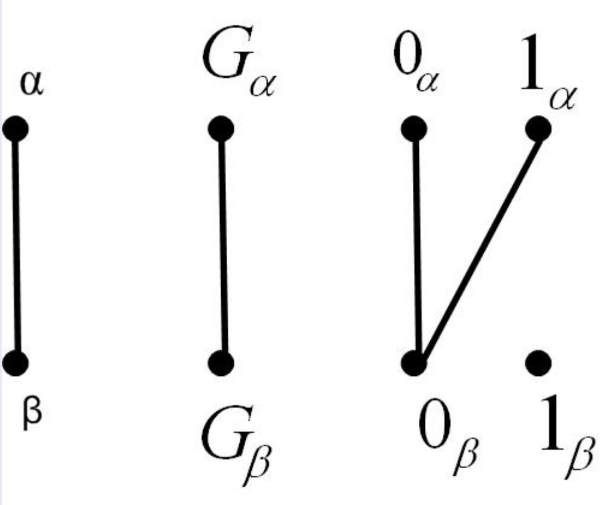

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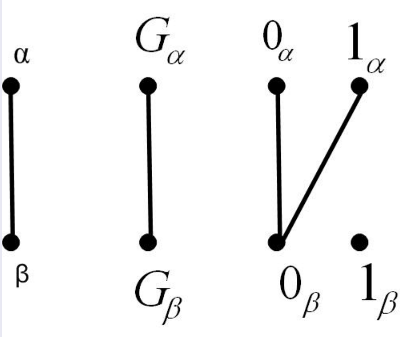

$$0_\alpha * 1_\alpha = (0_\alpha)\varphi_{\alpha,\alpha}(1_\alpha)\varphi_{\alpha,\alpha} = 0_\alpha 1_\alpha = 1_\alpha$$

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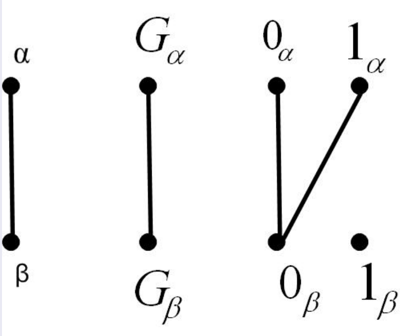

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Thus $(S = \{0_\alpha, 1_\alpha, 0_\beta, 1_\beta\}, *)$ *is a semigroup.*

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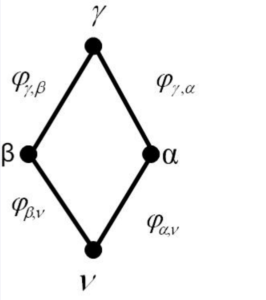

$$\varphi_{\gamma,\alpha}\varphi_{\alpha,\nu} = \varphi_{\gamma,\beta}\varphi_{\beta,\nu}$$

$$(a_\gamma)\varphi_{\gamma,\alpha}\varphi_{\alpha,\nu} = (a_\gamma)\varphi_{\gamma,\beta}\varphi_{\beta,\nu}$$

We begin by describing an important construction.

Definition

Let S be a semigroup, I, Λ be nonempty sets, and P be a $\Lambda \times I$ matrix with entries $p_{\lambda,i}$ from S. Define

$$M = M(S; I, \Lambda; P)$$

with the multiplication

$$(i, s, \lambda)(j, t, \mu) = (i, sp_{\lambda,j}t, \mu)$$

Then M is a semigroup called a Rees matrix semigroup over S. This is an important technique in semigroup theory for constructing new semigroups from old ones

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Definition

Let e be an idempotent in a semigroup S. Then e is a primitive Idempotent if for each idempotent f in S such that $f \leq e$ then $f = e$.

Clifford

Clifford proved that completely regular semigroups are semilattices of completely simple semigroups and conversely.

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Lallement

Using Clifford's representation of completely regular semigroups reduced the problem of their structure to structure of completely simple semigroups and certain functions among them and their translation hulls.

Theorem (Lallement theorem)

Let Y be a semilattice. For each $\alpha \in Y$ let S_α be a completely simple semigroup such that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For each piar $\alpha, \beta,\ \alpha \geq \beta$, let $\{\Phi_{\alpha,\beta} : S_\alpha \to \Omega(S_\beta)$ be a function satisfying the following conditions:

1) $\Phi_{\alpha,\alpha} : a \to \pi_\alpha\ (a \in S_\alpha)$,

2) $(S_\alpha \Phi_{\alpha,\alpha\beta})(S_\beta \Phi_{\beta,\alpha\beta}) \subseteq \Pi(S_{\alpha,\beta})$,

