On Key Relations Preserved by a Weak Near-Unanimity Operation

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Algebras & Clones fest
June 30 - July 3, 2014, Prague, Czech Republic

Outline

Relations.

Definitions

Let A be a finite set. A subset $\rho \subseteq A^n$ is called a n-ary relation.

Usually we assume that $A = \{0, 1, \ldots, k-1\}$.

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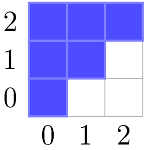
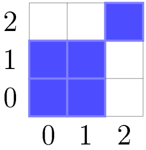
These binary relations can be represented in the following way





Vector-functions

A tuple $\Psi = (\psi_1, \psi_2, \ldots, \psi_h)$, where $\psi_i : A \to A$, is called a vector-function.

We say that Ψ preserves a relation ρ of arity h if

$$\Psi \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_h \end{pmatrix} := \begin{pmatrix} \psi_1(a_1) \\ \psi_2(a_2) \\ \vdots \\ \psi_h(a_h) \end{pmatrix} \in \rho \text{ for every } \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_h \end{pmatrix} \in \rho.$$

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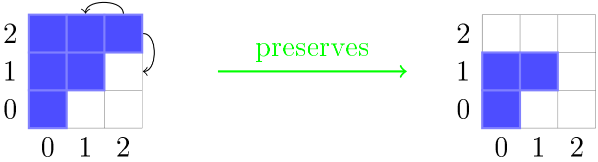

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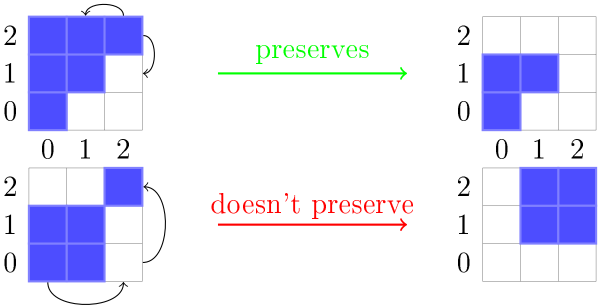

Definition

A relation ρ of arity h is called a key relation if there exists a tuple $\beta \in A^n \setminus \rho$ such that for every $\alpha \in A^h \setminus \rho$ there exists a vector-function Ψ which preserves ρ and gives $\Psi(\alpha) = \beta$.

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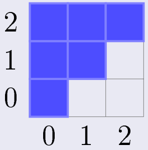

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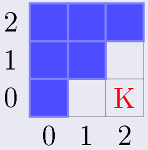

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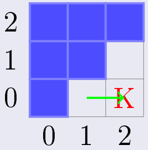

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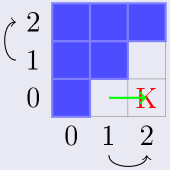

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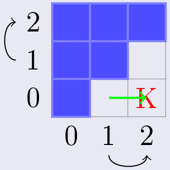

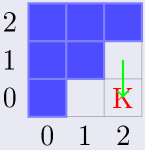

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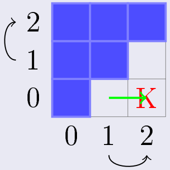

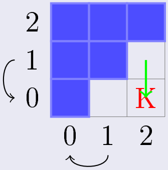

Example 2

Let $A = \{0, 1, 2\}$,
$\rho = \{(x, y, z) \mid x + y + z = 0\}$
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Lemma

Every $\beta \in \{0, 1, 2\}^3 \setminus \rho$ is a key tuple.

Sketch of the proof

For every $\alpha \in \{0, 1, 2\}^3 \setminus \rho$ we need a vector-function Ψ preserving ρ such that $\Psi(\alpha) = \beta$. Combining the following vector-functions $(2x, 2y, 2z)$, $(x + 1, y, z - 1)$, $(x, y + 1, z - 1)$, preserving ρ we can easily construct Ψ.


Why do we need these key relations???

Relational Clones and Galois Connection

Definition

$[S]$ is the set of all relations $\rho \in R_k$ that can be represented by a positive primitive formula over the set S:

$$\rho(x_1, \ldots, x_n) = \\ \exists y_1 \ldots \exists y_l \; \rho_1(z_{1,1}, \ldots, z_{1,n_1}) \wedge \ldots \wedge \rho_s(z_{s,1}, \ldots, z_{s,n_s}).$$

- Closed sets of relations containing equality and empty relations are called relational clones.
- There is a natural one-to-one correspondence between clones and relational clones, which reverses the partial order $\subseteq$.

Thus every clone can be defined by some set of relations.

We need key relations because

- Every relation ρ can be represented as $\rho_1 \cap \rho_2 \cap \ldots \cap \rho_s$ of some key relations $\rho_1, \ldots, \rho_s \in [\rho]$.
- $[\rho] = [\{\rho_1, \ldots, \rho_s\}]$.
- Every clone can be defined by only key relations.

Key Relations on Two-Element Set

Let $A = \{0, 1\}$. An equation $a_1x_1 + \ldots + a_sx_s = c_s$ is called a linear equation ("+" is addition modulo 2).

Theorem

ρ is a key relation if and only if $\rho(x_1, \ldots, x_n) = L_1 \vee L_2 \vee \ldots \vee L_m$ for some linear equations $L_1, L_2, \ldots, L_m$.

