Models of nonlinear elasticity: Questions and progress

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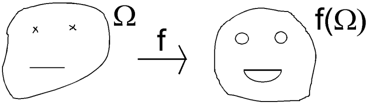
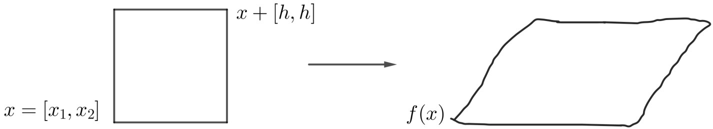
Models in Nonlinear Elasticity - Deformation

Object of study: $\Omega \subset \mathbf{R}^n$ is a body, $n = 2, 3, ..,$ $f : \Omega \to \mathbf{R}^n$ is a mapping (deformation of the body)



$\exists Df(x)$: f close to linear $f(y) \approx f(x) + Df(x)(y - x)$.



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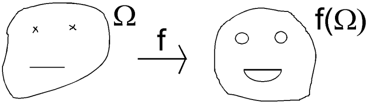

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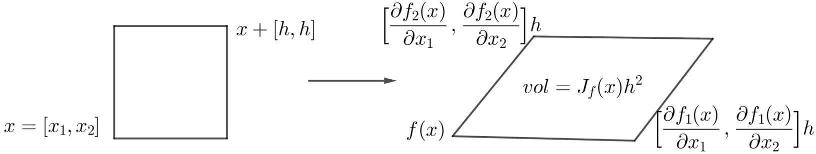

$Df(x)$ is $n \times n$ matrix of derivatives - deformation of segments
$J_f(x) = \det Df(x)$ is Jacobian - deformation of volume
$\int_A |J_f(x)|\, dx = |f(A)|$ if f is 1-1.

Models in Nonlinear Elasticity - Assumptions

Object of study: $\Omega \subset \mathbf{R}^n$ is a body, $n = 2, 3, ..,$ $f : \Omega \to \mathbf{R}^n$ is a mapping (deformation of the body)
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Motivation: J. Ball, V. Šverák - mathematical model for Nonlinear Elasticity. The mapping minimizes the elastic energy

$$\min_f \int_\Omega W(Df(x))\, dx$$

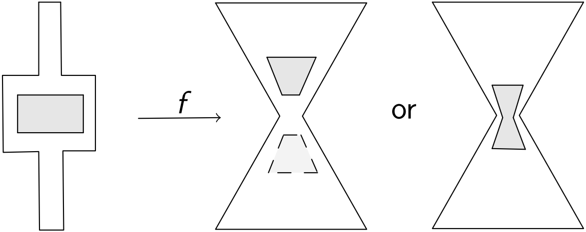
$W(A) \to \infty$ for $|A| \to \infty$, $W(A) \to \infty$ for $\det A \to 0$.

Naturally $|W(A)| \geq |A|^p$, i.e. $\int_\Omega |Df(x)|^p dx < \infty$ and $J_f(x) > 0$ a.e. (=mapping does not change orientation).

Sobolev space $W^{1,p}(\Omega, \mathbf{R}^n) = \{f : \int_\Omega |Df(x)|^p dx < \infty\}$.

Some natural questions

- Is there a minimizer of energy? Is it unique?



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- Is there a minimizer of energy? Is it unique?
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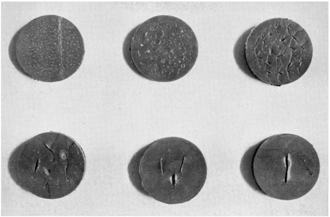

Cavities in rubber

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- Is the mapping one-to-one? Does there exist inverse map f^{-1}? (interpenetration of the matter)

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- Does f map sets of zero measure to sets of zero measure? (Is a new material created from 'nothing'? Is some material 'lost' during the deformation?). Does $\int_A J_f = |f(A)|$ hold?

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- Does f preserve orientation, i.e. $J_f \geq 0$ a.e.? (Can the body turn over?)
- Can we approximate it by piecewise linear homeomorphisms?
- What are the properties of f^{-1}? (Can we deform the body back to its original state?)

Hajlasz problem ($\sim$ 2000)

Problem: Let $\Omega \subset \mathbf{R}^n$ be a domain, $f : \Omega \to \mathbf{R}^n$ be a homeomorphism such that $f \in W^{1,1}(\Omega, \mathbf{R}^n)$. Is it true that $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.?

Motivation: a) change of variables formula - replace $|J_f|$ by J_f
b) assumption $J_f \geq 0$ in models superfluous
c) approximation

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Obstacles: $\exists f$ homeomorphism and Lipschitz, but $J_f = 0$ on a set of positive measure.

Theorem (H., Malý (2010))

Let $\Omega \subset \mathbf{R}^n$ be an open set, $n \geq 2$. Suppose that $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ is a homeomorphism for some $p > [n/2]$ ($p \geq 1$ for $n = 2, 3$). Then $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.

