Category Equivalences of Clones

Worakrit Supaporn

Institute of Mathematics, University of Potsdam, Germany

Let $O^{(n)}(A)$ denote the set of all n-ary operations on A and let $O(A)$ denote the set of all finitary operations on A.

For $f \in O^{(k)}(A)$, $g_1, \ldots, g_k \in O^{(n)}(A)$,

we can define a new operation $f(g_1, \ldots, g_k) : A^n \to A$ by

$$f(g_1, \ldots, g_k)(a_1, \ldots, a_n)$$

$$= f\Big(g_1(a_1, \ldots, a_n), \ldots, g_k(a_1, \ldots, a_n)\Big)$$

for $(a_1, \ldots, a_n) \in A^n$

and we will say that $f(g_1, \ldots, g_k)$ is a **superposition** of $f, g_1, \ldots, g_k$.

An n-ary i^{th}-**projection** on A is a mapping $e_i^{n,A} : A^n \to A$ which is defined by $e_i^{n,A}(a_1, \ldots, a_n) = a_i$.

A **clone** on a set A is a subset of $O(A)$ which is closed under superposition and contains all projections.

For an n-ary operation f on A and an h-ary relation ρ on A,

we say that f **preserves** ρ ($f \triangleright \rho$) if
for all

$$
\begin{array}{ccccccccccc}
(& a_1^1 & , & a_2^1 & , & \dots & , & a_h^1 &) & \in \rho. \\
(& a_1^2 & , & a_2^2 & , & \dots & , & a_h^2 &) & \in \rho. \\
 & & & & & & & & & \vdots \\
(& a_1^n & , & a_2^n & , & \dots & , & a_h^n &) & \in \rho,
\end{array}
$$

$$
\left(f(a_1^1, a_1^2, \dots, a_1^n), f(a_2^1, a_2^2, \dots, a_2^n), \dots, f(a_h^1, a_h^2, \dots, a_h^n)\right) \quad \in \rho.
$$

For a relation ρ on A and a set Q of relations on A,

let $\quad \mathrm{Pol}_A\{\rho\} := \{f \in O(A) \mid f \triangleright \rho\}$

and $\quad \mathrm{Pol}_A Q := \{f \in O(A) \mid \forall \rho \in Q, f \triangleright \rho\}$.

Remark

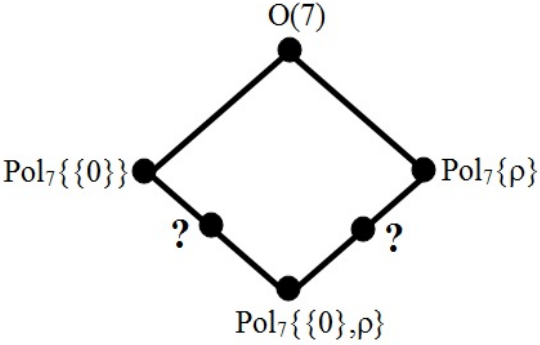
- $\mathrm{Pol}_A Q = \bigcap\{\mathrm{Pol}_A\{\rho\} \mid \rho \in Q\}$.
- C is a clone iff $C = \mathrm{Pol} Q$ for some set Q of relations.

The set of all clones on a set A forms a complete lattice.

The lattice of clones on 2-elements set was described by E.L.Post.

The descriptions of lattice of clones on n-elements set where $n \geq 3$ are **open problems**!

Let $\rho \in Eq(7)$ where $7/_{\rho} = \{\{0\}, \{1,2,3\}, \{4,5\}, \{6\}\}$.



We will describe this lattice

by using

"Category equivalence of clones".

Definition 1 Two varieties $\mathcal{V}$ and $\mathcal{W}$ are category equivalent if there is an equivalence functor from $\mathcal{V}$ to $\mathcal{W}$.

Definition 2([De-L]) Two clones C_A and C_B on sets A and B, respectively, are category equivalent if there are an algebra $\mathbf{A}$ with the universe A and the clone C_A of term operations and an algebra $\mathbf{B}$ with the universe B and the clone C_B of term operations such that there is an equivalence functor F from $V(\mathbf{A})$ to $V(\mathbf{B})$ which $F(\mathbf{A}) = \mathbf{B}$.

