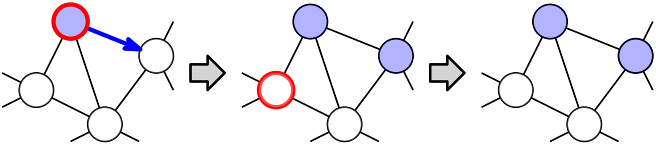
Evolutionary graph theory

Josef "Pepa" Tkadlec

Mathematical Forum, March 2026



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$$x = 1 + \frac{1}{6} \cdot 0 + \frac{5}{6} \cdot x.$$

This gives $x = 6$.

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Jug lemma: "On average, an event with probability p happens after $1/p$ steps."

How does stuff propagate through networks?

COVID pandemic

- Nodes: people. Edges: spending time together
- Process: people getting Susceptible → Infected → Recovered

Questions

- Which region gets hit next? When?
- Is there enough hospital capacity?
- Who should we vaccinate first?
- ...

How does stuff propagate through networks?

- epidemiology: coronavirus among humans
- social dynamics: influence (opinion, gossip, fake news) on social media
- evolutionary biology: genetic mutations in populations
- statistical mechanics, percolation, distributed computing, ...

System = a bunch of nodes of different types.

Overview

- PUSH protocol for rumor spreading
- Moran process on graphs

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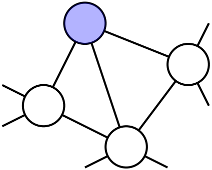

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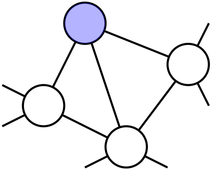

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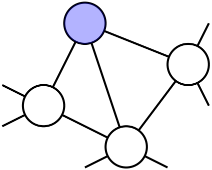

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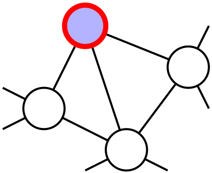

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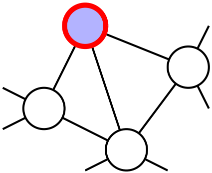

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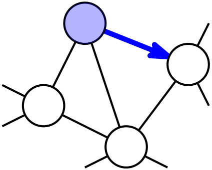

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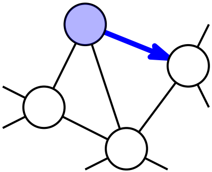

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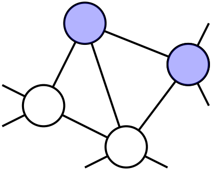

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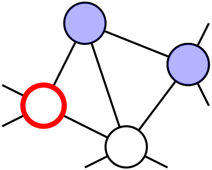

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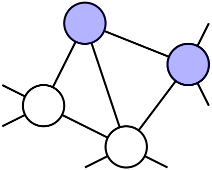

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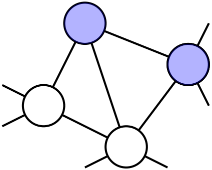
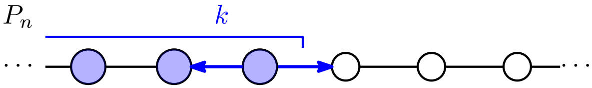


Question: How long until all nodes are informed?

Denote the expected number of rounds by $T(G_n, v)$.

Path graph P_n



- Idea: Wait for $n-1$ stages.
- Let $p_k =$"chance to make progress, with k nodes informed".
- On average, progress after $T_k = \frac{1}{p_k}$ rounds [Puzzle].
- In total:

$$T(P_n, v) = \cdots + \frac{1}{\frac{1}{n} \cdot \frac{1}{2}} + \ldots$$

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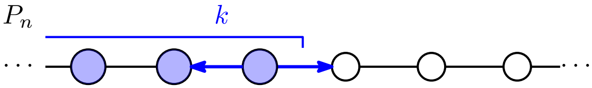

