Poset Loops

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the **path extension** f_p or $f_{x_0, x_1, \ldots, x_r}$ is $f(x_0, x_1)f(x_1, x_2) \ldots f(x_{r-1}, x_r)$ in S.

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$(f \cdot g)(x, y) = f_{x,y} + g_{x,y} + \sum_{x<z<y} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,y)} f_p g_q$

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Theorem: $Z_S^P = C(I_S^P, \cdot, e)$; a subloop of $(I_S^P, \cdot, e)$.

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Theorem: Suppose that locally finite poset $P' = (X, \leq')$ extends $P = (X, \leq)$.

Then I_S^P is a subloop of $I_S^{P'}$.

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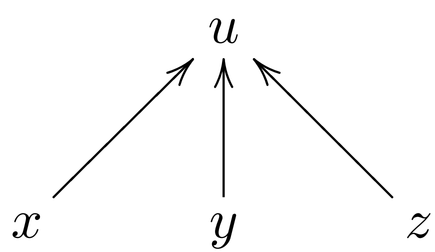

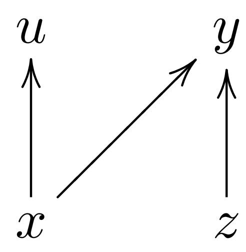

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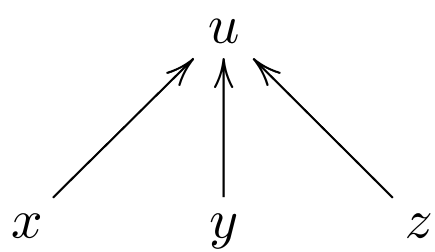

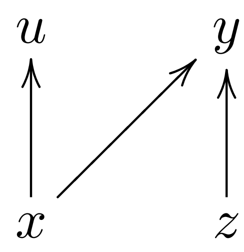

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Corollary: Each abelian group is the incidence loop of a locally finite poset.

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Then if $4l > k^2$, there is no locally finite poset P with $I_{\mathbb{Z}}^P \cong G$.

Corollary: If group G is free nilpotent of class 2, finite rank $k > 2$,

then there is no locally finite poset P with $I_{\mathbb{Z}}^P \cong G$.

Nilpotent loops

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of normal subloops of L with $L_i / L_{i+1} \subseteq Z(L/L_{i+1})$ for $0 \leq i < c$.

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Remark: For locally finite P of infinite height, I_S^P need not be nilpotent.

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$\vdots$

Find $\mathbf{V}_n$ as large as possible such that

for $|S^n| > 1$, have height of P at most n iff I_S^P in variety $\mathbf{V}_n$.

Thank you for your attention!