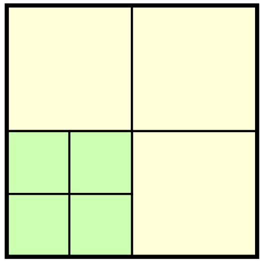
LECTURE SERIES

Numerical Simulation of Viscous Flow: Discretization, Optimization and Stability



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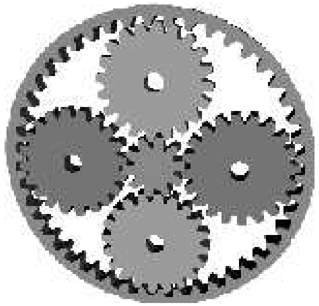


12th School "Mathematical Theory in Fluid Mechanics",
Kacov, May 27 - June 3, 2011

Lecture 1. Numerics of Incompressible Viscous Flow

We discuss the Galerkin finite element method for the discretization of the Navier-Stokes equations. Particular emphasis is put on the aspects of local and global error analysis, pressure stabilization, and truncation to bounded domains.

Lecture 2. Goal-Oriented Adaptivity

We introduce the concept underlying the DWR (Dual Weighted Residual) method for goal-oriented residual-based adaptivity in solving the Navier-Stokes equations. This approach is presented for stationary as well as nonstationary situations.

Lecture 3. Optimal Flow Control

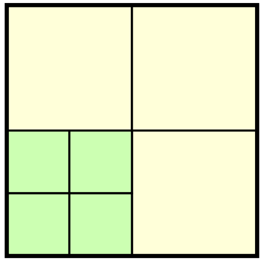
We discuss the use of the DWR metrhod for adaptive discretization in flow control and parameter estimation.

Lecture 4. Numerical Stability Analysis

We consider the numerical stability analysis of stationary flows employing the concepts of linearized stability and pseudospectra.

LECTURE 1

Numerical Computation of Viscous Incompressible Flow



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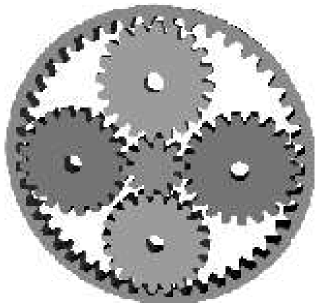

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1.0 References

- J. Heywood and R. Rannacher: *Finite element approximation of the nonstationary Navier-Stokes problem*,

  Part I: Regularity of solutions and second-order error estimates for spatial discretization, SIAM J. Numer. Anal. 19, 275–311 (1982);

  Part II: Stability of solutions and error estimates uniform in time, SIAM J. Numer. Anal. 23, 750–777 (1986);

  Part III: Smoothing property and higher-order error estimates for spatial discretization, SIAM J. Numer. Anal. 25, 489–512 (1988);

  Part IV: Error analysis for second-order time discretization, SIAM J. Numer. Anal. 27, 353–384 (1990).

- J. Heywood, R. Rannacher, and S. Turek: *Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations*, Int. J. Numer. Meth. Fluids 22, 1–28 (1996);

- R. Rannacher: *Finite methods for the incompressible Navier-Stokes equations*, in Fundamental Directions in Mathematical Fluid Mechanics (G.P. Galdi, J. Heywood, and R. Rannacher, eds), pp. 191–216, Birkhäuser, Basel-Boston-Berlin, 2000.

1.1 The mathematical model

The continuum mechanical model for the flow of a viscous Newtonian fluid is the system of conservation equations for mass, momentum and energy:

$$\partial_t \rho + \nabla \cdot [\rho v] = 0 \tag{1}$$

$$\partial_t(\rho v) + \rho v \cdot \nabla v - \nabla \cdot [\mu \nabla v + \tfrac{1}{3}\mu \nabla \cdot v I] + \nabla p_{tot} = \rho f \tag{2}$$

$$\partial_t(c_p \rho T) + c_p \rho v \cdot \nabla T - \nabla \cdot [\lambda \nabla T] = h \tag{3}$$

A state equation models the connection of pressure and density. In the following, the fluid is assumed to be *incompressible* and the density to be homogeneous, $\rho \equiv \rho_0 =$ const., so that (1) reduces to the constraint $\nabla \cdot v = 0$. Further, in the isothermal case, setting $\rho_0 = 1$, the Navier-Stokes system can be written in short as

$$\partial_t v + v \cdot \nabla v - \nu \Delta v + \nabla p = f \tag{4}$$

$$\nabla \cdot v = 0 \tag{5}$$

with the kinematic viscosity $\nu > 0$.

This system is supplemented by initial and boundary conditions

$$v_{|t=0} = v^0, \quad v_{|\Gamma_{\rm rigid}} = 0, \quad v_{|\Gamma_{\rm in}} = v^{in}, \quad (\mu\partial_n v + pn)_{|\Gamma_{\rm out}} = 0 \qquad (6)$$

where Γ_{rigid}, Γ_{in}, Γ_{out} are the rigid part, the inflow part and the outflow part, respectively, of the flow domain's boundary. The role of the natural outflow boundary condition on $\Gamma_{\rm out}$ will be discussed in more detail below. In this formulation the flow domain may be two- or three-dimensional.

