QCSP on semicomplete digraphs

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[Quantified] Constraint Satisfaction Problem - definition

Definition

Let $\mathbb{A}$ be a finite relational structure. The decision problem $QCSP(\mathbb{A})$ takes as input any first order sentence in prenex form such that the unquantified part is a conjunction of atoms. The accepted sentences are the ones which hold true in $\mathbb{A}$.

Definition

The problem $CSP(\mathbb{A})$ additionally stipulates that sentences are using only existential quantifiers.

$CSP(\mathbb{A})$ is at worst NP-complete, while $QCSP(\mathbb{A})$ is at worst Pspace-complete (for polynomial time many-one reductions).

Applications in non-monotonic reasoning and planning, yadda, yadda.

Algebraic approach - a Galois connection

An operation compatible with all relations of the relational structure $\mathbb{A}$ is a polymorphism of $\mathbb{A}$. All polymorphisms of $\mathbb{A}$ form the clone $Pol(\mathbb{A})$. The subset of surjective polymorphisms is denoted as $s - Pol(\mathbb{A})$.

A relation compatible with all operations of an algebra $\mathbf{A}$ is an invariant relation (or subalgebra of power, subpower) of $\mathbf{A}$. The set of all subpowers of $\mathbf{A}$ is the relational clone $Inv(\mathbf{A})$.

Algebraic approach - theorems

Theorem (Jeavons, 1998)

If $Pol(\Gamma_1) \subseteq Pol(\Gamma_2)$ *and* Γ_2 *is finite, then* $CSP(\Gamma_2)$ *logspace-reduces to* $CSP(\Gamma_1)$.

Theorem (BBCJK, 2009)

If $s - Pol(\Gamma_1) \subseteq s - Pol(\Gamma_2)$ *and* Γ_2 *is finite, then* $QCSP(\Gamma_2)$ *logspace-reduces to* $QCSP(\Gamma_1)$

Algebraic approach - practicalities

Problems with the analogy:

- $s - Pol(\mathbb{A})$ is not a clone;
- no reduction to cores (important!);
- We do not have a Mal'cev characterization of the 'nice' class,
- or expect one.

Good thing that we know complete graphs are Pspace-complete for QCSP and this implies:

Theorem (BBCJK, 2009)

If $s - Pol(\mathbb{A})$ consists only of essentially unary operations and $\mathbb{A}$ is finite, then $QCSP(\mathbb{A})$ is Pspace-complete.

Idea of the proof: all polymorphisms of a complete graph are essentially unary and they are the clone generated by all permutations of the universe.

The importance of being a core (O. Wilde)

When the structure $\mathbb{A}$ is a core, then $s - Pol(\mathbb{A}) = Pol(\mathbb{A})$ (for $f \in Pol(\mathbb{A})$, just look at $f(x, x, \ldots, x)$ and it better be surjective), so $s - Pol(\mathbb{A})$ is a clone.

Essentially unary polymorphisms are permutations of a variable.

So, for a core structure $\mathbb{A}$, all surjective polymorphisms are essentially unary iff all polymorphisms are essentially unary iff the only idempotent polymorphism is the identity.

That last bit: use $g(x) := f(x, x, \ldots, x)$ and then $f^* := g^{n!-1} \circ f$ is an idempotent operation of the same essential arity as f (where $|A| = n$).

Semicomplete digraphs

Definition

A digraph is semicomplete if it is irreflexive and any two distinct vertices are connected by at least one directed edge.

Why semicomplete?

- You gotta start someplace (Bang-Jensen, Hell and MacGillivray, 1988);
- Barny Martin posed the problem in Dagstuhl 2012;

Why *really* semicomplete?

- They are easy (= cores), and have no weak nus (except for the tractable few),
- They are even easier: sinks are hit by everybody and sources hit everybody, which helps with dealing with $\forall$,
- They are EVEN easier: when in trouble deduce that there's a loop and you've got a contradiction.

