On universal homogeneous polymorphisms and automatic homeomorphicity for clones

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Algebras & Clones Fest, Prague 2014

(joint work with Maja Pech)

Relational structures

Relational signatures

A relational signature is a pair $\underline{\Sigma} = (\Sigma, \mathrm{ar})$, where

- Σ is a set of relational symbols,
- $\mathrm{ar} : \Sigma \to \mathbb{N} \setminus \{0\}$.

Relational structures

A $\underline{\Sigma}$-structure is a pair $\mathbf{A} = (A, (\varrho^{\mathbf{A}})_{\varrho \in \Sigma})$, where

- A is a set,
- $\varrho^{\mathbf{A}} \subseteq A^{\mathrm{ar}(\varrho)}$, for each $\varrho \in \Sigma$.

Clones

$$O_A^{(n)} := A^{(A^n)}, \qquad O_A := \bigcup_{n \in \mathbb{N} \setminus \{0\}} O_A^{(n)},$$

Projections

$e_i^n \in O_A^{(n)} : (x_1, \ldots, x_n) \mapsto x_i$ (where $n \in \mathbb{N} \setminus \{0\}$, $1 \leq i \leq n$).

J_A denotes the set of all projections on A.

Clones

$C \subseteq O_A$ is called clone if

1. $J_A \subseteq C$,
2. it is closed with respect to composition.

Clone isomorphisms

A clone isomorphism between clones C and D is a bijection that preserves projections and composition.

Polymorphism clones

Given a relational signature $\underline{\Sigma}$, and a $\underline{\Sigma}$-structure $\mathbf{A}$.

Polymorphisms

$f \in O_A^{(n)}$ is called n-ary polymorphism of $\mathbf{A}$ if

$$f : \mathbf{A}^n \to \mathbf{A}.$$

The set of n-ary polymorphisms of $\mathbf{A}$ is denoted by $\mathrm{Pol}^{(n)}(\mathbf{A})$.

Polymorphism clones

$\mathrm{Pol}(\mathbf{A}) := \bigcup_{n\in\mathbb{N}\setminus\{0\}} \mathrm{Pol}^{(n)}(\mathbf{A})$ is a clone.

It is called the polymorphism clone of $\mathbf{A}$.

Topology on Clones

Given a set A, equipped with the discrete topology.

Topology on $O_A^{(n)}$

- for every finite $M \subseteq A^n$ and for every $h : M \to A$:

$$\Phi_{M,h} := \{f \in O_A^{(n)} \mid f\restriction_M = h\}.$$

- together all $\Phi_{M,h}$ form the basis of the Tychonoff topology on $O_A^{(n)}$,

Topology on O_A

- O_A can be considered as the topological sum of the $O_A^{(n)}$.
- Composition of functions is continuous.

Topology on clones (cont.)

Topology on clones

- Every clone $C \leq O_A$ can be considered as topological subspace of O_A.
- Thus, every clone is canonically equipped with a topology, with respect to which the composition is continuous.

Metrization of Tychonoff topology on $O_A^{(n)}$ when $|A| = \omega$

- Let $\overline{w} = (\overline{a}_i)_{i<\omega}$ be an enumeration of A^n.
- Define $D_{\overline{w}} : O_A^{(n)} \times O_A^{(n)} \to \omega + 1$:

$$D_{\overline{w}}(f,g) := \begin{cases} \min\{i \in \omega \mid f(\overline{a}_i) \neq g(\overline{a}_i)\} & f \neq g \\ \omega & f = g. \end{cases}$$

- Then the following defines an ultrametric on $O_A^{(n)}$:

$$d_{\overline{w}}(f,g) := \begin{cases} 2^{-D_n(f,g)} & f \neq g \\ 0 & f = g. \end{cases}$$

Automatic homeomorphicity for clones

Definition (Bodirsky, Pinsker, Pongrácz)

Let $C \leq O_A$ be a closed clone, and let $\mathcal{K}$ be a set of closed clones on A. We say that C has automatic homeomorphicity with respect to $\mathcal{K}$ if every clone isomorphism from C to a clone from $\mathcal{K}$ is a homeomorphism.