3) if $\alpha\beta > \gamma$ *and* $a \in S_\alpha,\ b \in S_\beta$, *then*

$$[(a\Phi_{\alpha,\alpha\beta})(b\Phi_{\beta,\alpha\beta})]\Phi^{-1}_{\alpha\beta,\alpha\beta}\Phi_{\alpha\beta,\gamma} = (a\Phi_{\alpha,\gamma})(b\Phi_{\beta,\gamma}).$$

On $S = \bigcup_{\alpha \in Y} S_\alpha$ *define a multiplication* $*$ *by*

$$a * b = [(a\Phi_{\alpha,\alpha\beta})(b\Phi_{\beta,\alpha\beta})]\Phi^{-1}_{\alpha\beta,\alpha\beta}\Phi_{\alpha\beta,\gamma}\ (a \in S_\alpha, b \in S_\beta).$$

Then S is a completely regular semigroup. Conversely, every completely regular semigroup can be so constructed.

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Theorem

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Theorem

(David Rees 1940) A semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup over a group.

There are also many research on the generalizations of Rees theorem with applications to some other classes of semigroups.

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Theorem

Let Y be a semilattice. For each $\alpha \in Y$ let E_α be a rectangular band such that $E_\alpha \cap E_\beta = \emptyset$ if $\alpha \neq \beta$. For each $\alpha, \beta \in Y$ with $\beta \leq \alpha$, let $\{\phi_{\alpha,\beta} : E_\alpha\} \to E_\beta\}$ be a collection of homomorphisms such that

1) if $\gamma \leq \beta \leq \alpha$ in Y then $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$.

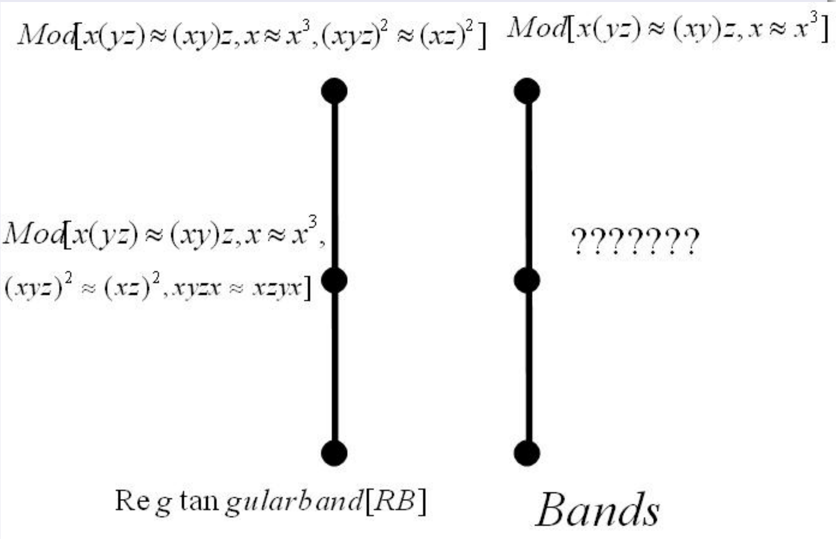
2) for each $\alpha \in Y$ in $\phi_{\alpha,\alpha}$ is the identity on E_α.

Then $S = \bigcup\limits_{\alpha \in Y} E_\alpha$ is a semigroup where for $x \in E_\alpha, y \in E_\beta$ the product in S is given by

$$xy = (x)(\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta}).$$

Then S is a band and conversely.

Example



$Mod[x(yz) \approx (xy)z, x \approx x^3, (xyz)^2 \approx (xz)^2]$
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???????
Re g tan gularband[RB]
Bands

Lemma

Let $S \in Mod[(xy)z \approx x(yz),\ x \approx x^3, (xyz)^2 \approx (xz)^2, xyzx \approx xzyx]$.
Then
1) $(xyzx)^2 = xy^2x$
2) $E(S) = \{x^2 \mid x \in S\}$
3) S *is completely simple*

Proof.1) Take $x, y, z \in S$.

$$
\begin{aligned}
(xyzx)^2 &= (xyzx)(xyzx) \\
&= x(yzxx)yzx) \\
&= x(yz)(xxyz)x \\
&= xyz(xyxz)x \\
&= xyzx(yzx)x \\
&= x(yzx)^2x \\
&= x(yx)^2x \\
&= xyxyxx \\
&= x^3y^2x \\
&= xy^2x
\end{aligned}
$$

2) Obviously, by $(x^2)^2 = x^3x = x^2$
3) By the property $x = x^3$ implies that S is regular. Now we consider for each $e, f \in E(S)$ such that $f = ef = fe$, then

$$\begin{aligned}
ef &= fe \\
(ef)ef &= (fe)ef \\
(eeff) &= feef \\
efefe &= fefee \\
ef^2e^2 &= fe^3 \\
efe &= f^2e \\
(efe)^2 &= (fe)^2 \\
(ee)^2 &= f^2 \\
e &= f.
\end{aligned}$$

Hence S is completely simple.