Open problem: 1. How about $p \leq [n/2]$.
2. Is there positively oriented f with $J_f \leq 0$?

Homeomorphisms with $J_f \equiv 0$

Area Formula : $\exists N \subset \Omega$ such that $\mathcal{L}_n(\Omega \setminus N) = \mathcal{L}_n(\Omega)$ but

$$0 = \int_{\Omega \setminus N} J_f(x) = \int_{f(\Omega \setminus N)} 1 = \mathcal{L}_n\big(f(\Omega \setminus N)\big)$$

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Theorem (H. (2011))

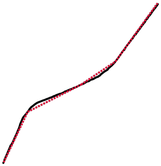
Let $n \geq 2$ and $1 \leq p < n$. There is a homeomorphism $f \in W^{1,p}((0,1)^n, (0,1)^n)$ such that $J_f(x) = 0$ a.e.

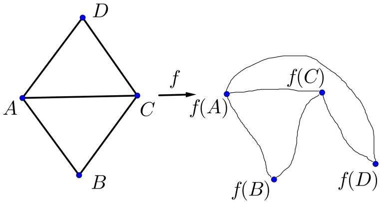
D'Onofrio, H., Schiattarella: $n \geq 3$ also $f^{-1} \in W^{1,1}$
Liu, Malý: f can be a gradient mapping, using laminates
Faraco, Mora-Corall, Oliva: laminates, sharp conditions also for $f \in W^{1,p}$, $f^{-1} \in W^{1,q}$ or $(\det Df)_k = 0$ a.e.

Ball-Evans Problem

Problem [Ball-Evans]: $\Omega \subset \mathbf{R}^n$ domain, $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ homeomorphism. Can we find f_k piecewise affine (or diffeomorphisms) such that $f_k \to f$ in $W^{1,p}$?

$n = 1$ easy: $n \geq 2$ hard:





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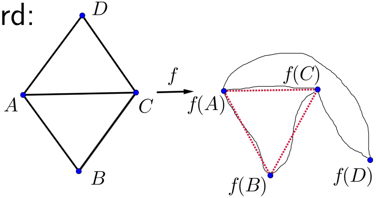

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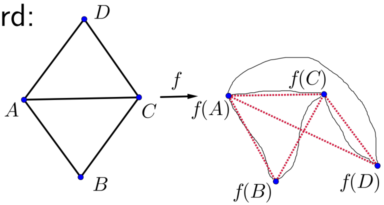

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It is not easy: triangulation or mollification destroy injectivity

$$\exists f_k \text{ smooth} \overset{\text{easy}}{\Rightarrow} \overset{\text{Pratelli\&Mora-Corral}}{\Leftarrow} \exists f_k \text{ piecewise affine}$$

Motivation

- Regularity for models in Nonlinear Elasticity
  Ball models $\min \int W(Du)$ where $E(u) \to \infty$ as $J_u \to 0$
- Numerics - finite elements method
- Easier proofs of known (and new) statements

Known results

C. Mora-Corral: f smooth up to one point

Theorem (Iwaniec, Kovalev, Onninen (2011))

Let $n = 2$ and $1 < p < \infty$. Given a homeomorphism $f \in W^{1,p}(\Omega, \mathbf{R}^2)$ there are diffeomorphisms f_k with $f_k \to f$ in $W^{1,p}$, $f_k \rightrightarrows f$ and $f_k - f \in W_0^{1,p}$

Theorem (H., Pratelli (2018))

Let $n = 2$. Given a homeomorphism $f \in W^{1,1}(\Omega, \mathbf{R}^2)$ there are diffeomorphisms f_k with $f_k \to f$ in $W^{1,1}$, $f_k \rightrightarrows f$ and $f_k - f \in W_0^{1,p}(\Omega, \mathbf{R}^2)$.

Open problems

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- $n = 2, p = 2,\ f \in W^{1,2},\ f^{-1} \in W^{1,2}$ - Can we approximate? Are the minimizers of $\int |Df|^2 + \frac{|Df|^2}{J_f} (= \int |Df|^2 + \int |Df^{-1}|^2)$ smooth?

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  Is there a minimization where the minimizer is a diffeomorphism?
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- Anything about the approximation in $n = 3$?
  Is there a minimization where the minimizer is a diffeomorphism?
  Is there some improved construction by hand?
- Is there some counterexample in dimension $n \geq 4$ in $W^{1,p},\ p > [n/2]$?

Dimension $n \geq 4$ - Hajlasz problem

Theorem (Campbell, H., Tengvall, Vejnar (2016, 2018))

Let $n \geq 4$ and $1 \leq p < [\frac{n}{2}]$. There is a homeomorphism in the Sobolev space $f \in W^{1,p}((0,1)^n, \mathbf{R}^n)$ such that $\mathcal{L}^4(\{x : \ J_f(x) > 0\}) > 0$ and $\mathcal{L}^4(\{x : \ J_f(x) < 0\}) > 0$.

