Maltsev Constraints Via LFP+Rank (joint work with A. Habte)

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Motivation

- $\mathbb{A}$: a finite relational structure in finite vocabulary
- HOM($\mathbb{A}$) - the homomorphism problem for $\mathbb{A}$ (or, equivalently, we can look at CSP($\mathbb{A}$))

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Motivation: Try to find a common logical framework for seemingly disparate algorithms for solving HOM($\mathbb{A}$)

Digraph Canonization Problem

- Consider all *finite* structures in a fixed finite relational vocabulary (may assume that the vocabulary is $\{E\}$, E-binary.)
- For a logic (i.e., a description or query language) $\mathcal{L}$, we ask for which properties P, there is a sentence φ of the language such that
$$\mathbb{A} \in P \Longleftrightarrow \mathbb{A} \models \varphi.$$
- Of particular interest is the case when $P \in$**P**, the class of all properties decidable in polynomial time (Canonization Problem)
- Clearly, the first-order logic cannot capture **P** on digraphs (e.g. weak/strong connectedness)

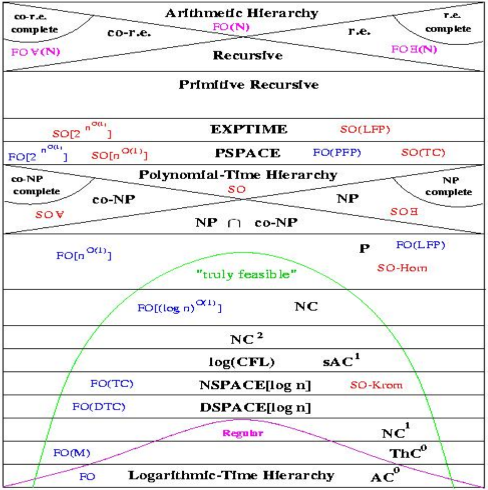

Figure: Descriptive Complexity Hierarchy

Least Fixed Point Logic (LFP)

- LFP: logic obtained from the first-order logic by closing it under formulas computing the least fixed points of monotone operators defined by positive formulas.
- On structures that come equipped with a linear order, LFP expresses precisely those properties that are in **P**.
- LFP cannot express evenness of a digraph (pebble games.)
- Datalog $<$ LFP so, the structures of bounded width have HOM definable in LFP.

- Immermann: proposed LFP+C, a two sorted extension of LFP with a mechanism that allows counting.
- We expand $\mathbb{A}$ into a two-sorted structure $(\omega, \mathbb{A})$; ω carries its standard arithmetical operations along with $<$.
- Numerical terms that count the number of elements of the structure satisfying a formula ϕ:

$$\# x \phi(x)$$

- FO quantifiers are bounded over the non-negative integer sort.

- There are polynomial time properties of digraphs not definable in LFP+C (Cai-Fürer-Immermann graphs; Bijection games)
- Atserias, Bulatov, Dawar (2007): LFP+C cannot express solvability of linear equations over $\mathbb{F}_2$.

Expressibility of HOM($\mathbb{A}$) in extensions of LFP

- **Problem:** Is there an extension of first-order logic $\mathcal{L}$ for which the model checking problem is in **P** such that $\mathcal{L}$ captures (whatever that means in the precise sense...) HOM($\mathbb{A}$)?
- LFP+C is not such a logic, by the Atserias-Bulatov-Dawar result.
- What is lacking?

- What can be expressed in LFP+C?
- Over a finite field $\mathbb{F}_p$, we can express matrix multiplication, non-singularity of matrices, the inverse of a matrix, determinants, the characteristic polynomial... (Dawar, Grohe, Holm, Laubner, 2010)
- What cannot be expressed?

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- What cannot be expressed? The rank of the matrix.

Logic LFP+Rank

- LFP+Rank is the logic obtained from LFP by adding the ability to compute the rank of a matrix over a finite field $\mathbb{F}_q$. It is a proper extension of LFP+C.
- We still have the sort of ω with same restrictions
- The model checking for LFP+Rank is in **P**
- LFP+Rank is closed under negations.
- All known examples of non-expressible properties in LFP+C can be handled in this logic. (Dawar, Grohe, Holm, Laubner, 2010)

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$M(\phi; x_1, \ldots, x_n)$ - the $n \times n$-matrix over $\mathbb{F}_2$ defined by:

$$M(\phi; x_1, \ldots, x_n)[i, j] = 1 \quad \Leftrightarrow \quad \phi(x_i, x_j) \text{ holds in } \mathbb{A};$$

otherwise, the entry is 0.

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This can be generalized in several ways: we can use tuples of any fixed length instead of individual variables x_i's (consequently, we may end up with non-square matrices) or, we can work with any finite number of formulas instead of a single formula ϕ (consequently, we no longer get $\{0, 1\}$-valued matrices only; instead, we work over some $\mathbb{F}_p$, $(p \geq 2)$)

Expressing CSPs in LFP+Rank

- We can convert an instance of HOM($\mathbb{A}$) problem into a an instance of CSP (variables + domain + constraints);
- For the sake of simplicity, we may assume that we are dealing with structures with a single binary relation;
- We end up with a 2-sorted structure:

  Variable sort V: $\{x_1, \ldots, x_n\}$, Domain sort D: $\{a_1, a_2, \ldots, a_m\}$

  (we treat all domain sort elements as constants).
- We also have the ω-sort so that we can e.g. count, compute ranks of matrices, etc.

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Problem: Does there exist an LFP+Rank sentence $\phi_{\mathbb{A}}$ such that on every 3-sorted structure $\mathbb{S}$ encoding an instance of the CSP($\mathbb{A}$) problem,

$$\mathbb{S} \models \phi_A \text{ iff } \mathbb{S} \text{ codes a solvable instance ?}$$

- We need to construct such a sentence by encoding the Dalmau algorithm into the LFP+Rank logic.
- Technical, but not intrinsically difficult.
- Various “modules" of the Dalmau algorithm (detecting nonempty intersection, detecting witnesses, updating constraints) need to be realized;
- The compact representation (along with signature) is maintained using a matrix and a $2n + 3$-ary relation; if the matrix attains zero rank at any point there is no solution.

The Compact Representations

- In the presence of a Maltsev polymorphism, we can represent the intermediate solution sets of the CSP in an economical way.
- The solution set can be reconstructed from its compact representation; all we need to keep track of are triples (i, a, b) which tell us for which $i \leq n$ and $a, b \in A$ we have two intermediate solution n-tuples which agree up to the i-th position, where they fork with values a and b, respectively.
- Triples (i, a, b) give us the signature of the intermediate solution set; we also need to keep track of the pairs of tuples witnessing such triples.
- This bookkeeping can be done with formulas in LFP+Rank, using matrices and $(2n + 3)$-ary relations.

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Theorem

If a finite relational template $\mathbb{A}$ has an edge polymorphism, then HOM($\mathbb{A}$) *can be defined in LFP+Rank.*

Next Step: Given a finite template with the structure

X on top of Y

where $X, Y \in \{$ Maltsev, bounded width $\}$, can its homomorphism problem be defined in LFP+Rank?

More ambitiously:

Question: Are tractable finite templates $\mathbb{A}$ precisely those for which HOM($\mathbb{A}$) can be defined in LFP+Rank?