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Examples

- $(x \leq y) = (x = 0) \vee (y = 1)$
- $(x \neq y) = (x + y = 1)$

Let $\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Is it a key relation?

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$$\rho(x_1, x_2, x_3) = \rho_1(x_1, x_2, x_3) \wedge \rho_2(x_1, x_2) \wedge \rho_2(x_2, x_3) \wedge \rho_2(x_1, x_3),$$

where $\rho_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $\rho_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

- $\rho \in [\{\rho_1, \rho_2\}]$.
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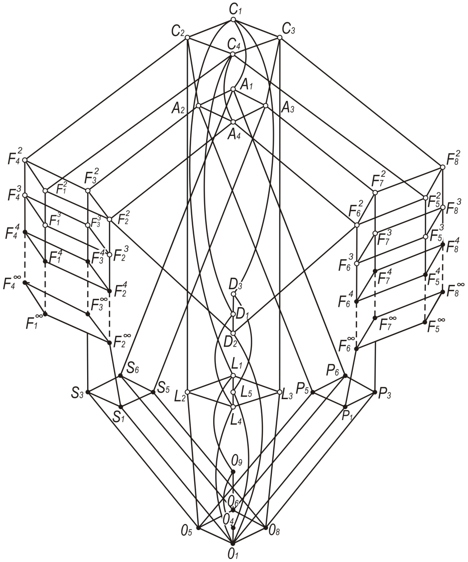
$\rho_1(x_1, x_2, x_3) = (x_1 + x_2 + x_3 = 1)$,
$\rho_2(x_1, x_2) = (x_1 = 0) \vee (x_2 = 0)$.

The lattice of all clones on two elements(for $|A| = 2$)



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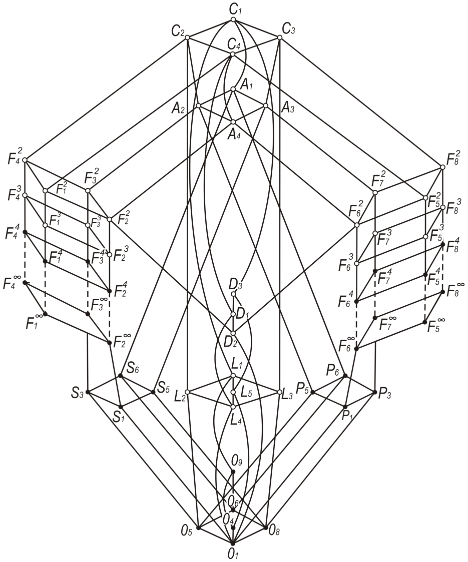

We can generalize this!

Definition

Suppose projection of ρ onto every coordinate is a two-element set,
$\varphi_i : A \to \{0, 1\}$ for $i = 1, 2, \ldots, n$.
An equation $a_1\varphi_1(x_1) + \ldots + a_n\varphi_n(x_n) = c_n$ is called a linear equation ("+" is addition modulo 2).

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Suppose projection of ρ onto every coordinate is a two-element set, then ρ is a key relation if and only if
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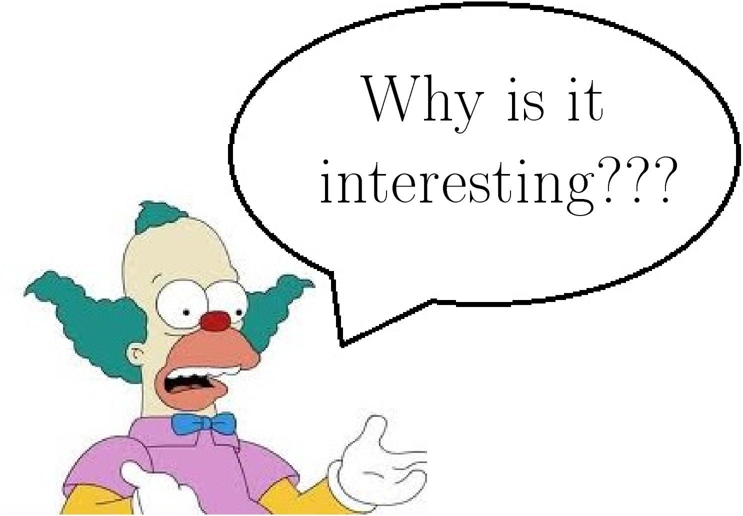
Why is it interesting???

Example for $|A| = 3$

Let $s(x, y, z) = \begin{cases} x, & |\{x, y, z\}| < 3 \\ y, & |\{x, y, z\}| = 3 \end{cases}$,

Task

Describe all clones containing s.

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Theorem

Suppose ρ is a key relation preserved by s then

$$\rho \in \left\{ \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \right\},$$

or projection of ρ onto every coordinate is a two-element set (or a one-element set).