This model is made dimensionless through a scaling transformation with the Reynolds number $Re = UL/\nu$ as the characteristic parameter, where U is the reference velocity and L the characteristic length, e.g., $U \approx \max|v^{\rm in}|$ and $L \approx {\rm diam}(\Omega)$.

We concentrate on *laminar* flows in which all relevant spatial and temporal scales can be resolved, and no additional modeling of turbulence effects is required. The numerical solution of this differential (-algebraic) system is complicated mainly because of the incompressibility constraint, which enforces the use of implicit methods.

The Galerkin FEM is based on a variational formulation of the problem and determines "discrete" approximations of the primitive variables velocity and pressure in certain finite dimensional trial spaces consisting of piecewise polynomial functions. This discretization inherits most of the structure of the continuous problem, which results in high computational flexibility and provides the basis of a solid mathematical analysis.

Notation:

$$L_0^2(\Omega) = \{\varphi \in L^2(\Omega) : \ (v,1) = 0\}, \quad H^1(\Omega) = \{v \in L^2(\Omega), \, \nabla v \in L^2(\Omega)^d\}$$

and $H_0^1(\Gamma;\Omega) = \{v \in H^1(\Omega), \, v_{|\Gamma} = 0\}$ for some (non-trivial) part Γ of the boundary $\partial\Omega$, as well as the corresponding inner products and norms

$$(u,v) = \int_\Omega uv \, \mathrm{dx}, \quad \|v\| = (v,v)^{1/2}, \qquad (\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{dx}.$$

These are all spaces of $\mathbb{R}$-valued functions. Spaces of $\mathbb{R}^d$-valued functions $v = (v_1, \dots, v_d)$ are denoted by boldface-type, e.g., $\mathbf{H}_0^1(\Gamma;\Omega) = H_0^1(\Gamma;\Omega)^d$.

The pressure p in the Navier-Stokes equations is uniquely (possibly up to a constant) determined by the velocity field v. There holds the stability estimate ("inf-sup" stability)

$$\inf_{q\in L^2(\Omega)} \sup_{\varphi\in \mathbf{H}_0^1(\Gamma;\Omega)} \frac{(q,\nabla\cdot\varphi)}{\|q\|\,\|\nabla\varphi\|} \geq \gamma_0 > 0 \tag{7}$$

where $L^2(\Omega)$ has to be replaced by $L_0^2(\Omega)$ in the case $\Gamma = \partial\Omega$. Finally, we introduce the bilinear and trilinear forms

$$a(u,v) := \nu(\nabla u, \nabla v)\,, \quad b(p,v) := -(p, \nabla\cdot v)\,, \quad n(u,v,w) := (u\cdot\nabla v, w)$$

and the abbreviations

$$\mathbf{H} := \mathbf{H}_0^1(\Gamma;\Omega), \quad L := L^2(\Omega) \quad \big(L := L_0^2(\Omega) \text{ in the case } \Gamma = \partial\Omega\big)$$

where $\Gamma = \Gamma_{\text{in}} \cup \Gamma_{\text{rigid}}$.

The variational formulation of the Navier-Stokes problem (4), (5) reads as follows: *Find functions* $v(\cdot,t) \in v^{\rm in} + \mathbf{H}$ *and* $p(\cdot,t) \in L$, *where* $v(\cdot,t)$ *is continuous on* $[0,T]$ *and continuously differentiable on* $(0,T]$, *such that* $v_{|t=0} = v^0$, *and*

$$(\partial_t v, \varphi) + a(v,\varphi) + n(v,v,\varphi) + b(p,v) = (f,\varphi) \quad \forall \varphi \in \mathbf{H} \tag{8}$$

$$(\nabla \cdot v, \chi) = 0 \quad \forall \chi \in L \tag{9}$$

It is well known that in **two** space dimensions the pure Dirichlet problem (8), (9), with $\Gamma_{out} = \emptyset$, possesses a unique solution on any time interval $[0,T]$, which is also a classical solution if the data of the problem are smooth enough. For small viscosity, i.e., large Reynolds number, this solution may be unstable.

In **three** dimensions, the existence of a unique solution is known only for sufficiently small data, e.g., $\|\nabla v^0\| \approx \nu$, or on sufficiently short intervals of time, $0 \le t \le T$, with $T \approx \nu$. The proof or disproof of the general result would be worth a million Dollar.

1.2 Regularity of solutions

Formally, even for large Reynolds number, the Navier-Stokes equations are of elliptic type in the stationary and of parabolic type in the nonstationary case. This means that *qualitatively* the solution of the problem should be as regular as the data and the boundary of the flow domain permit. However, numerical approximation depends on *quantitative* regularity properties of the solution. For example, the presence of a strong boundary layer or part of the boundary with large curvature may strongly effect the accuracy on realistic meshes, though the solution is *qualitatively* smooth. The regularity estimates for the solutions of the Navier-Stokes equations provided in the mathematical literature are mostly of such a *qualitative* character and have only limited value in predicting the performance of a concrete numerical computation. Therefore, we do not list these results but rather concentrate on some aspects of regularity which are peculiar to the *incompressible* Navier-Stokes equations.