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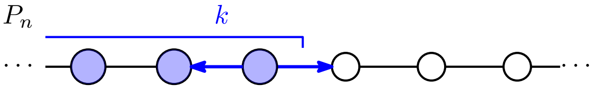

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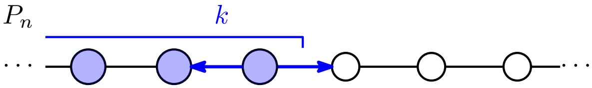

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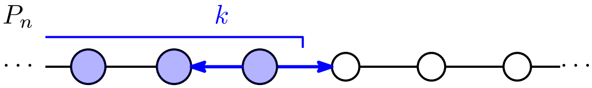

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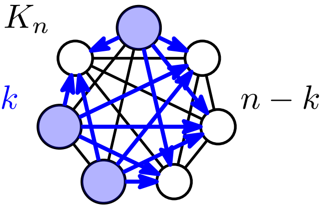
On a Cycle graph C_n: $T(C_n, v) = \dots = n(n-1) \approx n^2$.

Complete graph K_n



- Same idea: $p_k = \dots$

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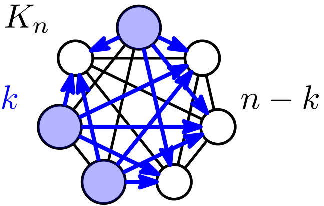

- Same idea: $p_k = \ldots \frac{k}{n} \cdot \frac{n-k}{n-1}$
- Total time

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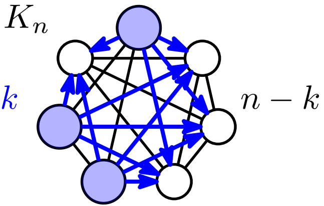

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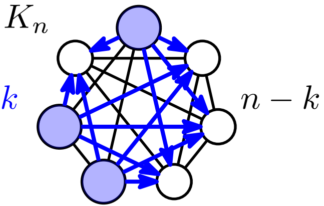

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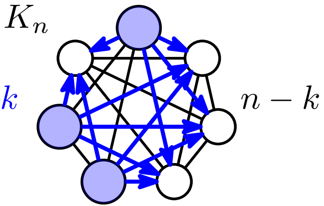

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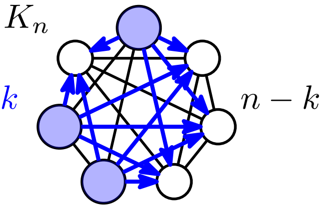

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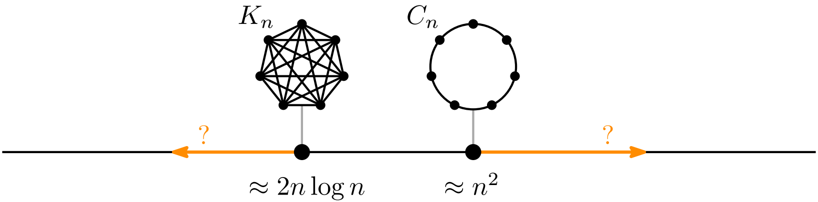
How fast / slow can a graph be?



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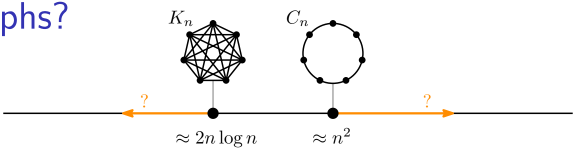

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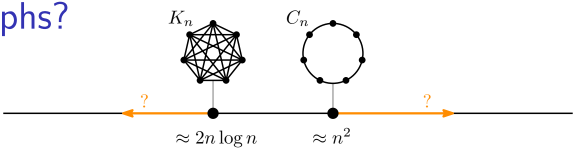