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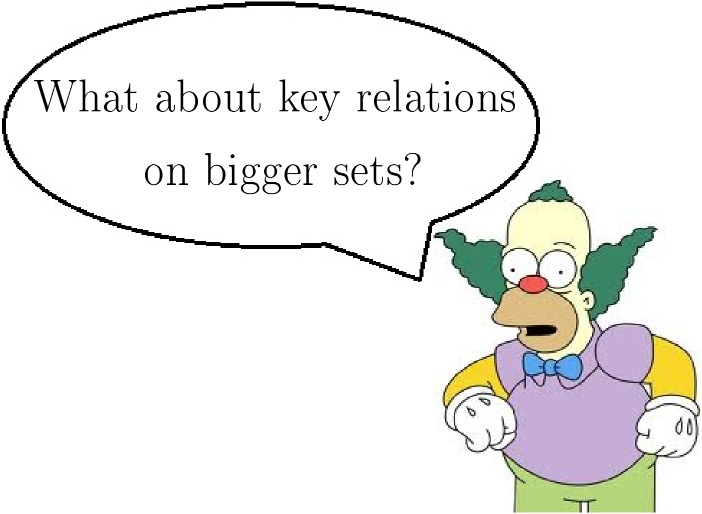
or $\rho(x_1,\dots,x_n) = L_1 \vee L_2 \vee \dots \vee L_m$ for some linear equations $L_1, L_2, \dots, L_m$.


What about key relations on bigger sets?

Key relations on bigger sets

- For a prime p the relation $a_1x_1 + \ldots + a_nx_n = a_0$ and $a_1x_1 + \ldots + a_nx_n \neq a_0$ define key relations, where "+" is addition modulo p.
- For any two key relations ρ_1 and ρ_2, the relation defined by $\rho_1(x_1, \ldots, x_m) \vee \rho_2(x_{m+1}, \ldots, x_{m+n})$ is a key relation.

Key relations on bigger sets

An example of a key relation on $A = \{0, 1, 2\}$

$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 2 & 0 & 1 & 2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 \end{pmatrix}$ is key and $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is a key tuple.

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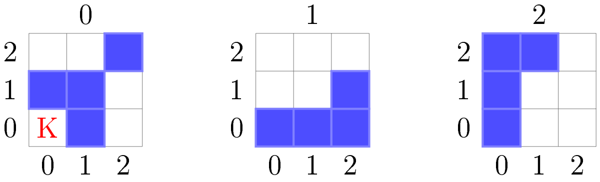

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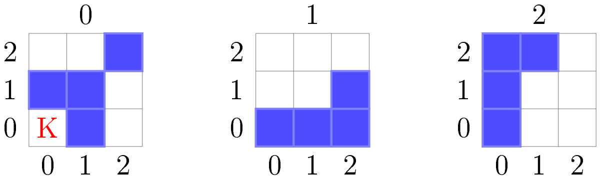

- Only projections preserve this relation.

Weak Near-Unanimity Operation

- To avoid considering such relations we consider just relations preserved by a weak near-unanimity operation.

Definition

A weak near unanimity operation (WNU) is an operation f satisfying $f(x, x, \ldots, x) = x$ and

$$f(x, \ldots, x, y) = f(x, \ldots, x, y, x) = \cdots = f(y, x, \ldots, x).$$

Key relations on 2 elements with WNU

Fact

Key relations on $A = \{0, 1\}$ preserved by a WNU can be represented as follows

$$(x_1 + x_2 + \cdots + x_m = c) \vee (x_{m+1} = d_{m+1}) \vee \cdots \vee (x_n = d_n)$$

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- We will try to generalize this fact for $|A| > 2$.
- Every variable occurs just in one equation.

Pattern of a Key Relation

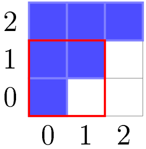
Definition

The pattern of a relation ρ of arity n is a binary relation $\sim$ on the set $\{1, 2, \ldots, n\}$ defined as follows:
$i \sim j$ if for every $a_1, \ldots, a_n, b_i, b_j \in A$ we have

$$\begin{pmatrix} a_1 \\ \cdots \\ a_{i-1} \\ a_i \\ a_{i+1} \\ \cdots \\ a_{j-1} \\ a_j \\ a_{j+1} \\ \cdots \\ a_n \end{pmatrix} \notin \rho, \quad \begin{pmatrix} a_1 \\ \cdots \\ a_{i-1} \\ a_i \\ a_{i+1} \\ \cdots \\ a_{j-1} \\ b_j \\ a_{j+1} \\ \cdots \\ a_n \end{pmatrix}, \begin{pmatrix} a_1 \\ \cdots \\ a_{i-1} \\ b_i \\ a_{i+1} \\ \cdots \\ a_{j-1} \\ a_j \\ a_{j+1} \\ \cdots \\ a_n \end{pmatrix} \in \rho \Longrightarrow \begin{pmatrix} a_1 \\ \cdots \\ a_{i-1} \\ b_i \\ a_{i+1} \\ \cdots \\ a_{j-1} \\ b_j \\ a_{j+1} \\ \cdots \\ a_n \end{pmatrix} \notin \rho.$$

We set $i \sim i$ for every i.

Pattern of a Key Relation



$1 \not\sim 2$

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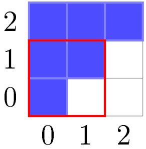

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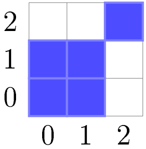

$1 \sim 2$ because there is no such square.

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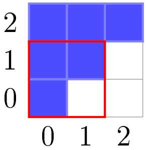

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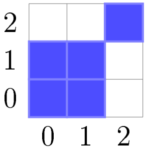

$1 \sim 2$ because there is no such square.