1.2.1 Domain with irregular boundary

For Reynolds number of small to moderate size, the solution of the Navier-Stokes equations on (nonconvex) polygonal or polyhedral domains exhibits “corner singularities”. Using polar coordinates (r, θ) around the corner in 2d, the solution can be written in the form

$$v(r,\theta) = Ar^{\alpha}\varphi(\theta) + \tilde{v}(r,\theta), \qquad p(r,\theta) = Br^{\alpha-1}\psi(\theta) + \tilde{p}(r,\theta)$$

with a certain $\alpha > 0$, some analytic functions φ, ψ, and more “regular” components $\tilde{v}$ and $\tilde{p}$. The exponent α characterizes the regularity of v and p at the corner, i.e., for $r \to 0$, and is mainly determined by the linear “Stokes part” of the equation.

Corner singularities cause a significant reduction of accuracy in the numerical approximation especially near the corner point but also in the interior of the computational domain where the solution itself is smooth. This so-called “pollution effect” can be suppressed, for instance, by adaptive local mesh refinement.

The structure of such corner singularities has been thoroughly analyzed in the literature. Although, the Stokes problem contains the Laplacian Δ in the momentum equation, the incompressibility constraint makes it behave quite differently. In fact, the common stream-function formulation of the 2d Stokes problem is a scalar fourth-order equation involving the biharmonic operator Δ^2. This equivalent formulation can be used for deriving regularity results and explicit representations of corner singularities for the 2d Navier-Stokes equations on arbitrary polygonal domains and general boundary conditions. For example, pipe flow over a backward-facing step, with inner angle $\omega = \frac{3}{2}\pi$, exhibits a corner singularity of the form

$$v(r,\theta) \approx Ar^{0.54\ldots}\varphi(\theta), \qquad p(r,\theta) \approx Br^{-0.45\ldots}\psi(\theta)$$

i.e., vorticity $\nabla \times v$ as well as pressure p become singular at the tip of the step.

Such singularities not only occur at corners, they can also be caused by changes in the boundary conditions. For example, the sudden change from Dirichlet to Neumann boundary conditions results in a singular behavior of the form $|p(r,\theta)| \approx Br^{-0.5}$. The extreme case is reached for the 'lid-driven cavity' where in the upper corners the boundary data is discontinuous resulting in a singularity $|p(r,\theta)| \approx Br^{-1}$, such that the solution does not belong to $(v^{\text{in}} + \mathbf{H}) \times L$.

The analysis of "corner and edge singularities" for 3d domains is rather difficult and not completely settled yet. The only general rules are that *corner* singularities are weaker than *edge* singularities, and 3d edge singularities look very much like the corresponding corner singularities in the 2d cross-sections.

1.2.2 Solution behavior as $t \to 0$

For the purposes of numerical analysis one needs regularity of the solution uniformly down to $t = 0$, which turns out to be a delicate requirement. To illustrate this, let us assume that the solution $\{v, p\}$ is uniformly smooth as $t \to 0$. Then, applying the divergence operator to the Navier-Stokes equations and letting $t \to 0$ implies:

$$\nabla \cdot (\partial_t v + v \cdot \nabla v) = \nabla \cdot (\nu \Delta v - \nabla p) \;\rightarrow\; \nabla \cdot (v^0 \cdot \nabla v^0) = -\Delta p^0, \quad \text{in } \Omega$$

$$\partial_t v + v \cdot \nabla v = \nu \Delta v - \nabla p \;\rightarrow\; \partial_t g_{|t=0} + v^0 \cdot \nabla v^0 = \nu \Delta v^0 - \nabla p^0, \quad \text{on } \partial\Omega$$

where g is the boundary data, v^0 the initial velocity and $p^0 := \lim_{t \to 0} p(t)$ the “initial pressure”. Hence, in the limit $t = 0$, we obtain an overdetermined Neumann problem for the initial pressure including the compatibility condition

$$\partial_\tau p^0{}_{|\partial\Omega} = \tau \cdot (\nu \Delta v^0 - \partial_t g_{|t=0} - v^0 \cdot \nabla v^0) \tag{10}$$

where τ is the tangent direction along $\partial\Omega$.

If this compatibility condition is violated, then the solution cannot remain smooth as $t \to 0$, particularly,

$$\|\partial_t^2 v(t)\| + \|\nabla^3 v(t)\| + \|\nabla^2 p(t)\| \to \infty \quad (t \to 0).$$

Since p^0 is not known, (10) is a *global* condition which generally cannot be verified for given data. An example for such a situation is flow between constantly accelerated concentric spheres ("Taylor problem").