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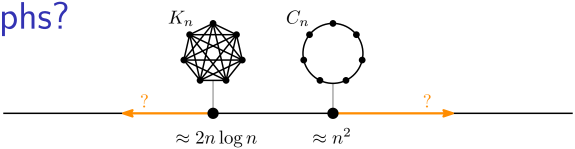

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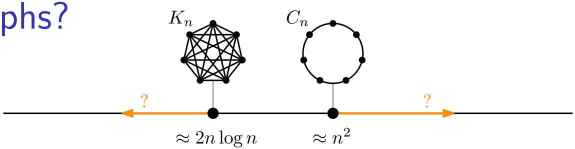

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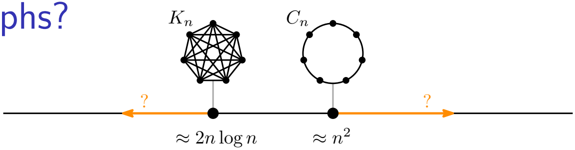

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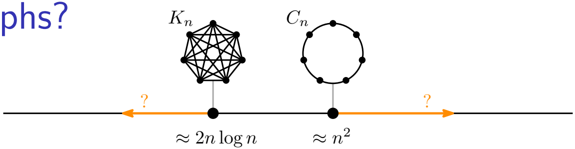

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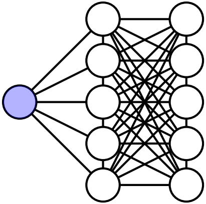

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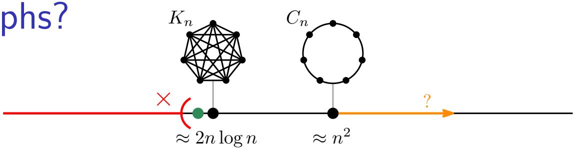

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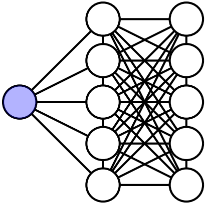

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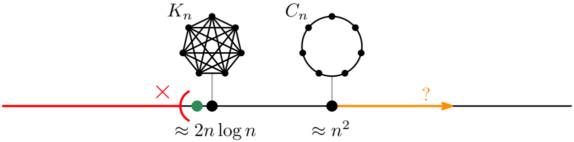

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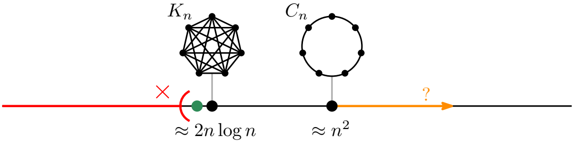

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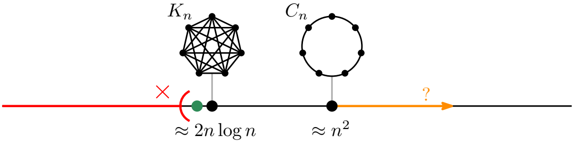
Slowest graphs?



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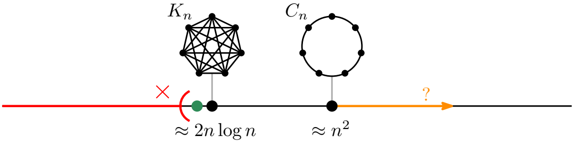

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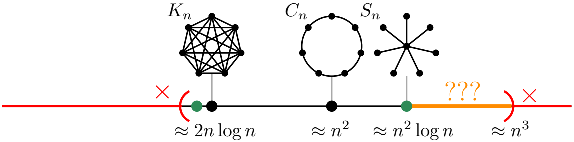

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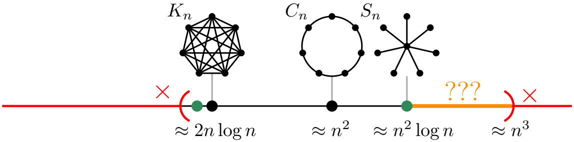