The result

Theorem

Let $\mathbb{G}$ be a semicomplete digraph. Then

- *Either $\mathbb{G}$ has at most one cycle, in which case QCSP($\mathbb{G}$) is tractable, or*
- *$\mathbb{G}$ has at least two cycles, a source and a sink, in which case QCSP($\mathbb{G}$) is NP-complete, or*
- *$\mathbb{G}$ has at least two cycles, but not both a source and a sink, in which case QCSP($\mathbb{G}$) is Pspace-complete.*

Proof took 50+ pages (and I said it was easy). We are not very smart. That is as it should be at this stage of the game.

Tractable and NP-complete cases

When there exists both a source and a sink, universal quantification (except formal) implies failure of the formula, so QCSP reduces to CSP.

$\mathbb{K}_2$ and $\mathbb{C}_3$ have majority polymorphisms, and thus are tractable by BBCJK.

Then we prove that $QCSP(\mathbb{K}_2^{\to j+1})$ reduces to $QCSP(\mathbb{K}_2^{\to j})$ and similarly for $QCSP(\mathbb{C}_3^{\to j+1})$ and $QCSP(\mathbb{C}_3^{\to j})$.

The Good

The Good

This is the case when we reduce from smooth but not strongly connected to strongly connected case.

Take a polymorphism which acts on each strong component which is not a cycle as a projection and prove it is a projection on the smooth digraph with more than one strong component.

Not much to say, mostly straightforward.

The Ugly

The Ugly

It is the strongly connected case.

Here we prove that the only strongly connected semicomplete digraphs with nontrivial idempotent polymorphisms are $\mathbb{K}_2$ and $\mathbb{C}_3$.

We first prove it for the locally transitive (strongly connected) tournaments.

Next we deal with the semicomplete digraphs which are obtained from strongly connected locally transitive tournaments by blowing up each vertex into a semicomplete graph. We call these P-graphs. (This bit we can shorten and will before the result is published).

Finally, we deal with the remaining cases, which is painfully long and requires several ideas.

The Bad

The Bad

Assume that a semicomplete digraph has more than one cycle and a sink, but no sources. Here we don't have only essentially unary polymorphisms, though we don't have a weak nu, either.

We reduce to a single-sink extension of a smooth semicomplete digraph.

Then we find a semicomplete digraph $\mathbb{G}'$ which is also a single-sink extension of a smooth semicomplete digraph, and such that $s - pol(\mathbb{G}) \subseteq s - Pol(\mathbb{G})'$, but this one has either

- a pp-definable subgraph $\mathbb{K}_n^{\rightarrow}$ where $n > 2$, or
- a pp-definable subgraph $\mathbb{K}_{2\rightarrow 2}^{\rightarrow}$, or
- a pp-definable subgraph $\overline{\mathbb{T}}_n^{\rightarrow}$, where $n > 2$.

Here $\mathbb{K}_{2\rightarrow 2} :=$ a copy of $\mathbb{K}_2$ beats another copy of $\mathbb{K}_2$, while $\overline{\mathbb{T}}_n^{\rightarrow} := \langle n; < \cup \{(n-1, 0)\} \rangle$.

The Bad, Part II

So we are forced to prove that $QCSP(\mathbb{K}_n^{\rightarrow})$, where $n > 2$, $QCSP(\mathbb{K}_{2\rightarrow 2}^{\rightarrow})$ and $QCSP(\overline{\mathbb{T}}_n^{\rightarrow})$, where $n > 2$, are all Pspace-complete.

And here we ran out of polymorphism ideas. So we proved it directly by reducing to known Pspace-complete problems via gadgets and similar complexity-theoretic ideas. Actually, the first was proved to be Pspace-complete by (BBCJK, 2009), and we did the other two. The first two admit reductions from $QNAE3 - SAT$, while the last one admits a reduction from $Q - 1 - in - 3 - SAT$.

PGP/EGP

Also, we note that the polymorphism algebras of semicomplete digraphs with PSPACE-complete $QCSP$ have exponentially generated powers ($d_{\mathbf{A}}(n)$ is bounded from below by an exponential function), while polymorphism algebras of semicomplete digraphs with $QCSP$ in the class NP have polynomially generated powers ($d_{\mathbf{A}}(n)$ is bounded from above by a polynomial function). This condition is speculated by H. Chen to imply that $QCSP$ reduces to CSP.

Further work

If I got this far, I'm probably over time. So

THANK YOU FOR YOUR ATTENTION!