Proposition (Howie)

Let e be an idempotent of a semigroup S. Then

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Lemma

Let $S \in Mod[x(yz) \approx (xy)z, x \approx x^3, (xyz)^2 \approx (xz)^2, xyzx \approx xzyx]$. Then the set

$$G_{x^2} = \{y \in S \mid y^2 = x^2\}$$

is an abelian group.

Lemma

Let $S \in Mod[x(yz) \approx (xy)z, x \approx x^3, (xyz)^2 \approx (xz)^2, xyzx \approx xzyx]$ *and* $x^2 \in E(S)$. *Then the set*

$$L_{x^2} = \{(ax^2)^2 \mid a \in S\}$$

is a left zero semigroup and $lx^2 = l$ *for every* $l \in L_{x^2}$.

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$$R_{x^2} = \{(x^2a)^2 \mid a \in S\}$$

is a right zero semigroup and $x^2r = r$ *for every* $r \in R_{x^2}$.

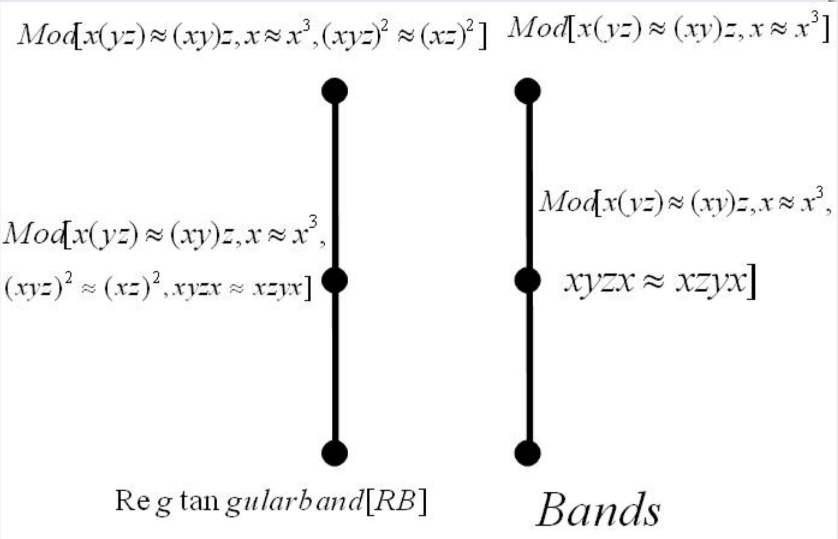
Theorem

Let $S \in Mod[x(yz) \approx (xy)z, x \approx x^3, (xyz)^2 \approx (xz)^2, xyzx \approx xzyx]$. *Then*

$$S \cong M[G_{x^2}; L_{x^2}, R_{x^2}; P]$$

where $G_{x^2}, L_{x^2}, R_{x^2}$ *are described above and* P *is a sandwich Rees Matrix.*

Example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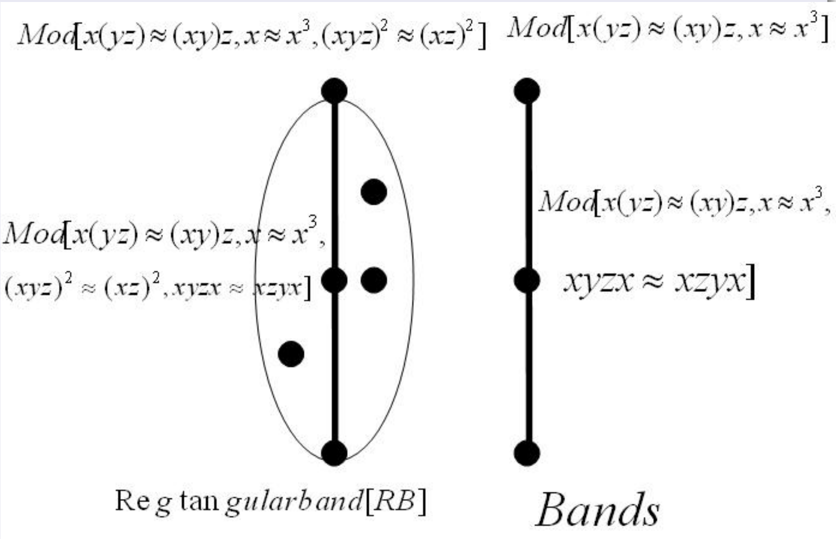
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Thank You !