If $\mathcal{K}$ is the set of all closed clones on A. Then we say that C has automatic homeomorphicity.

Theorem (Bodirsky, Pinsker, Pongrácz)

The following clones have automatic homeomorphicity:

1. *every closed clone on A that contains $O_A^{(1)}$,*
2. *the polymorphism clone of the Rado graph,*
3. *the Horn-clone*

Here the Horn clone is the smallest clone on a countable set A that contains all injective functions from O_A.

New results

Recall:

- $(\mathbb{P}, \leq)$ is the Fraïssé-limit of the class of finite partial orders — a.k.a. the countable universal homogeneous poset,
- $\mathbb{U}_{\mathbb{Q}}$ is the Fraïssé-limit of the class of finite metric spaces with rational distances — a.k.a. the rational Urysohn space.

Theorem (MP+CP)

1. $\mathrm{Pol}(\mathbb{P}, \leq)$ *has automatic homeomorphicity with respect to the class of polymorphism clones of* ω*-categorical structures.*
2. $\mathrm{Pol}(\mathbb{U}_{\mathbb{Q}})$ *has automatic homeomorphicity.*

Strong gate coverings

Definition

Let A be countable, let $C \leq O_A$ be a clone, and let G be the group of units in $C^{(1)}$.

A strong gate covering of C consists of

- an open covering $\mathcal{U}$ of C,
- functions $f_U \in U$, for each $U \in \mathcal{U}$,

such that for all $U \in \mathcal{U}$ and for all Cauchy-sequences $(g^j)_{j\in\omega}$ of elements of U of the same arity n there exist

- a Cauchy-sequence $(\alpha^j)_{j\in\omega}$ in $\overline{G}$,
- Cauchy-sequences $(\beta_i^j)_{j\in\omega}$ $(1 \leq i \leq n)$ in $\overline{G}$,

such that for all $(x_1, \dots, x_n) \in A^n$ we have

$$g^j(x_1, \dots, x_n) = \alpha_j(f_U(\beta_1^j(x_1), \dots, \beta_n^j(x_n))).$$

Remark

The original definition of gate coverings is due to Bodirsky, Pinsker, Pongrácz.

Our working horse

Proposition (MP+CP)

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be two countable relational structures,* $h : \mathrm{Pol}(\mathbf{A}) \to \mathrm{Pol}(\mathbf{B})$ *be an isomorphism, such that*

1. h *is open,*
2. $h\restriction_{\mathrm{Aut}(\mathbf{A})}$ *is continuous,*
3. $\mathrm{Pol}(\mathbf{A})$ *has a strong gate covering.*

Then h *is continuous.*

Corollary

Let $\mathbf{A}$ *be a countable relational structure such that*

1. $\mathrm{Aut}(\mathbf{A})$ *has automatic homeomorphicity,*
2. $\mathrm{Pol}(\mathbf{A})$ *has a strong gate covering,*
3. *every isomorphism from* $\mathrm{Pol}(\mathbf{A})$ *to another closed clone on* A *is open.*

Then $\mathrm{Pol}(\mathbf{A})$ *has automatic homeomorphicity.*

A criterion for automatically open clone isomorphisms

Proposition (Bodirsky, Pinsker, Pongrácz)

Let $\mathbf{A}$ *be a structure such that* $\mathrm{Pol}(\mathbf{A})$ *contains all constant functions. Then every isomorphism to another clone of functions is open.*

Our way to show automatic homeomorphicity

The (**rough**) road for revealing new clones with automatic homeomorphicity goes as follows:

1. start with a countable relational structure $\mathbf{A}$,
2. show that $\mathrm{Aut}(\mathbf{A})$ has automatic homeomorphicity,
3. show that $\mathrm{Pol}(\mathbf{A})$ has a strong gate covering (**hard!**),
4. show that every clone-isomorphism from $\mathrm{Pol}(\mathbf{A})$ to another closed clone on A is open,
5. conclude that $\mathrm{Pol}(\mathbf{A})$ has automatic homeomorphicity.