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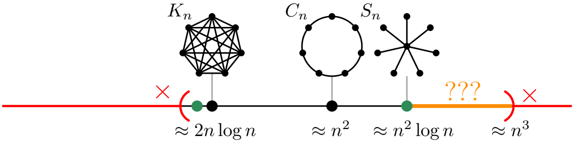

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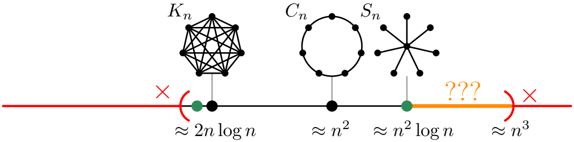
Claim [Feige '90]. In fact, $T(G_n, v) = \mathcal{O}(n^2 \log n)$.

Question. Is S_n the slowest graph?

Slowest graphs?



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Claim [Feige '90]. In fact, $T(G_n, v) = \mathcal{O}(n^2 \log n)$.

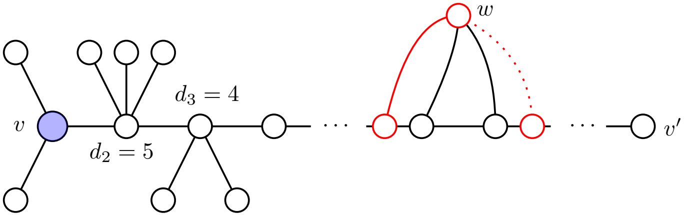
Question. Is S_n the slowest graph? This is Open.

Stronger upper bound

Key lemma. For any G_n, v, v' we have $T(v \to v') \leq 5n^2$.

Proof idea:

- Take the shortest path from v to v'.
- Then $T(v \to v') \leq nd_1 + nd_2 + \cdots = n \cdot \sum_i d_i$.



- Each w outside the path connects to at most 3 path nodes!
- So at most $3n$ edges connect to the path.
- So $\sum_i d_i \leq 5n$ and $T(v \to v') \leq n \cdot 5n = 5n^2$.

Stronger upper bound, cont'd

Recall:

Key Lemma. For any G_n, v, v' we have $T(v \to v') \leq 5n^2$.

Claim [Feige '90]. We have $T(G_n, v) = \mathcal{O}(n^2 \log n)$.

Proof idea:

- Split into phases of $10n^2$ steps each.
- Fix v'. By Lemma and Markov inequality, v' is informed with probability $\geq 1/2$.
- So after k phases, v' is informed unless $1/2^k$.

Stronger upper bound, cont'd

Recall:

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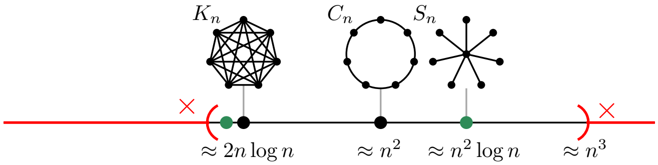
- Fix v'. By Lemma and Markov inequality, v' is informed with probability $\geq 1/2$.
- So after k phases, v' is informed unless $1/2^k$.
- By Union bound, k rounds suffice unless $n/2^k$, so

$$T(G_n, v) \leq 10n^2 \cdot \log_2 n + 10n^2 \cdot \sum_{k \geq \log_2 n} \frac{n}{2^k}$$

$$= 10n^2 \log_2 n + \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) = 20n^2 \log_2 n.$$

Landscape



Open Questions:

1. What is the fastest graph?
2. What is the slowest graph? Is it the Star S_n?
3. Efficient algorithm to compute $T(G_n, v)$, for any G_n?

Overview

- PUSH protocol for rumor spreading
- Moran process on graphs

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  - "progressive": only 1 color spreads
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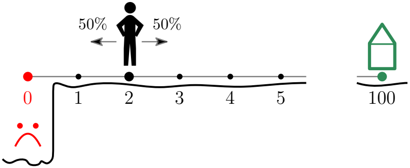
Overview

- PUSH protocol for rumor spreading
  - "progressive": only 1 color spreads
- Moran process on graphs
  - both colors fight!

