The lattice of quasivarietes of modules over a Dedekind ring

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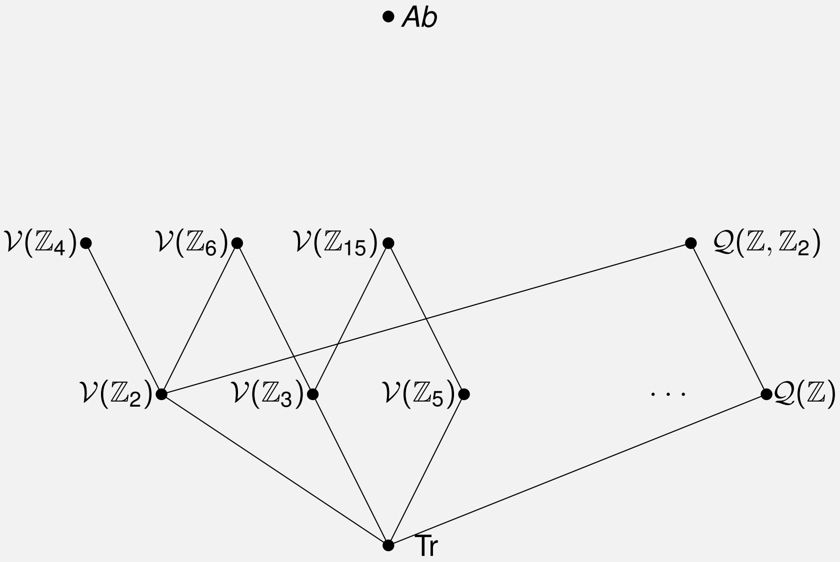
Quasivarieties of abelian groups (Vinogradov)



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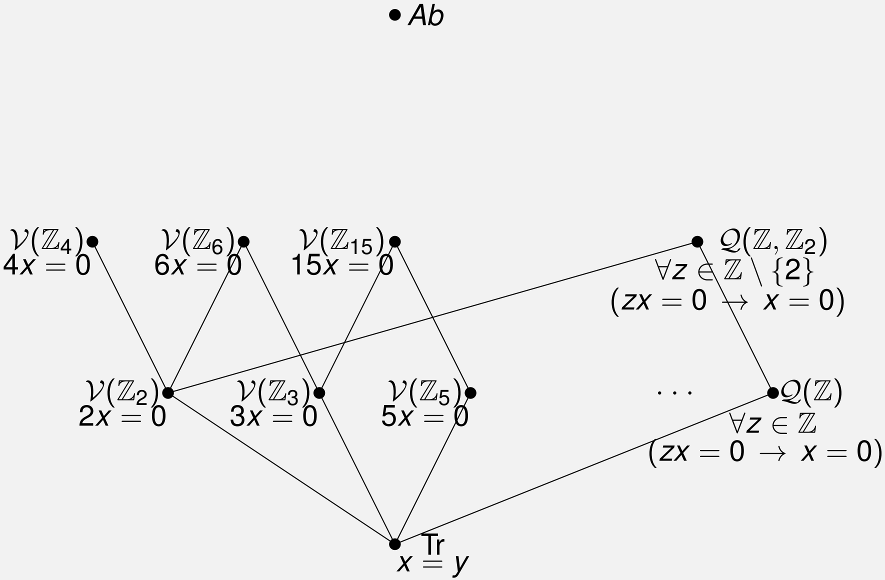

Quasivarieties of modules over $\mathbb{Z}$

Theorem

For each natural n the ring (group) $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Corollary