Proposition 3([De-L]) Two clones C_A and C_B are category equivalent iff two relational algebras $\mathbf{Inv_A C_A}$ and $\mathbf{Inv_B C_B}$ are isomorphic.

$$
\begin{aligned}
& C_A \widetilde{\equiv} C_B \\
& \Leftrightarrow \mathbf{Inv_A C_A} \cong \mathbf{Inv_B C_B} \\
& \Rightarrow (Sub(\mathbf{Inv_A C_A}); \subseteq) \cong (Sub(\mathbf{Inv_B C_B}); \subseteq) \\
& \Leftrightarrow (Sub(\mathbf{Inv_A C_A}); \subseteq)^{\partial} \cong (Sub(\mathbf{Inv_B C_B}); \subseteq)^{\partial} \\
& \Leftrightarrow ([C_A, O(A)]; \subseteq) \cong ([C_B, O(B)]; \subseteq).
\end{aligned}
$$

$$C_A \widetilde{\equiv} C_B \Rightarrow ([C_A, O(A)]; \subseteq) \cong ([C_B, O(B)]; \subseteq)$$

Proposition 4

If $\theta \in \mathit{Eq}(3)$ where $3/_\theta = \{\{0\}, \{1, 2\}\}$
and $\rho \in \mathit{Eq}(7)$ where $7/_\rho = \{\{0\}, \{1, 2, 3\}, \{4, 5\}, \{6\}\}$,

then $\mathrm{Pol}_3\{\{0\}, \theta\}$ and $\mathrm{Pol}_7\{\{0\}, \rho\}$ are category equivalent.

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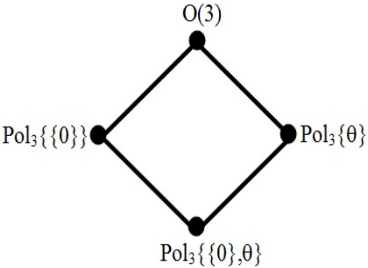

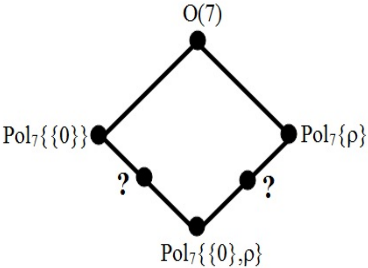

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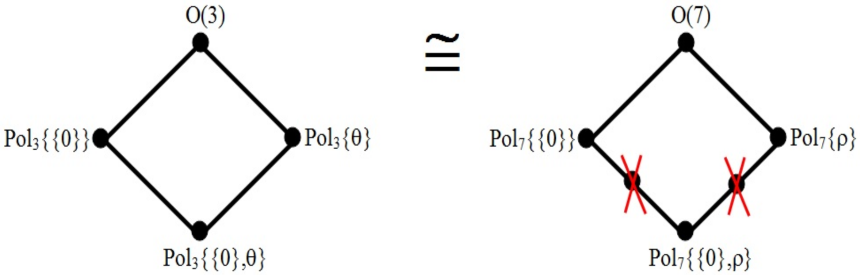

In this presentation, we will consider clone on arbitrary finite set A in the following forms.

- $\mathrm{Pol}_A\{\theta\}$ where $\theta \in Eq(A) \setminus \{\Delta_A, A^2\}$.
- $\mathrm{Pol}_A\{\{a\}\}$ where $a \in A$.
- $\mathrm{Pol}_A\{B\}$ where $B \subsetneqq A$ and $|B| \geq 2$.

Proposition 5([De-L])

Let C be a clone on a finite set A.

(1) $C \cong \mathrm{Pol}_3\{\theta\}$ where $3/_\theta = \{\{0\},\{1,2\}\}$ iff $C = \mathrm{Pol}_A\{\rho\}$ for some $\rho \in Eq(A) \setminus \{\Delta_A, A^2\}$ $(|A| \geq 3)$.

(2) $C \cong \mathrm{Pol}_2\{\{0\}\}$ iff $C = \mathrm{Pol}_A\{\{a\}\}$ for some $a \in A$ $(|A| \geq 2)$.

(3) $C \cong \mathrm{Pol}_3\{\{0,1\}\}$ iff $C = \mathrm{Pol}_A\{B\}$ for some $B \subsetneq A$ such that $|B| \geq 2$ $(|A| \geq 3)$.

Now, we will consider clones which are intersections of clones in the following forms:

- $\mathrm{Pol}_A\{\{a\}\}$ where $a \in A$,
- $\mathrm{Pol}_A\{B\}$ where $B \subsetneq A$ and $|B| \geq 2$.

Example

- $\mathrm{Pol}_3\{\{0\},\{1\},\{2\}\}$.
- $\mathrm{Pol}_4\{\{0\},\{1,2,3\}\}$.
- $\mathrm{Pol}_5\{\{0\},\{1,2,3\}\}$.

Lemma 6

Let A be a finite set with at least two elements,
let Q be a collection of nonempty proper subsets of A,
let $a \in A$.

If $a \notin \bigcup Q$, then

$\mathrm{Pol}_A Q \cong \mathrm{Pol}_{A \setminus \{a\}} Q$ if and only if $A \setminus \{a\} \notin Q$.

Example $\mathrm{Pol}_5 \{\{0\}, \{1, 2, 3\}\} \cong \mathrm{Pol}_4 \{\{0\}, \{1, 2, 3\}\}$.