In view of this generally unavoidable regularity defect of the solution of the Navier-Stokes equations at $t = 0$, the use of higher-order discretization appears questionable unless they are shown to possess the 'smoothing property', i.e. to exhibit their full order of convergence at positive time $t > 0$. This reflects the well-known "smoothing behavior" of the continuous solution $\{v, p\}$ as $t \to 0$. This smoothing property has been established for various discretization schemes for the Navier-Stokes equations.

1.2.3 Solution behavior as $t \to \infty$

The constants in the a priori error estimates available in the literature for space and time discretization of the Navier-Stokes equation generally depend exponentially on the Reynolds number $Re \approx \nu^{-1}$ and on time T. In general, this exponential dependence is unavoidable, due to the use of Gronwall's inequality in the derivation of these estimates, and may also be natural in the case of unstable solutions. Since even in the range of laminar flows $e^{20} \approx 10^8$, $e^{100} \approx 10^{43}$, the practical meaning of a priori error estimates containing such constants is questionable. If the solution to be computed were known to possess certain stability properties, for instance to be "exponentially stable", then the exponential dependence on T can be suppressed, so justifying long-time flow simulation. However, such stability assumptions are usually not verifiable by *a priori* analysis but may be justified in many situations by experimental evidence. Consequently, reliable flow simulation requires computable *a posteriori* error bounds in terms of the approximate solution.

1.3 Artificial boundary conditions: truncation to bounded domains

The variational formulation (8), (9) does not contain an explicit reference to any "outflow boundary condition". Suppose that the solution $v \in v^{in} + \mathbf{H}$, $p \in L$ is sufficiently smooth. Then, integration by parts and varying $\varphi \in \mathbf{H}$ in the terms

$$\nu(\nabla v, \nabla\varphi) - (p, \nabla \cdot \varphi) = (\nu\partial_n v - pn, \varphi)_{\Gamma_{\text{out}}} + (-\nu\Delta v + \nabla p, \varphi)$$

yields the already mentioned "natural" outflow boundary condition

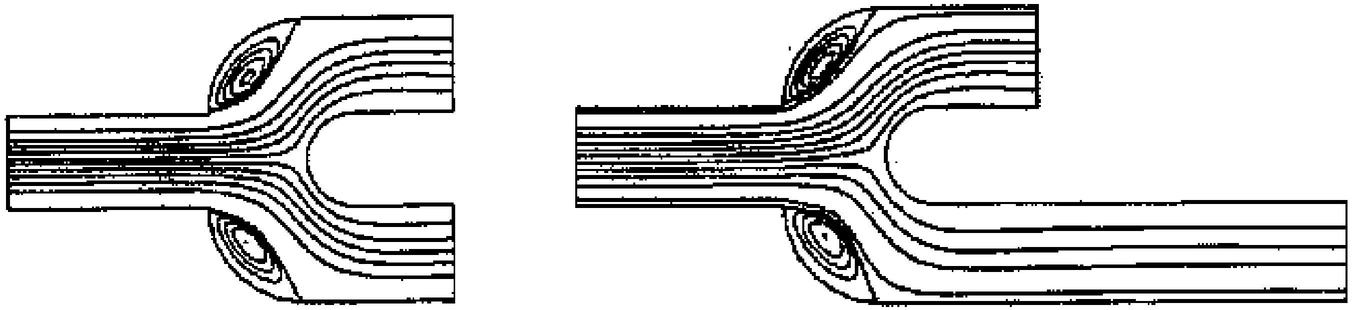
$$\nu\partial_n v - pn = 0 \quad \text{on} \;\; \Gamma_{\text{out}} \tag{11}$$

This condition has proven to be well suited in modeling almost parallel flows. It occurs in the variational formulation of the problem if one does not prescribe any boundary condition for the velocity at the outlet suggesting the name "do nothing" or "free" boundary condition.

However, this approach has to be used with care. For example, in 2d if the flow region contains more than one outlet, undesirable effects may occur, since the "do nothing" condition contains an additional hidden condition on the pressure. In fact, integrating (11) over any component S of the outflow boundary and using the incompressibility constraint $\nabla \cdot v = 0$ and $v_{|\Gamma_{\mathrm{rigid}}} = 0$ yields

$$\int_S pn \,\mathrm{do} = \nu \int_{\mathrm{S}} \partial_{\mathrm{n}} \mathrm{v} \,\mathrm{do} = -\nu \int_{\mathrm{S}} \partial_\tau \mathrm{v} \,\mathrm{do} = 0 \tag{12}$$

i.e., the mean pressure over S must be zero. To illustrate this, let us consider low Reynolds number flow through a junction in a system of pipes, again prescribing a Poiseuille inflow upstream.