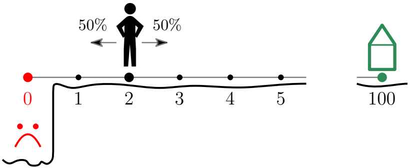
Puzzle: Random Walk

Puzzle. A drunk person, repeatedly stepping left/right 50 : 50.
How likely are they to make it home?



Puzzle: Random Walk

Puzzle. A drunk person, repeatedly stepping left/right 50 : 50.
How likely are they to make it home? Answer: 2%



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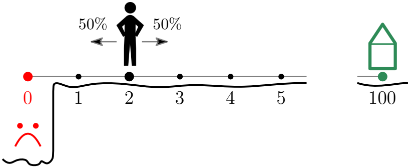
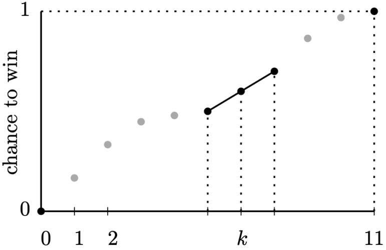


Proof idea: Let p_k = "win-chance from k".
Then $p_k = \frac{1}{2}p_{k-1} + \frac{1}{2}p_{k+1}$.



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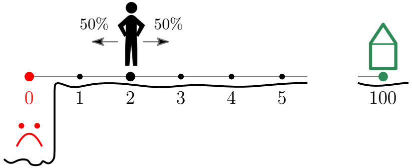

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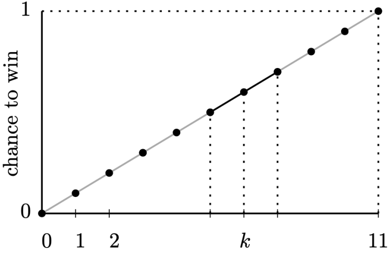
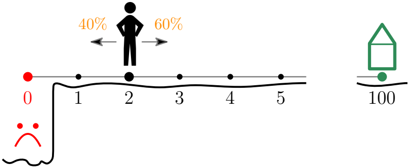


Puzzle: Biased Random Walk

Puzzle. Not completely drunk, steps left/right 40 : 60.

- How likely are they to make it home?



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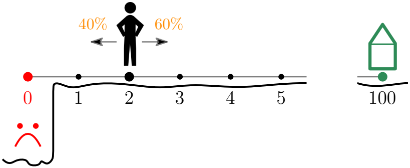

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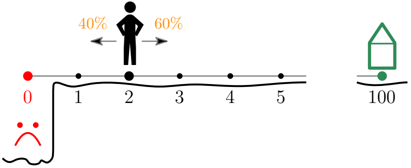
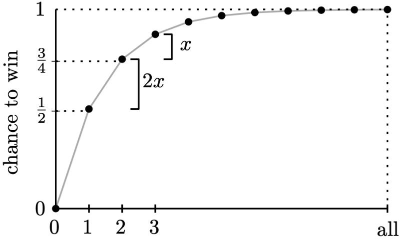


Proof idea: This time $p_k = 0.4 \cdot p_{k-1} + 0.6 \cdot p_{k+1}$.
If it were $\frac{1}{3} : \frac{2}{3}$, the answer would be $\approx 75\%$



Model: Moran process on a graph

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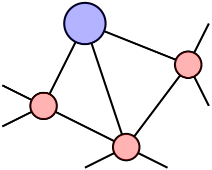

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mutant	r
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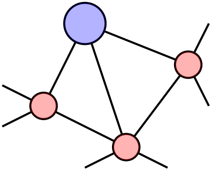

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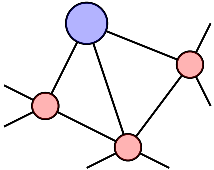