Instead of showing the existence of a strong gate covering, we generally show a much stronger property of $\mathrm{Pol}(\mathbf{A})$ — the existence of universal homogeneous polymorphisms of all arities.

Universal homogeneous polymorphisms

Let $\mathbf{A}$ be a relational structure. Then we define

$$\mathrm{Age}(\mathbf{A}) = \{\mathbf{B} \mid \mathbf{B} \text{ finite}, \exists \iota : \mathbf{B} \hookrightarrow \mathbf{A}\}$$
$$\overline{\mathrm{Age}}(\mathbf{A}) = \{\mathbf{B} \mid \mathbf{B} \text{ countable}, \mathrm{Age}(\mathbf{B}) \subseteq \mathrm{Age}(\mathbf{A})\}$$

Universal polymorphisms

Let $\mathbf{U}$ be a structure. Then $u \in \mathrm{Pol}^{(n)}(\mathbf{U})$ is called universal if for all $\mathbf{A} \in \overline{\mathrm{Age}}(\mathbf{U})$ and for every $f : \mathbf{A}^n \to \mathbf{U}$ there exist $\iota : \mathbf{A} \hookrightarrow \mathbf{U}$ such that for all $(a_1, \dots, a_n) \in \mathbf{A}^n$ holds

$$f(a_1, \dots, a_n) = u(\iota(a_1), \dots, \iota(a_n))$$

Universal homogeneous polymorphisms (cont.)

Homogeneous polymorphisms

Let $\mathbf{U}$ be a structure. Then $u \in \mathrm{Pol}^{(n)}(\mathbf{U})$ is called homogeneous if for all $\mathbf{A} \in \mathrm{Age}(\mathbf{U})$, for every $f : \mathbf{A}^n \to \mathbf{U}$, and for all $\iota_1, \iota_2 : \mathbf{A} \hookrightarrow \mathbf{U}$ with

$$\forall (a_1, \dots, a_n) \in A^n, i \in \{1,2\} : \qquad u(\iota_i(a_1), \dots, \iota_i(a_n)) = f(a_1, \dots, a_n)$$

there exists $h \in \mathrm{Aut}(\mathbf{U})$ such that

1. $h \circ \iota_1 = \iota_2$,
2. for all $(a_1, \dots, a_n) \in U^n$ we have

$$u(h(a_1), \dots, h(a_n)) = u(a_1, \dots, a_n).$$

From UH polymorphisms to strong gate coverings

Proposition

If a structure $\mathbf{U}$ *has universal homogeneous polymorphisms of all arities, then* $\mathrm{Pol}(\mathbf{U})$ *has a strong gate covering.*

- In the following we will characterize all homogeneous structures that have n-ary universal homogeneous polymorphisms.
- all conditions are properties of the age of the structure in question.

The amalgamated extension property

Definition

Let $\mathcal{C}$ be a class of structures of the same type, and let $n \in \mathbb{N} \setminus \{0\}$. We say that $\mathcal{C}$ has the [n]-AEP if $\forall \mathbf{A}, \mathbf{B}_i, \mathbf{T} \in \mathcal{C}, \ldots$

$$
\begin{tikzcd}
\mathbf{B}_1 & \\
\mathbf{A} \arrow[u, hook, "f_1"] \arrow[r, hook, "f_2"] & \mathbf{B}_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & \mathbf{T} \\
\mathbf{B}_1^n \arrow[urr, bend left, "h_1"] & & \\
\mathbf{A}^n \arrow[u, hook, "f_1^n"] \arrow[r, hook, "f_2^n"] & \mathbf{B}_2^n \arrow[uur, bend right, "h_2"'] &
\end{tikzcd}
$$