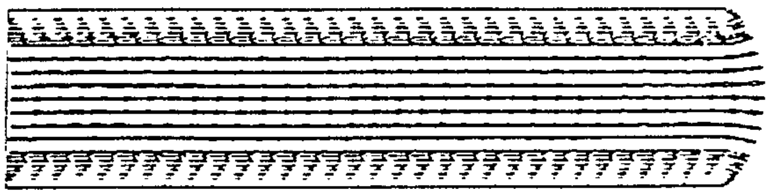


Obviously, making one leg of the pipe longer significantly changes the flow pattern. By the property (12) the pressure gradient is greater in the shorter of the two outflow sections, which explains why there is a greater flow through.

In the momentum equation (8) the Dirichlet form $(\nabla v, \nabla \varphi)$ may be replaced by the deformation form $(D[v], D[\varphi])$, with $D[v] = \frac{1}{2}(\nabla v + \nabla v^T)$. This change has no effect in the case of pure Dirichlet boundary conditions as then the two forms coincide. But in using the "do nothing" approach this modification leads to the outflow boundary condition

$$n \cdot D[v] - pn = 0 \quad \text{on } \Gamma_{out} \tag{13}$$

which results in a non-physical behavior of the flow with outward bending streamlines for simple Poiseuille flow.



Problem: Despite its remarkable success of the "do nothing" outflow boundary condition in modeling (almost) parallel flow, there is a theoretical problem with existence and uniqueness.

a) Existence of solutions (in 2d as well as in 3d) can be shown even in the stationary case only for very small Reynolds numbers. The problem is the lacking a priori bound for the Galerkin approximations in the Dirchlet norm due to the indefiniteness of the nonlinear term:

$$(v\cdot\nabla v, v) = \tfrac{1}{2}(v, \nabla v^2) = -\tfrac{1}{2}(v\cdot n, v^2)_{\Gamma_{\text{out}}} \geq 0\ ?$$

b) This dilemma can also be described in terms of uniqueness: Is $v \equiv 0$ the only solution to the homogeneous problem

$$-\nu\Delta v + v\cdot\nabla v + \nabla p = 0, \quad \nabla\cdot v = 0, \quad \text{in } \Omega,$$

$$v_{|\Gamma_{\text{rigid}}\cup\Gamma_{\text{in}}} = 0, \quad (\nu\partial_n v + pn)_{|\Gamma_{\text{out}}} = 0.$$

Extensive numerical tests did not lead to an affirmative answer to this question. Therefore, the use of the "do nothing" outflow boundary condition in practice may inherit some risk of ill-posedness.

1.4 Discretization in space

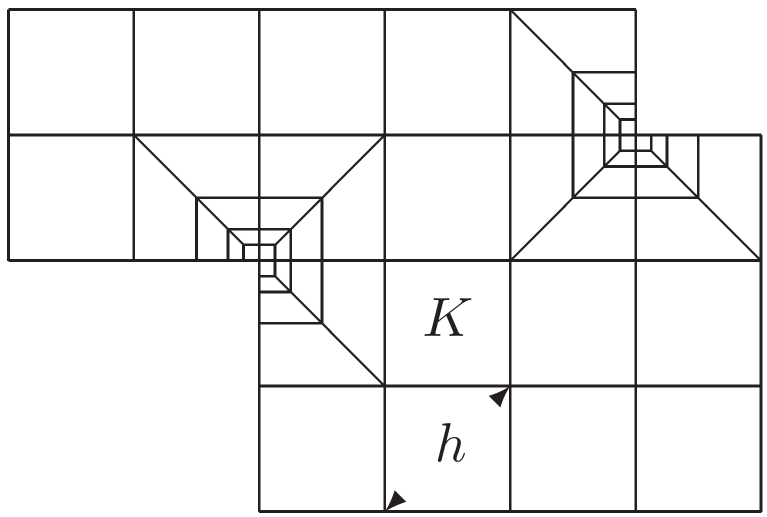


Figure 1: Admissible mesh with local refinement at corner points.

In setting up a finite element model, one starts from the variational formulation of the problem: *Find* $v \in v^{in} + \mathbf{H}$ *and* $p \in L$, *such that*

$$a(v,\varphi) + n(v,v,\varphi) + b(p,\varphi) = (f,\varphi) \quad \forall\, \varphi \in \mathbf{H} \tag{14}$$

$$b(\chi, v) = 0 \quad \forall\, \chi \in L \tag{15}$$

The choice of the function spaces $\mathbf{H} \subset \mathbf{H}^1(\Omega))$ and $L \subset L^2(\Omega)$ depends on the specific boundary conditions imposed in the problem to be solved. On a finite element mesh $\mathbb{T}_h$ on Ω, one defines spaces of "discrete" trial and test functions, $\mathbf{H}_h \subset' \mathbf{H}$ and $L_h \subset L$. The discrete analogues of (14), (15) then read as follows: *Find* $v_h \in v_h^{in} + \mathbf{H}_h$ *and* $p_h \in L_h$, *such that*

$$a_h(v_h, \varphi_h) + n_h(v_h, v_h, \varphi_h) + b_h(p_h, \varphi_h) = (f, \varphi_h) \quad \forall\, \varphi_h \in \mathbf{H}_h \qquad (16)$$