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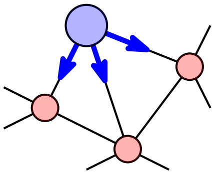

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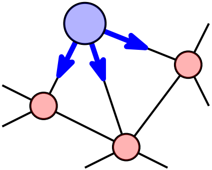

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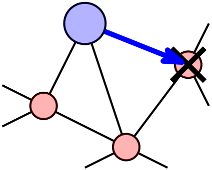

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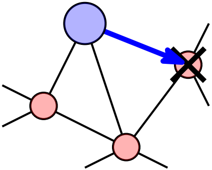

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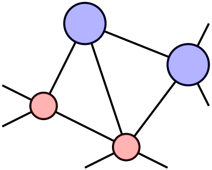

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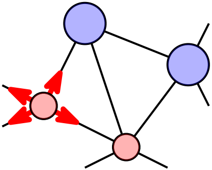

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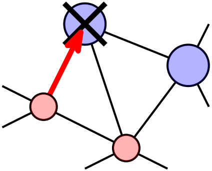

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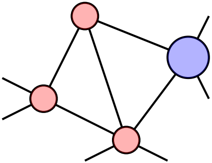

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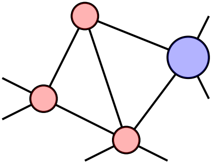
Some features of the Moran process



	fitness
mutant	r
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1. It is stochastic (random).
2. In some steps, nothing happens ("config" stays the same)
3. Nodes can toggle back and forth (more opinions than gossip).
4. Eventually, all nodes become the same type (no mutation).

Quantities of interest:

1. Fixation probability $\mathrm{fp}^r(G_n, v)$: How likely is it that blue wins?
2. Fixation time $\mathrm{T}^r(G_n, v)$: Average #steps until one type wins?
3. ...

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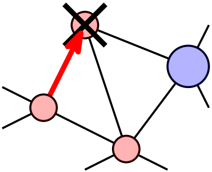

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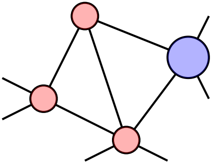

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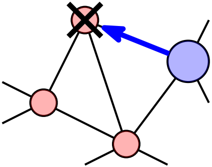

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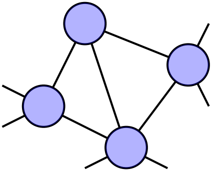

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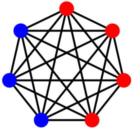
Moran process on K_n ($r > 1$)

Idea: Track #mutants and instead of p_k, have p_k^+ and p_k^-.



$$F = kr + (n-k)$$

$$\left.\begin{aligned} p_k^+ &= \tfrac{kr}{F} \cdot \tfrac{n-k}{n-1} \\ p_k^- &= \tfrac{n-k}{F} \cdot \tfrac{k}{n-1} \end{aligned}\right\} \quad \frac{p_k^+}{p_k^-} = r$$

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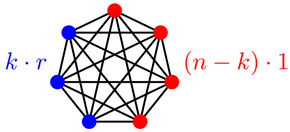

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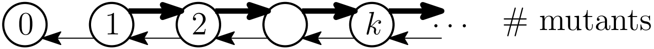
- In each step, we are r-times more likely to gain than to lose a mutant.
- So like the drunk with a sense of direction.

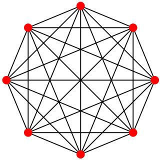


Claim. $\mathrm{fp}^r(K_n) = \frac{1-1/r}{1-1/r^n} \to_{n\to\infty} \boxed{1 - 1/r}$.