The equivalence relation $\{\{1, 2, \ldots, m\}, \{m + 1\}, \ldots, \{n\}\}$ is the pattern of a key relation on $\{0, 1\}$ defined by

$$(x_1 + x_2 + \cdots + x_m = c) \vee (x_{m+1} = d_{m+1}) \vee \cdots \vee (x_n = d_n)$$

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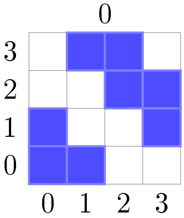

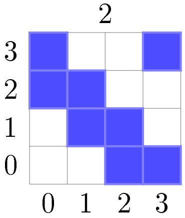

Then $1 \sim 3$, $2 \sim 3$ but $1 \not\sim 2$.

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0

3	K			K
2	K	K		
1		K	K	
0			K	K
	0	1	2	3

2

3		K	K	
2			K	K
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Then $1 \sim 3$, $2 \sim 3$ but $1 \not\sim 2$.

Pattern of a Key Relation with WNU

Theorem

Suppose ρ is a key relation preserved by a WNU. Then the pattern of ρ is an equivalence relation. Moreover, at most one equivalence class contains more than one element.

Linear and almost linear relation

Definition 1

A relation $\sigma \subseteq A_1 \times \cdots \times A_n$ is called linear if there exist a prime number p and bijective mappings $\varphi_i : A_i \to \mathbb{Z}_p$ for $i = 1, 2, \ldots, n$ such that σ is defined by

$$\varphi_1(x_1) + \ldots + \varphi_n(x_n) = 0.$$

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Definition 2

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$$(\varphi_1(x_1) + \ldots + \varphi_n(x_m) = 0) \vee (x_{m+1} = b_{m+1}) \vee \cdots \vee (x_n = b_n).$$

Main theorem

Theorem

Suppose ρ is a key relation of arity n preserved by a WNU, whose pattern is $\{\{1, 2, \ldots, m\}, \{m+1\}, \{m+2\}, \ldots, \{n\}\}$. Then for every key tuple α there exists $\widetilde{A} = A_1 \times A_2 \times \cdots \times A_n$ such that $\alpha \in \widetilde{A}$ and $\rho \cap \widetilde{A}$ is almost linear with m variables in the nontrivial linear equation.

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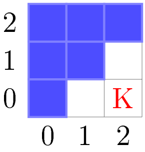

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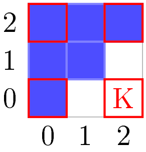

$Pattern : \{\{1\}, \{2\}\}$

$(x_1 = 0) \vee (x_2 = 2)$

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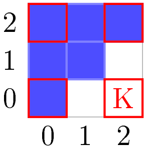

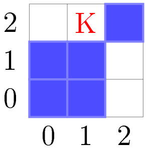

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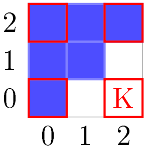

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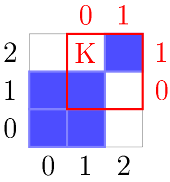
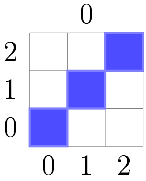


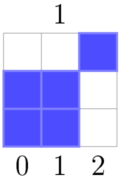
$Pattern : \{\{1, 2\}\}$

$x_1 + x_2 = 0$

Example

Let $A = \{0, 1, 2\}$ and $\rho = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.





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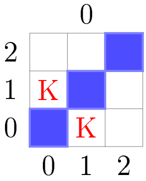

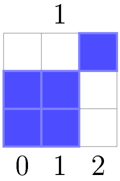

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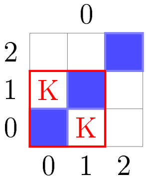

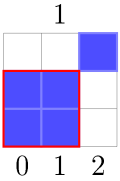

Pattern: $\{\{1,2\},\{3\}\}$

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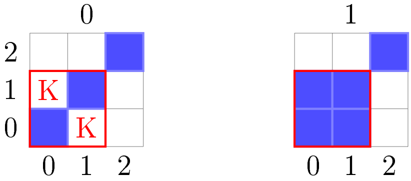

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$(x_1 + x_2 = 0) \vee (x_3 = 1)$

Pattern is a trivial equivalence relation: $\{\{1\},\{2\},\dots,\{n\}\}$

By the main theorem for any key relation ρ preserved by a WNU with the pattern $\{\{1\},\{2\},\{3\},\dots,\{n\}\}$

- we can find a part which is organized as follows: $(x_1 = b_1) \vee (x_2 = b_2) \vee \dots \vee (x_n = b_n)$,
- or equivalently there exist $(a_1, a_2, \dots, a_n) \notin \rho$ and $b_1, \dots, b_n \in A$ such that

$$(\{a_1, b_1\} \times \{a_2, b_2\} \times \dots \times \{a_n, b_n\}) \setminus \{(a_1, a_2, \dots, a_n)\} \subseteq \rho.$$

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Theorem

Suppose ρ is a relation preserved by a WNU with the pattern $\{\{1\},\{2\},\{3\},\ldots,\{n\}\}$. Then ρ is a key relation iff there exist $(a_1, a_2, \ldots, a_n) \notin \rho$ and $b_1, b_2, \ldots, b_n \in A$ such that

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Do we really
need WNU here?