$$b_h(\chi_h, v_h) = 0 \quad \forall\, \chi_h \in L_h \qquad (17)$$

where v_h^{in} is a suitable approximation of the inflow data v^{in}, defined on all of Ω. The notation $\mathbf{H}_h \subset' \mathbf{H}$ indicates that in this discretization the spaces $\mathbf{H}_h$ may be "nonconforming", i.e., the discrete velocities v_h are continuous across the interelement boundaries and zero along the rigid boundaries only in an approximate sense; in this case the discrete forms $a_h(\cdot,\cdot)$, $b_h(\cdot,\cdot)$, $n_h(\cdot,\cdot,\cdot)$ and the discrete "energy norm" $\|\nabla \cdot\|_h$ are defined in the piecewise sense,

$$a_h(\varphi,\psi) := \sum_{K\in\mathbb{T}_h} \nu(\nabla\varphi,\nabla\psi)_K\,, \quad b_h(\chi,\varphi) := -\sum_{K\in\mathbb{T}_h} (\chi,\nabla\cdot\varphi)_K$$

$$n_h(\varphi,\psi,\xi) := \sum_{K\in\mathbb{T}_h} (\varphi\cdot\nabla\psi,\xi)_K \quad \|\nabla\varphi\|_h := \Big(\sum_{K\in\mathbb{T}_h} \|\nabla\varphi\|_K^2\Big)^{1/2}$$

In order that (16), (17) is a stable approximation to (14), (15), as $h\to 0$, it is crucial that the spaces $\mathbf{H}_h\times L_h$ satisfy a compatibility condition, the so-called "inf–sup" or "Babuska-Brezzi" condition, which is the discrete analogue of the continuous inf-sup stability estimate (7):

$$\inf_{q_h\in L_h}\ \sup_{\varphi_h\in\mathbf{H}_h} \frac{b_h(q_h,\varphi_h)}{\|q_h\|\,\|\nabla\varphi_h\|_h} \geq \gamma > 0 \qquad (18)$$

Here, the constant γ is required to be independent of h. This ensures that the problems (16), (17) possess solutions which are uniquely determined in $\mathbf{H}_h\times L_h$, stable and of optimal order convergent.

1.4.1 Examples of Stokes elements

(0) The Q_1/P_0 Stokes element

For completeness, we mention the simplest (quadrilateral) Stokes element, the so-called Q_1/P_0 element, which uses continuous bilinear velocity and piecewise constant pressure approximation; the triangular counterpart of this element is not consistent. This element pair is formally second-order accurate but lacks stability since the uniform inf-sup condition (18) is not satisfied on general meshes. Yet, this approximation has successfully been used in nonstationary flow computations. The explanation for the good performance of this "unstable" spatial discretization within a nonstationary computation is that the time-stepping, a pressure correction scheme, introduces sufficient stabilization as long as the time step is not too small.

(1) The nonconforming "rotated" d-linear $\tilde{Q}_1/P_0$ Stokes element

This is the natural quadrilateral analogue of the well-known triangular nonconforming finite element of Crouzeix/Raviart. Its two- as well as three-dimensional versions have been implemented in state-of-the-art Navier-Stokes codes. In two space dimensions, this nonconforming element uses piecewise "rotated" bi-linear reference shape functions for the velocities, spanned by $\{1, x, y, x^2 - y^2\}$, and piecewise constant pressures.

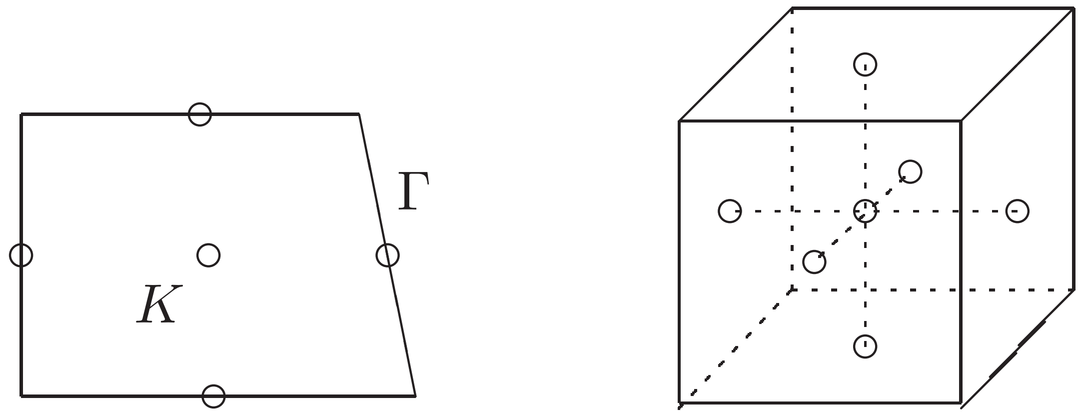


Figure 2: Nonconforming 'rotated d-linear' Stokes element.