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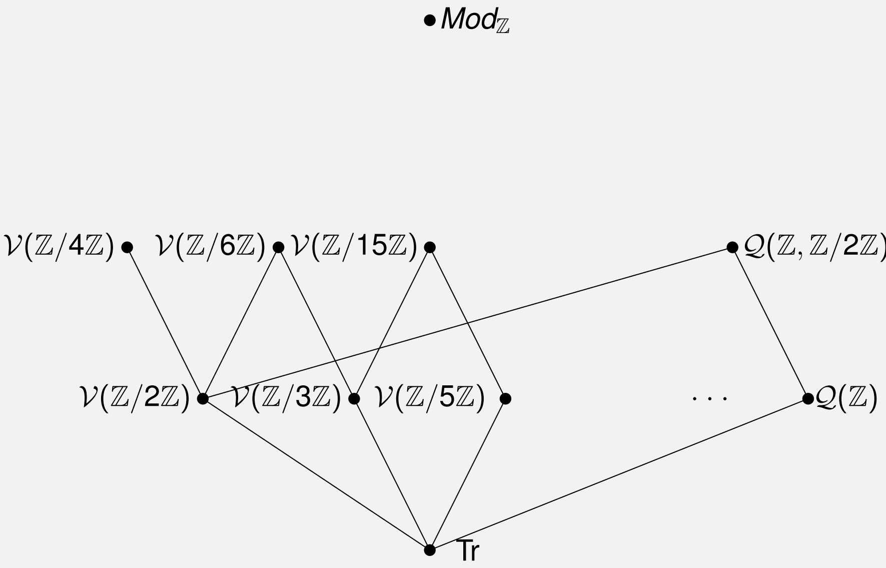

The Bielkin Lattice $\mathcal{L}(\alpha)$

For each ordinal number α , let α^+ denote the sum $\alpha \cup \{\infty\}$. Let $\mathcal{L}(\alpha)$ be the set of functions

$$f : \alpha^+ \to \omega^+,$$

where $f(\infty) \in \{0, \infty\}$ and $f(\infty) = 0$ implies that $f(\alpha) < \infty$ and $f(i) = 0$ for almost all $i \in \alpha$. Then $\mathcal{L}(\alpha)$ is a distributive lattice with respect to the following operations:

$$(f \vee g)(i) = max\{f(i), g(i)\}, \ (f \wedge g)(i) = min\{f(i), g(i)\}.$$

Quasivarieties of modules over PID

Theorem [Bielkin]

Let the ring $\mathcal{R}$ be a principal ideal domain and $|\mathbb{P}| = \alpha$, where $\mathbb{P}$ is the set of prime elements in the ring $\mathcal{R}$. Then the lattice of quasivarieties of the variety of modules over the ring $\mathcal{R}$ is isomorphic to the lattice $\mathcal{L}(\alpha)$,

$$\mathcal{L}_q(\mathbf{Mod}_{\mathcal{R}}) \cong \mathcal{L}(\alpha).$$

Example

$$\mathcal{L}_q(\mathbf{Mod}_{\mathbb{Z}}) \cong \mathcal{L}(\omega).$$

If $\mathcal{R}$ is a finite PID then

$$\mathcal{L}_q(\mathbf{Mod}_{\mathcal{R}}) \cong \mathcal{L}(\mathbf{0}) \cong \underline{\mathbf{2}},$$

where $\underline{2}$ is the two-element lattice.

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Deductive variety

We say that a variety $\mathcal{V}$ is *deductive* if each subquasivariety of $\mathcal{V}$ is a variety.

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PID and UDF

UFD

A unique factorization domain (UFD) is a commutative ring in which every non-zero non-unit element can be written as a product of prime elements (or irreducible elements), uniquely up to order and units.

Theorem

If $\mathcal{R}$ is PID, then it is UFD

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Dedekind rings

The ring $\mathbb{Z}[\sqrt{-5}]$ is not a UFD because

$$9 = 3 \cdot 3 = (2 + \sqrt{-5}) \cdot (2 - \sqrt{-5}).$$

Definition

A ring $\mathcal{R}$ is said to be *a Dedekind ring* if it is an integral domain and if every nonzero proper ideal of $\mathcal{R}$ is a finite product of prime ideals.

Theorem

If $\mathcal{R}$ is a Dedekind ring then the product decomposition is unique.

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We say that an element $x \in \mathcal{R}$ is *integral* over $\mathcal{P}$ if there exists a positive natural number n and elements $a_1, \ldots, a_n \in \mathcal{P}$ such that

$$x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0.$$

The set of all elements of the ring $\mathcal{R}$ integrals over $\mathcal{P}$ is called *integral closure* of $\mathcal{P}$ in $\mathcal{R}$ and is denoted by $C_{\mathcal{R}}(\mathcal{P})$. We say that a ring $\mathcal{R}$ is *integrally closed* if $C_{\mathcal{F}}(\mathcal{R}) = \mathcal{R}$, where $\mathcal{F}$ is the fraction field of $\mathcal{R}$.