The existence of a WNU preserving a relation is a necessary condition

Example

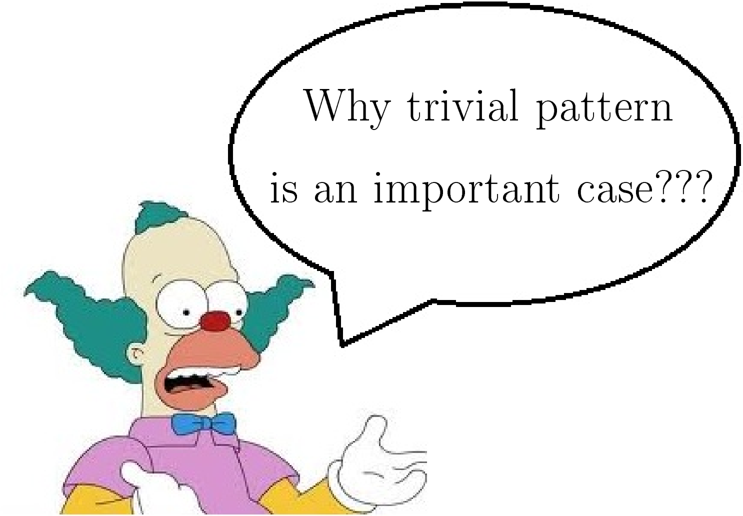
$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 2 & 0 & 1 & 2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 \end{pmatrix} \text{ is a key relation.}$$

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the only key tuple in ρ.

The pattern of this relation is a trivial equivalence relation.

We cannot find $b_1, b_2, b_3 \in \{1, 2\}$ such that

$$(\{0, b_1\} \times \{0, b_2\} \times \{0, b_3\}) \setminus \{(0, 0, 0)\} \subseteq \rho.$$


Why trivial pattern
is an important case???

Near-unanimity operation

A near unanimity operation is an operation f satisfying

$$f(x,\ldots,x,y) = f(x,\ldots,x,y,x) = \cdots = f(y,x,\ldots,x) = x.$$

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Theorem

Suppose ρ is a key relation of arity greater than 2 preserved by a near-unanimity operation. Then the pattern of ρ is a trivial equivalence relation.

Corollary

Suppose ρ (of arity greater than 2) is preserved by a NU. Then ρ is a key relation if and only if there exist $(a_1,a_2,\dots,a_n) \notin \rho$ and $b_1,\dots,b_n \in A$ such that

$$(\{a_1,b_1\} \times \{a_2,b_2\} \times \dots \times \{a_n,b_n\}) \setminus \{(a_1,a_2,\dots,a_n)\} \subseteq \rho.$$

Pattern is an almost trivial equivalence relation: $\{\{1,2\},\{3\},\dots,\{n\}\}$

By the main theorem for any key relation ρ preserved by a WNU with the pattern $\{\{1,2\},\{3\},\dots,\{n\}\}$

- we can find a part which is organized as follows: $(x_1 + x_2 = 0) \vee (x_3 = b_3) \vee \dots \vee (x_n = b_n)$,
- therefore there exist $(a_1, a_2, \dots, a_n) \notin \rho$ and $b_1, \dots, b_n \in A$ such that

$$(\{a_1, b_1\} \times \dots \times \{a_n, b_n\}) \setminus \{(a_1, a_2, a_3, \dots, a_n), (b_1, b_2, a_3, \dots, a_n)\} \subseteq \rho.$$

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Theorem

Suppose ρ is a relation preserved by a WNU, the pattern of ρ is $\{\{1,2\},\{3\},\dots,\{n\}\}$. Then ρ is a key relation iff there exist $(a_1,a_2,\dots,a_n)\notin\rho$ and $b_1,b_2,\dots,b_n\in A$ such that

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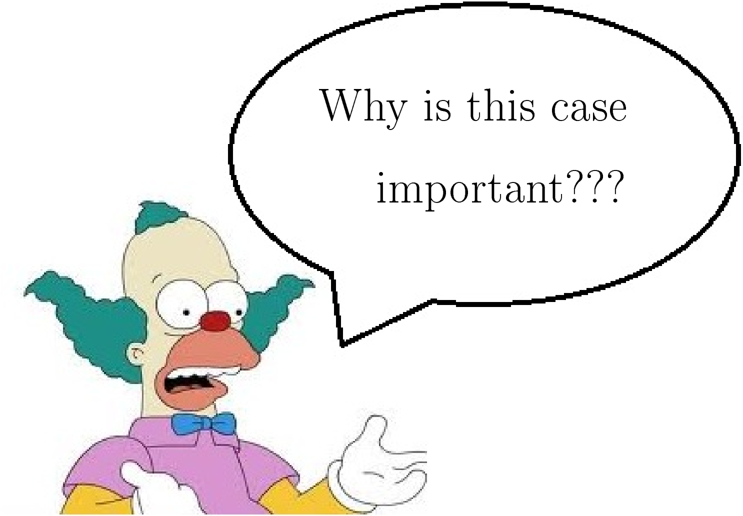
Why is this case important???

Semilattice operation

A semilattice operation is a binary associative commutative idempotent operation.
A 2-semilattice operation is a binary commutative idempotent operation satisfying $f(x, f(x, y)) = f(x, y)$.

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Suppose ρ is a key relation preserved by a semilattice (2-semilattice) operation. Then the pattern of ρ is either trivial, or almost trivial.

Corollary

Suppose ρ is a relation preserved by a semilattice (2-semilattice) operation. Then ρ is a key relation if and only if there exist $(a_1, a_2, \ldots, a_n) \notin \rho$ and $b_1, \ldots, b_n \in A$ such that

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Pattern is a full equivalence relation: $\{\{1, 2, \ldots, n\}\}$

By the main theorem for any key relation ρ preserved by a WNU with the pattern $\{\{1, 2, \ldots, n\}\}$ we can find a part which is a linear relation: $\varphi_1(x_1) + \varphi_2(x_2) + \ldots + \varphi_n(x_n) = 0$.