(2) The d-linear Q_1/Q_1 Stokes element with pressure stabilization

This Stokes element uses continuous isoparametric d-linear shape functions for both the velocity and the pressure approximations. The nodal values are just the function values of the velocity and the pressure at the vertices of the cells, making this approximation particularly attractive in three dimensions.

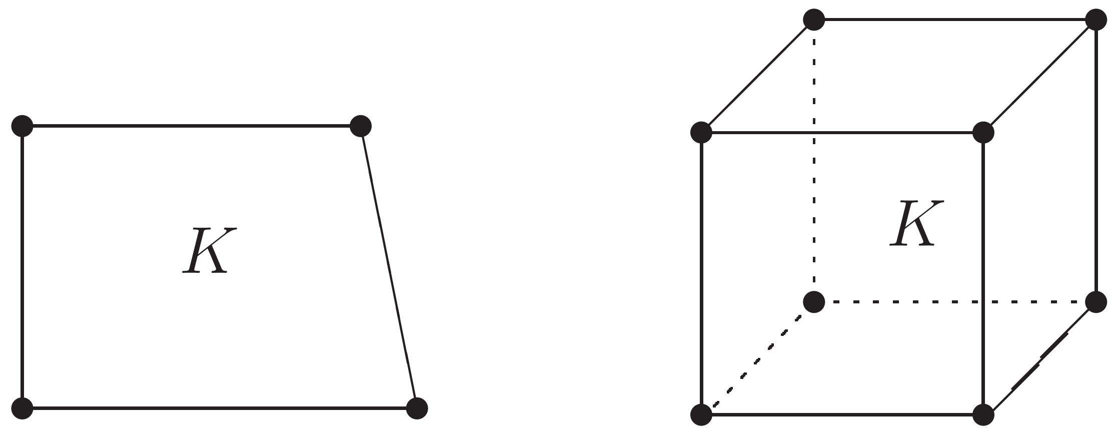


Figure 3: Conforming d-linear Stokes element, with nodal values $v_h(a), p_h(a)$.

This combination of spaces, however, would be unstable, i.e., it would violate the condition (18), if used together with the variational formulation (16), (17). We add certain least squares terms in the continuity equation (17) (so-called "pressure stabilization method"),

$$b(\chi_h, v_h) + c_h(\chi_h, p_h) = g_h(v_h; \chi_h) \tag{19}$$

where

$$c_h(\chi_h, p_h) = \frac{\alpha}{\nu} \sum_{K \in \mathbb{T}_h} h_K^2 (\nabla \chi_h, \nabla p_h)_K$$

$$g_h(v_h; \chi_h) = \frac{\alpha}{\nu} \sum_{K \in \mathbb{T}_h} h_K^2 (\nabla \chi_h, f + \nu \Delta v_h - v_h \cdot \nabla v_h)_K$$

The correction terms on the right hand side have the effect that this modification is fully consistent, since the additional terms cancel out if the exact solution $\{v, p\}$ of problem (14), (15) is inserted. On regular meshes, one obtains a stable and consistent approximation of the Navier-Stokes problem (14), (15) with optimal-order accuracy.

However, from a practical point of view, this stabilization has several short-comings. First, being used together with the "free" outflow condition described above it induces a non-physical numerical boundary layer along the outflow boundary. Second, the evaluation of the various terms in the stabilization forms $c_h(\chi_h, p_h)$ and $g_h(v_h; \chi_h)$ is very expensive, particularly in 3d. These problems can be resolved by using instead so-called "stabilization by pressure projection". Here, the stabilization forms are $g_h = 0$ and

$$c_h(\chi_h, p_h) := (\nabla(\chi_h - \pi_{2h}\chi_h), \nabla(p_h - \pi_{2h}p_h)) \tag{20}$$

where π_{2h} denotes the projection into a space L_{2h} defined on a coarser mesh $\mathbb{T}_{2h}$. The resulting scheme is of second-order accurate, the evaluation of the system matrices is cheap and the consistency defect at the outflow boundary is avoided. Further, the resulting discretization has conceptional advantages in solving flow control problems by the indirect approach (based on the KKT system).

(3) Higher-order Stokes elements

One of the main advantages of the Galerkin finite element method is that it provides systematic ways of dealing with complex geometries and of constructing higher-order approximation. Popular examples are:

- The triangular $P_2^{\#}/P_1$ element with continuous quadratic velocity (augmented by a cubic bulb) and discontinuous linear pressures.
- The quadrilateral Q_2/Q_2 element with continuous biquadratic velocity and pressure approximation and pressure stabilization.
- The triangular P_2/P_1 or the quadrilateral Q_2/Q_1 Taylor-Hood element with continuous quadratic or biquadratic velocity and linear or bilinear pressure approximation.

All these Stokes elements are inf-sup stable and third-order accurate for the velocity and second-order for the pressure, the errors measured in the repective L^2 norms. Practical experience shows that they yield much better approximations than the lower-order elements.