Theorem

An integral domain $\mathcal{R}$ is a *Dedekind ring* if and only if the following conditions are satisfied:

(DR1) $\mathcal{R}$ is Noetherian;

(DR2) every non-zero prime ideal is maximal;

(DR3) R is integrally closed.

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The following rings are Dedekind rings:

1. A field,
2. A principal ideal domain,
3. The ring $\mathbb{Z}[\sqrt{-n}]$ where n is a square-free integral number and $n \equiv 1, 2 \pmod 4$.

Lemma

Let $\mathfrak{a}$ be a nonzero ideal of the Dedekind domain, and let r be any nonzero element of $\mathfrak{a}$. Then $\mathfrak{a}$ can be generated by two elements, one of which is r.

Theorem

Let $\mathcal{M}$ be a finitely generated non-trivial $\mathcal{R}$-module, and $\mathcal{M}_T$ be its submodule consisting of all torsion elements. Then $\mathcal{M}$ is isomorphic to a direct sum

$$\mathcal{M} \cong \mathcal{R}^n \oplus \mathfrak{a} \oplus \mathcal{M}_T,$$

where $n \in \mathbb{N}$ and $\mathfrak{a}$ is an ideal of $\mathcal{R}$.

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Lemma

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{R}$. Let $a \in \mathfrak{p} \setminus \mathfrak{p}^2$. Then $a^k \in \mathfrak{p}^k \setminus \mathfrak{p}^{k+1}$, for each $k \in \mathbb{N}$.

Let $\mathcal{R} = \mathbb{Z}[\sqrt{-5}]$. The prime ideal $(3; 2+\sqrt{-5})$ can be represented by many different pairs of generators, e.g. $(9; 5+\sqrt{-5})$. But here the number 9 is not good for our purposes since $9 \in (9; 5+\sqrt{-5})^2 = (3; 2+\sqrt{-5})^2$.

Lemma

For every Dedekind domain $\mathcal{R}$ and a finitely generated torsion-free $\mathcal{R}$-module $\mathcal{M} = \mathcal{R}^n \oplus \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $\mathcal{Q}(\mathcal{R}^n \oplus \mathfrak{a}) = \mathcal{Q}(\mathcal{R})$.

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Proposition

Let $\mathcal{R}$ be a Dedekind domain. Each proper subvariety of the variety $Mod_{\mathcal{R}}$ is deductive.

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Let $\mathcal{R}$ be a Dedekind domain and let $\mathfrak{a}$ be a nonzero ideal of this ring. The lattice $\mathcal{L}_q(\mathcal{V}(\mathcal{R}/\mathfrak{a})) = \mathcal{L}(\mathcal{V}(\mathcal{R}/\mathfrak{a}))$ is isomorphic to the lattice of divisors $\mathfrak{a}$ under divisibility.

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$$\mathcal{L}_q(\mathbf{Mod}_{\mathcal{R}}) \cong \mathcal{L}(\alpha).$$

The main theorem

Sketch of proof: The isomorphism $\varphi : \mathcal{L}(\alpha) \to \mathcal{L}_q(\mathbf{Mod}_{\mathcal{R}})$ is defined as follows:

$$\varphi(f) = \mathcal{Q}_f,$$

where $\mathcal{Q}_f$ is the subquasivaiety of $\mathbf{Mod}_{\mathcal{R}}$ containing the following modules:

(a) if $f(\infty) = \infty$, then $\mathcal{Q}_f$ contains the $\mathcal{R}$-module $\mathcal{R}$ and if $f(i) = \infty$ where $i \in \alpha$, then $\mathcal{Q}_f$ contains the $\mathcal{R}$-module $\mathcal{R}/\mathfrak{p}_i^n$ for $n \in \mathbb{N}$. If $f(i) = k$, where $k < \infty$, then $\mathcal{Q}_f$ contains the $\mathcal{R}$-module $\mathcal{R}/\mathfrak{p}_i^k$ and does not contain the $\mathcal{R}$-module $\mathcal{R}/\mathfrak{p}_i^{k+1}$.

(b) if $f(\infty) = 0$, then $\mathcal{Q}_f$ does not contain the $\mathcal{R}$-module $\mathcal{R}$ and $\mathcal{Q}_f$ is a variety generated by the finite set of torsion modules $\mathcal{R}/\mathfrak{p}_i^k$, where $f(i) = k$.

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THANK YOU FOR YOUR ATTENTION.