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The structure of key relations with full pattern preserved by a WNU

- The relation can be reduced to a core by a vector-function preserving the relation which doesn't change at least one key tuple.
- A core can be divided into blocks of the form $B_1 \times B_2 \times \dots \times B_n$. A block is called trivial if it contains only tuples from the relation.
- Different blocks cannot contain common tuples and tuples that differ in one component.
- Every nontrivial block is a linear relation.
- All nontrivial blocks are «isomorphic».

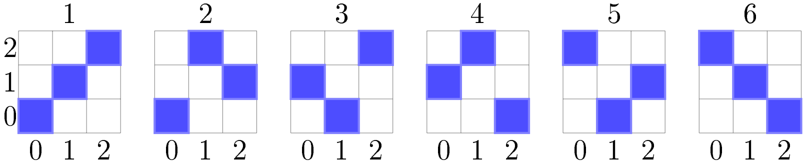

Do we really
need WNU here?

We really need WNU here!

The existence of a WNU preserving a relation is a necessary condition.

Counter Example

Let $s_1, s_2, \ldots, s_6$ be all premutations on the set $\{0, 1, 2\}$.
$\rho = \{(a, b, i) \mid i \in \{1, 2, \ldots, 6\}, a, b \in \{0, 1, 2\}, s_i(a) = b\}$.



- ρ is not a linear relation.
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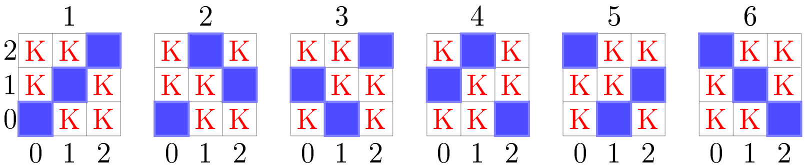

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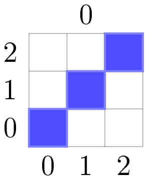
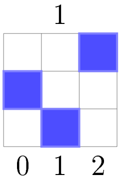
Let $A = \{0, 1, 2\}$ and $\rho = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$.





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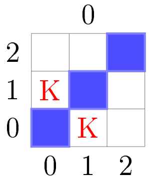

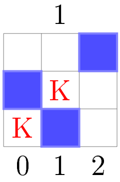

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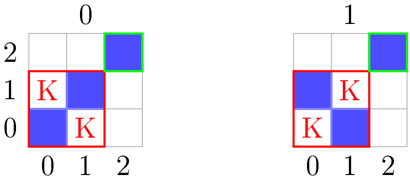


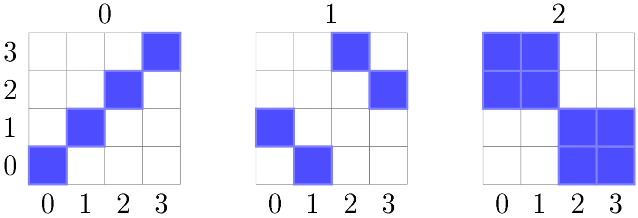
- trivial block

- nontrivial block

Example 2

Let $A = \{0,1,2,3\}$ and

$$\rho = \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 0 & 1 & 2 & 2 & 3 & 3 \\ 0 & 1 & 2 & 3 & 1 & 0 & 3 & 2 & 2 & 2 & 3 & 3 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$



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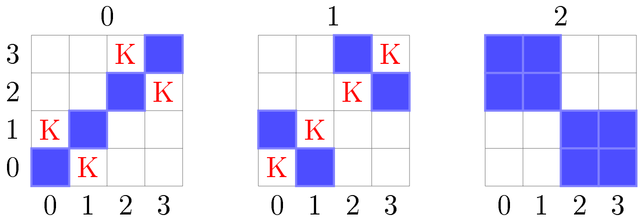

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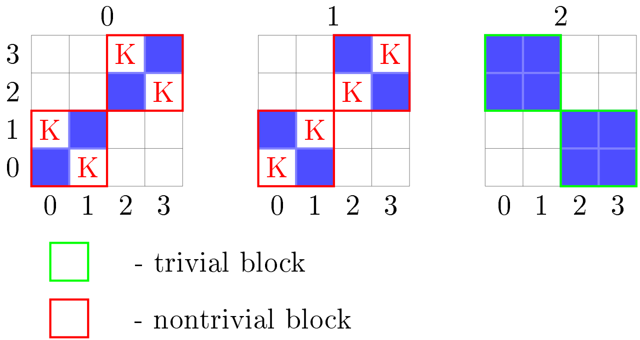

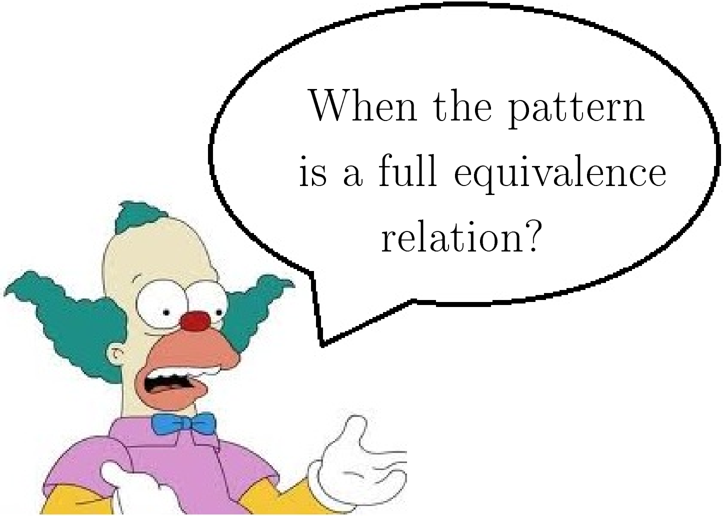
When the pattern
is a full equivalence
relation?