Lemma 7

Let A be a finite set with at least two elements,
let Q be a collection of nonempty proper subsets of A,
let $a \in A$.

If $a \in \bigcup Q$, then

$\mathrm{Pol}_A Q \cong\!\!\!\!\equiv \mathrm{Pol}_{A\setminus\{a\}}\{B \setminus \{a\} \mid B \in Q\}$ if and only if

$|\bigcap \mathcal{B}_a| \geq 3$ and $(\bigcap \mathcal{B}_a) \setminus \{a\} \not\subseteq B$ for all $B \in Q \setminus \mathcal{B}_a$.

Note $\mathcal{B}_a = \{B \mid B \in Q \text{ and } a \in B\}$.

Example $\mathrm{Pol}_4\{\{0\},\{1,2,3\}\} \cong\!\!\!\!\equiv \mathrm{Pol}_3\{\{0\},\{1,2\}\}$.

A clone that is category equivalent to a given clone $\mathrm{Pol}_A Q$ by using Lemma 6 and Lemma 7, is a clone on a set that get by removing some element of A. Then the cardinality of this set must less than the cardinality of A.

By applying all elements in A with Lemma 6 and Lemma 7, we will get a clone on a set of the smallest cardinality which is category equivalent to $\mathrm{Pol}_A Q$.

Theorem 8

Let A be a finite set with at least two elements and let Q be a collection of nonempty proper subsets of A.

The clone $\mathrm{Pol}_A Q$ is a clone on a set of the smallest cardinality with respect to category equivalence if and only if
for each $a \in A$,

1. if $a \notin \bigcup Q$, then $A \setminus \{a\} \in Q$;
2. if $a \in \bigcup Q$, then $|\bigcap \mathcal{B}_a| < 3$ or $(\bigcap \mathcal{B}_a) \setminus \{a\} \subseteq B$ for some $B \in Q \setminus \mathcal{B}_a$.

Next, we will consider clones which are intersections of clones in the following forms:

- $\mathrm{Pol}_A\{\theta\}$ where $\theta \in Eq(A) \setminus \{\Delta_A, A^2\}$,
- $\mathrm{Pol}_A\{B\}$ where $\emptyset \neq B \subsetneq A$.

Proposition 9

Let $\emptyset \neq B \subsetneqq A$,
let $\theta \in Eq(A) \setminus \{\triangle_A, A^2\}$,
let $a \in A \setminus \left(\bigcup\{[b]_\theta \mid b \in B\}\right)$.

Then

$\mathrm{Pol}_A\{B, \theta\} \widetilde{\equiv} \mathrm{Pol}_{A\setminus\{a\}}\{B, \theta_a\}$ if and only if θ_a is nontrivial and

$\bigcup\{[b]_\theta \mid b \in B\} \neq A \setminus \{a\}$.

Note $\theta_a := \theta \cap (A \setminus \{a\})^2$.

Example $\mathrm{Pol}_7\{\{0\}, \rho\} \widetilde{\equiv} \mathrm{Pol}_3\{\{0\}, \theta\}$ where
$7|_\rho = \{\{0\}, \{1, 2, 3\}, \{4, 5\}, \{6\}\}$ and $3|_\theta = \{\{0\}, \{1, 2\}\}$.

Proposition 10

Let $\emptyset \neq B \subsetneqq A$,
let $\theta \in Eq(A) \setminus \{\triangle_A, A^2\}$,
let $a \in \left(\bigcup\{[b]_\theta \mid b \in B\}\right) \setminus B$.

Then

$\mathrm{Pol}_A\{B,\theta\} \stackrel{\sim}{\equiv} \mathrm{Pol}_{A\setminus\{a\}}\{B,\theta_a\}$ if and only if

$B \cup \{a\} \neq \bigcup\{[b]_\theta \mid \ b \in B\}$.

References

- B. A. Davey and H. Werner : *Dulities and Equivalences for Varieties of Algebras*, Contributions to lattice theory, Colloq. Math. Soc. János Bolyai, 33, North-Holland, Amsterdam, New York, 1983, pp. 101-275.
- D. Lau : *Function algebras on finite sets*, A basic course on many-valued logic and clone theory, Springer, 2006.
- J. Adamek, H. Herrlich and G. E. Strecker : *Abstract and Concrete Categories : The Joy of Cats*, John Wiley and Sons, 1990.
- J. W. Grabowski : *Varieties and Clones of Relational Structures*, Dissertation, Technische Universität Dresden, 2002.
- K. Denecke and O. Lüders : *Category equivalences of clones*, Algebra Universalis, 34, 1995, 608-618.

References

K. Denecke and O. Lüders : *Categorical Equivalence of Varieties and Invariant Relations*, Algebra Universalis, 46 (2001) 105-118.

R. McKenzie : *An Algebraic version of categorical equivalence for varieties and more general algebraic theories*, Logic and Algebra, Vol. 180(1996), 211-243.