Regime $r > 1$ fixed, n large

1. Baseline: Complete graph K_n. We have $\mathrm{fp}^r(K_n) \to 1 - 1/r$.

K_n



$1 - \frac{1}{r}$

Regime $r > 1$ fixed, n large

1. Baseline: Complete graph K_n. We have $\text{fp}^r(K_n) \to 1 - 1/r$.
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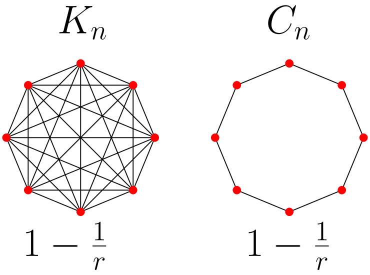

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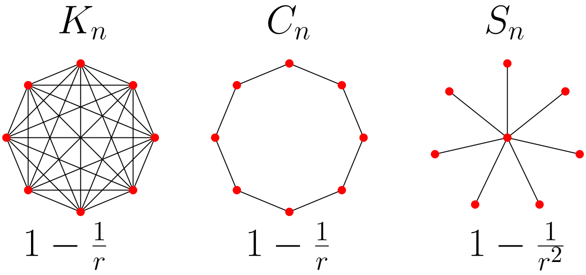

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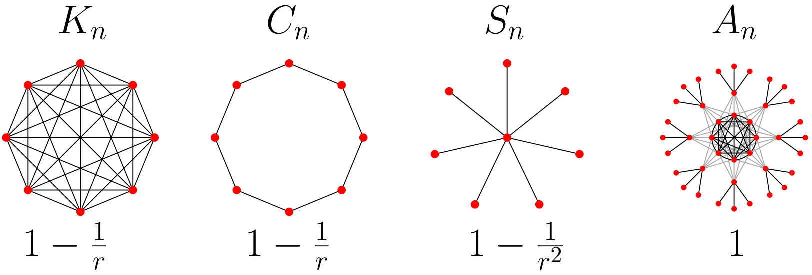
4. There exist strong amplifiers: graphs A_n with $\text{fp}^r(A_n) \to_{n\to\infty} 1$, for any $r > 1$.



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  - Do we know anything about $\mathrm{T}^r(G_n, v)$? Yes!

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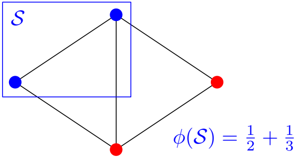

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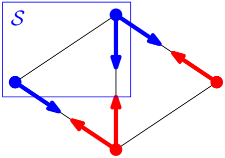

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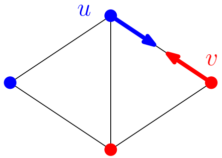

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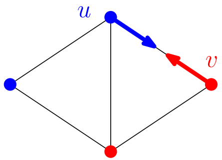

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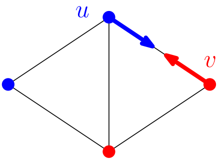

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3. But $\phi(\mathcal{S}) \in [0, n]$, so ϕ "can't keep increasing for too long".
4. $\Rightarrow T^r(G_n) = \mathcal{O}(n^4)$.

Summary

- Fixation probability can be approximated by simulations.
- Fixation time is between $\Omega(n \log n)$ and $\mathcal{O}(n^{3+\varepsilon})$ [GLR '19].

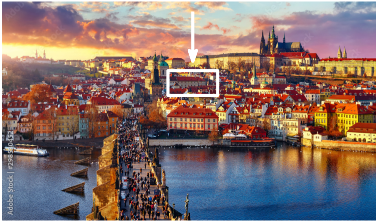
Open questions:

1. Computing $\mathrm{fp}^r(G_n, v)$ exactly?
2. Approximations on directed graphs?
3. Adversarial amplifier?
4. Location-dependent fitness?
5. #mutants at time T?
6. More colors?
7. ...

Wrap-up

- How does stuff spread through networks?
- Even for the simplest model, basic questions are open.
- Puzzles: Jug lemma, Coupon collector, (biased) Random walk

EGT-Prague



Computer Science Institute, Charles Uni

Thank you