Mal'cev operation

A Mal'cev operation is a ternary operation f satisfying

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Lemma

Suppose ρ is a key relation preserved by a Mal'cev operation.

- The pattern of ρ is full equivalence relation.
- Every block in a core of ρ is nontrivial.

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- The pattern of ρ is full equivalence relation.
- Every block in a core of ρ is nontrivial.

Lemma [Keith A.Kearnes and Ágnes Szendrei]

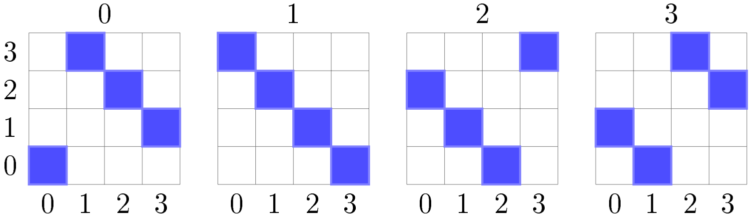
A relational clone admits a Mal'cev operation if and only if the pattern of each key relation in it is a full equivalence relation.

Mal'cev operation

Let $A = \{0, 1, 2, 3\}$ and $\rho = \{(x, y, z) \mid x + y + z = 0(\mod 4)\}$.

Mal'cev operation: $f(x, y, z) = x - y + z$.

WNU: $g(x_1, x_2, x_3, x_4, x_5) = x_1 + x_2 + x_3 + x_4 + x_5$.



Mal'cev operation

Let $A = \{0, 1, 2, 3\}$ and $\rho = \{(x, y, z) \mid x + y + z = 0(\mod 4)\}$.

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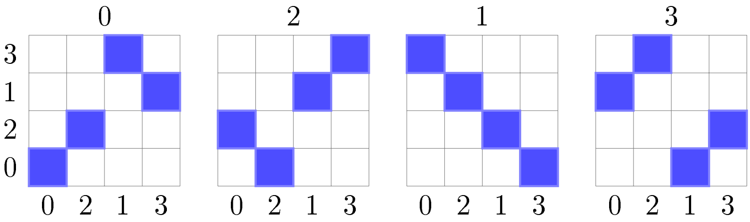

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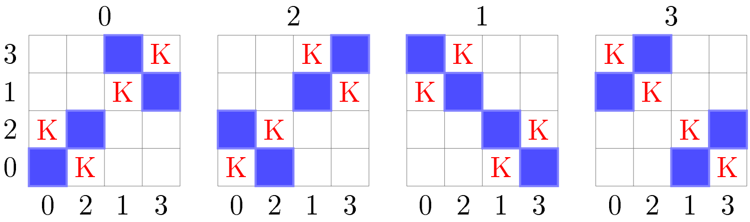

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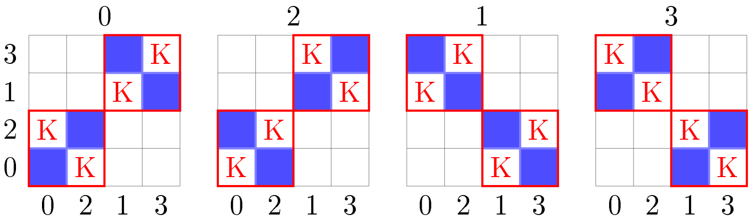

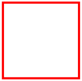 - nontrivial block

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Let $A = \{(0,0),(0,1),(1,0),(1,1)\}$ and
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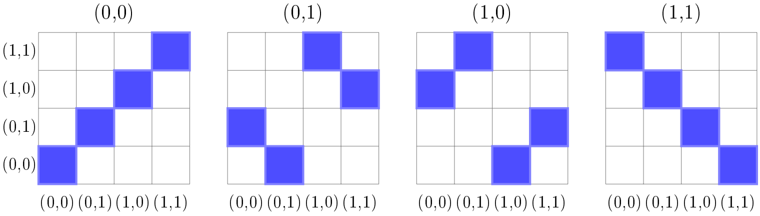
Mal'cev operation and WNU: $f(x,y,z) = x+y+z$.