The amalgamated extension property

Definition

Let $\mathcal{C}$ be a class of structures of the same type, and let $n \in \mathbb{N} \setminus \{0\}$. We say that $\mathcal{C}$ has the $[n]$-AEP if $\forall \mathbf{A}, \mathbf{B}_i, \mathbf{T} \in \mathcal{C}, \ldots$

$$
\begin{tikzcd}
 & & & & & & \mathbf{T}' \\
 & & & & & \mathbf{T} \arrow[ur, hook, "k"'] & \\
\mathbf{B}_1 \arrow[r, hook, "g_1"] & \mathbf{C} & & \mathbf{B}_1^n \arrow[r, hook, "g_1^n"] \arrow[urr, bend left, "h_1"] & \mathbf{C}^n \arrow[uurr, bend left, "h"] & & \\
\mathbf{A} \arrow[u, hook, "f_1"] \arrow[r, hook, "f_2"] & \mathbf{B}_2 \arrow[u, hook, "g_2"'] & & \mathbf{A}^n \arrow[u, hook, "f_1^n"] \arrow[r, hook, "f_2^n"] & \mathbf{B}_2^n \arrow[u, hook, "g_2^n"'] \arrow[uur, bend right, "h_2"'] & &
\end{tikzcd}
$$

Definition

Let $\mathcal{C}$ be a class of structures of the same type, and let $n \in \mathbb{N} \setminus \{0\}$. We say that $\mathcal{C}$ has the [n]-HAP if for all $\mathbf{A}, \mathbf{B} \in \mathcal{C}$, $g : \mathbf{A} \hookrightarrow \mathbf{B}$, $\mathbf{T}_1 \in \mathcal{C}$, $a : \mathbf{A}^n \to T_1 \ldots$

$$\begin{tikzcd} \mathbf{B}^n & \\ \mathbf{A}^n \arrow[u, hook, "g^n"'] \arrow[r, "a"] & \mathbf{T}_1. \end{tikzcd}$$

The homo amalgamation property

Definition

Let $\mathcal{C}$ be a class of structures of the same type, and let $n \in \mathbb{N} \setminus \{0\}$. We say that $\mathcal{C}$ has the [n]-HAP if for all $\mathbf{A}, \mathbf{B} \in \mathcal{C}$, $g : \mathbf{A} \hookrightarrow \mathbf{B}$, $\mathbf{T}_1 \in \mathcal{C}$, $a : \mathbf{A}^n \to T_1$ ...

$$\begin{tikzcd}
\mathbf{B}^n \arrow[r, "b"] & \mathbf{T}_2 \\
\mathbf{A}^n \arrow[u, hook, "g^n"'] \arrow[r, "a"] & \mathbf{T}_1 . \arrow[u, hook, "h"']
\end{tikzcd}$$

Definition

Let $\mathcal{C}$ be a class of structures of the same type, and let $n \in \mathbb{N} \setminus \{0\}$. We say that $\mathcal{C}$ has the [n]-HAP if for all $\mathbf{A}, \mathbf{B} \in \mathcal{C}$, $g : \mathbf{A} \hookrightarrow \mathbf{B}$, $\mathbf{T}_1 \in \mathcal{C}$, $a : \mathbf{A}^n \to T_1 \ldots$

$$
\begin{tikzcd}
\mathbf{B}^n \arrow[r, "b"] & \mathbf{T}_2 \\
\mathbf{A}^n \arrow[u, hook, "g^n"'] \arrow[r, "a"] & \mathbf{T}_1 . \arrow[u, hook, "h"]
\end{tikzcd}
$$

$\mathcal{C}$ has the HAP if it has the [1]-HAP.

Homogeneous structures with UH polymorphisms

Proposition

Let **U** *be a countable homogeneous structure and let* $n \in \mathbb{N} \setminus \{0\}$*. Then* **U** *has an n-ary universal homogeneous polymorphism if and only if* Age(**U**) *has the* [*n*]-AEP *and the* [*n*]-HAP*.*

Theorem

The following structures have universal homogeneous polymorphisms of all arities:

1. *the polymorphism clone of the countable universal homogeneous poset* $(\mathbb{P}, \leq)$*,*
2. *the polymorphism clone of the rational Urysohn-space (polymorphisms are non-expansive functions).*
3. *every free homogeneous structure whose age is closed with respect to finite products and has the* HAP *(e.g., the Rado graph).*