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Corollary: Let $n \geq 4$. For this $f \in W^{1,p}$ there are no diffeomorphisms (or piecewise affine homeomorphisms) with $f_k \to f$ in $W^{1,p}_{\text{loc}}((-1,1)^n, \mathbf{R}^n)$.

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Proof by contradiction: $f_k \overset{W^{1,p}}{\to} f \overset{\text{subsequence}}{\Rightarrow} Df_k(x) \to Df(x)$ for a.e. $x \Rightarrow J_{f_k}(x) \to J_f(x)$ for a.e. x. As f_k smooth $\Rightarrow J_{f_k} \geq 0$ a.e. or $J_{f_k} \leq 0$ a.e. Hence their pointwise limit does not change sign - contradiction.

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Example: $\exists$ homeomorphism $f(x) = x$ on $\partial[0,1]^n$ and $J_f < 0$ a.e. for $n = 4$ and $1 \leq p < 3/2$.

Sobolev regularity of the inverse

Problem: Let $f \in W^{1,p}(\Omega, \mathbf{R}^n)$, $p \geq 1$, be a homeomorphism. When $f^{-1} \in W^{1,1}_{\text{loc}}(f(\Omega), \Omega)$? (or BV)?

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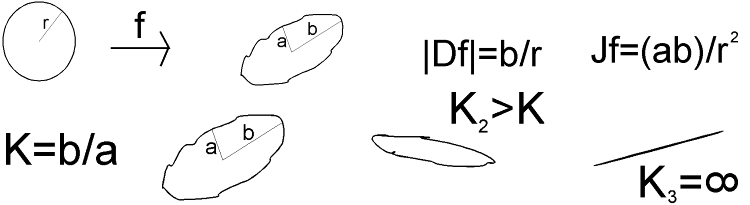
Elementary example: There is a Lipschitz homeomorphism $f : [0,2] \times [0,1]^{n-1} \to [0,1]^n$ with $f^{-1} \notin W^{1,1}_{\text{loc}}([0,1]^n, \mathbf{R}^n)$.
Proof ($n = 2$): $h(x) = x + C(x)$, $f(x,y) = [h^{-1}(x), y]$.

Definition: Homeomorphism $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbf{R}^n)$ has finite distortion if $|Df(x)|^n \leq K(x) J_f(x)$ holds a.e., where $1 \leq K(x) < \infty$ a.e. Especially if $J_f > 0$ a.e. then f has finite distortion.



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Suppose that $f \in W^{1,n-1}(\Omega, \mathbf{R}^n)$ is a homeomorphism of finite distortion. Then $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbf{R}^n)$ and has finite distortion.

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Theorem

Suppose that $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbf{R}^n)$ is a homeomorphism of finite distortion and moreover we assume that $K \in L^{n-1}(\Omega)$. Then $f^{-1} \in W^{1,n}_{\text{loc}}(f(\Omega), \mathbf{R}^n)$.

BV regularity of the inverse

Theorem (Csörnyei+H.+Malý (2010))

Suppose that $f \in W^{1,n-1}(\Omega\ \mathbf{R}^n)$ is a homeomorphism. Then $f^{-1} \in BV_{\mathrm{loc}}(f(\Omega), \mathbf{R}^n)$.

Definition

We say that $h \in BV(\Omega)$ if $h \in L^1(\Omega)$ and $D_i h = \mu_i$ are signed Radon measures with finite total variation:
$\int_\Omega h D_i\varphi\ dx = -\int_\Omega \varphi\ d\mu_i,$ for all $\varphi \in C_0^\infty(\Omega)$.
We say that $f \in BV(\Omega; \mathbf{R}^n)$ if $f_i \in BV(\Omega)$.

BV regularity of the inverse

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Suppose that $f \in W^{1,n-1}(\Omega\ \mathbf{R}^n)$ *is a homeomorphism. Then* $f^{-1} \in BV_{\text{loc}}(f(\Omega), \mathbf{R}^n)$.

Theorem

Let $0 < \varepsilon < 1$ *and* $n \geq 3$. *There is a homeomorphism of finite distortion* $f \in W^{1,n-1-\varepsilon}((-1,1)^n; \mathbf{R}^n)$ *such that* $f^{-1} \notin BV_{\text{loc}}(f(\Omega); \mathbf{R}^n)$. *('because* $|\nabla f^{-1}| \notin L^1_{\text{loc}}$*')*

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$$
\begin{aligned}
\int_{f(\Omega)} |Df^{-1}(y)|\, dy &= \int_\Omega |Df^{-1}(f(x))| J_f(x)\, dx \\
&\overset{f^{-1}\circ f = \text{id}}{=} \int_\Omega |(Df(x))^{-1}| J_f(x)\, dx \\
&\overset{A\,\text{adj}\,A = \det A I}{=} \int_\Omega |\operatorname{adj} Df(x)|\, dx \leq \int_\Omega |Df(x)|^{n-1}\, dx
\end{aligned}
$$