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(0,0)

	(0,0)	(0,1)	(1,0)	(1,1)
(1,1)	K	K	K	
(1,0)	K	K		K
(0,1)	K		K	K
(0,0)		K	K	K

(0,1)

	(0,0)	(0,1)	(1,0)	(1,1)
(1,1)	K	K		K
(1,0)	K	K	K	
(0,1)		K	K	K
(0,0)	K		K	K

(1,0)

	(0,0)	(0,1)	(1,0)	(1,1)
(1,1)	K		K	K
(1,0)		K	K	K
(0,1)	K	K	K	
(0,0)	K	K		K

(1,1)

	(0,0)	(0,1)	(1,0)	(1,1)
(1,1)		K	K	K
(1,0)	K		K	K
(0,1)	K	K		K
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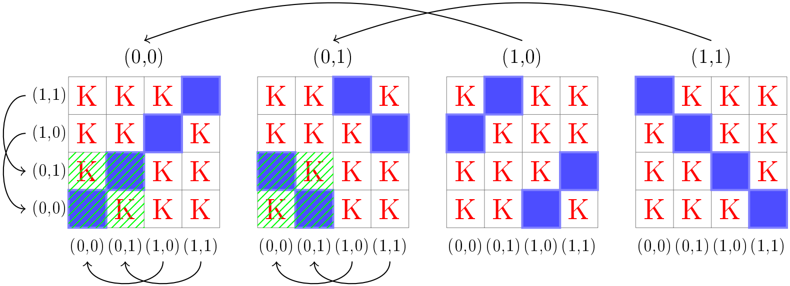

- a core

Edge operation

A k-edge operation is an operation f of arity $k+1$ satisfying

$$f(y,y,x,x,\ldots,x,x)=x$$
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$$\ldots$$
$$f(x,x,x,x\ldots,x,y)=x.$$

Edge operation

Lemma

Suppose a key relation ρ of arity n is preserved by a k-edge operation, then

- If $n \geq k$ then the pattern of ρ is full.
- If $n < k$ then the pattern of ρ is either trivial, or full.
- If the pattern of ρ is full, then every block in a core of ρ is nontrivial.

I still don't like this.
What the hell is core?
Can you prove
something in general?

Conjecture (Theorem)

Every key relation with full pattern preserved by a WNU can be divided into blocks such that

- Different blocks cannot contain common tuples and tuples that differ in one component.
- Every nontrivial block is a linear relation(with respect to some equivalence relation).
- All nontrivial blocks are «similar».


Tell me again
why it is useful!

Corollaries

- Mal'cev operation + near-unanimity operation $\Longrightarrow$ majority operation.

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- edge operation + semilattice (2-semilattice) operation $\Longrightarrow$ near-unanimity operation.
- edge operation $\Longleftrightarrow$ cube operation.

An operation preserving $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$ is called self-dual.

Task

Describe all clones of self-dual operations on 3 elements.

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Solution

Find all minimal clones of self-dual operations and for each clone describe all clones containing it.

- $x + 1 (mod 3)$.
- $2x + 2y (mod 3)$
- semiprojection s.
- self-dual majority operation.
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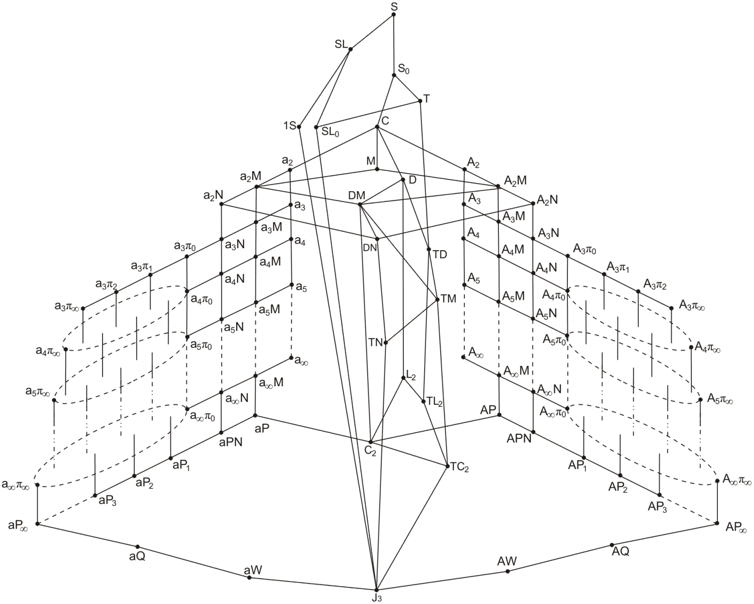
S
SL
S_0
T
1S
SL_0
C
a_2
M
A_2
a_2M
D
A_2M
a_2N
DM
A_3
A_2N
a_3
A_3M
a_3M
a_4
DN
A_4
A_3N
$a_3\pi_0$
a_3N
TD
A_4M
$A_3\pi_0$
$a_3\pi_1$
a_4M
A_4N
$A_3\pi_1$
a_4N
A_5
a_5
$a_3\pi_2$
$A_4\pi_0$
A_5M
$A_3\pi_2$
$a_3\pi_\infty$
$a_4\pi_0$
a_5M
TM
$A_3\pi_\infty$
A_5N
a_5N
$A_5\pi_0$
$a_4\pi_\infty$
$a_5\pi_0$
TN
$A_4\pi_\infty$
A_∞
a_∞
$a_\infty M$
$A_\infty M$
L_2
$a_5\pi_\infty$
$A_\infty N$
$a_\infty N$
TL_2
$A_5\pi_\infty$
$a_\infty\pi_0$
$A_\infty\pi_0$
AP
aP
APN
aPN
C_2
TC_2
aP_1
AP_1
$a_\infty\pi_\infty$
aP_2
AP_2
$A_\infty\pi_\infty$
aP_3
AP_3
aP_∞
AP_∞
aQ
AQ
aW
AW
J_3


Thank you for your attention