1.4.2 Treating dominant transport

In the case of higher Reynolds number (e.g., $\mathrm{Re} > 500$ for the 2d driven cavity, and $\mathrm{Re} > 50$ for pipe-flow around a cylinder) the finite element models (16), (17) and (16), (19) may become unstable since they essentially use central-differences-like discretization of the advection term. This instability most frequently occurs in form of a drastic slow-down or even break-down of the iteration processes for solving the algebraic problems; in the extreme case the possibly existing "mathematical" solution contains nonphysical oscillatory components. In order to avoid these effects some additional numerical damping is required. The use of simple first-order artificial viscosity or upwinding is not advisable since it introduces too much numerical diffusion.

The Galerkin structure of the FEM is fully respected by the so-called "streamline diffusion method". The idea of streamline diffusion is to introduce artificial diffusion acting only in the transport direction while maintaining the second-order consistency of the scheme.

This can be achieved either by augmenting the test space by direction-oriented terms resulting in a “Petrov-Galerkin method” with streamline-oriented upwinding (“SUPG method”), or by adding certain least-squares terms to the discretization including streamline-oriented diffusion (“LSSD method”).

In the following, we describe the full LSSD method for the Navier-Stokes system. To this end, we introduce the product Hilbert-spaces $\mathbf{V} := \mathbf{H} \times L$ of pairs $u := \{v, p\}$ and $\varphi = \{\psi, \chi\}$ as well as their discrete analogues $\mathbf{V}_h := \mathbf{H}_h \times L_h$ of pairs $u_h := \{v_h, p_h\}$ and $\varphi_h = \{\psi_h, \chi_h\}$. On these spaces, we define the semi-linear form

$$A(u;\varphi) := a_h(v,\psi) + n_h(v,v,\psi) + b_h(p,\psi) - b(\chi,v)$$

and the linear functional $F(\varphi) := (f,\psi)$. Then, the variational formulation (8) of the Navier-Stokes equations is written in the following compact form: *Find* $u \in \mathbf{V} + (v_h^{in}, 0)^T$, *such that*

$$A(u;\varphi) = F(\varphi) \quad \forall \varphi \in \mathbf{V} \tag{21}$$

For defining the stabilization, we introduce the matrix differential operator $A(u)$, depending on $u = \{v, p\}$, and the forcing vector F by

$$A(u) := \begin{bmatrix} -\nu\Delta + v\cdot\nabla & \mathrm{grad} \\ \mathrm{div} & 0 \end{bmatrix}, \quad F := \begin{bmatrix} f \\ 0 \end{bmatrix}$$

Then, with the weighted L^2-bilinear form

$$(v, w)_h := \sum_{K\in\mathbb{T}_h} \delta_K (v, w)_K$$

the LSSD stabilized finite element approximation reads: *Find* $u_h \in \mathbf{V}_h + (v^{in}, 0)^T$ *such that*

$$A(u_h; \varphi_h) + (A(u_h)u_h, A(u_h)\varphi_h)_h = (F, \varphi_h) + (F, A(u_h)\varphi_h)_h \quad \forall \varphi_h \in \mathbf{V}_h$$

This discretization contains several features. The stabilization form contains the terms

$$\sum_{K\in\mathbb{T}_h} \delta_K(\nabla p_h, \nabla\chi_h)_K, \quad \sum_{K\in\mathbb{T}_h} \delta_K(v_h\cdot\nabla v_h, v_h\cdot\nabla\psi_h)_K, \quad \sum_{K\in\mathbb{T}_h} \delta_K(\nabla\cdot v_h, \nabla\cdot\psi_h)_K$$

where the first one stabilizes the pressure-velocity coupling for the conforming Q_1/Q_1 Stokes element, the second one stabilizes the transport operator, and the third one enhances mass conservation. The other terms introduced in the stabilization are correction terms which guarantee second-order consistency for the stabilized scheme. Practical experience and analysis suggest the following choice of the stabilization parameters:

$$\delta_K = \alpha\Big(\frac{\mu}{h_K^2} + \frac{\beta|v_h|_{K;\infty}}{h_K}\Big)^{-1} \tag{22}$$

with values $\alpha \approx \frac{1}{12}$ and $\beta \approx \frac{1}{6}$. In practice, the terms $-\nu\Delta v_h$ as well as $-\nu\Delta\psi_h$ in the stabilization are usually dropped, since they vanish or almost vanish on the low-order elements considered.

1.4.3 The algebraic problems

The discrete Navier-Stokes problem including simultaneously pressure and streamline diffusion stabilization has to be converted into an algebraic system. To this end, we choose local "nodal bases" $\{\psi_h^i,\ i=1,\dots,N_v\}$ of the "velocity space" $\mathbf{H}_h$, and $\{\chi_h^i,\ i=1,\dots,N_p\}$ of the "pressure space" L_h, and expand the unknown solution $\{v_h,p_h\}$ in the form $v_h - v_h^{in} = \sum_{j=1}^{N_v} x_j\psi_h^j$ and $p_h = \sum_{j=1}^{N_p} y_j\chi_h^j$. Accordingly, we introduce

$$A = \big(a_h(\psi_h^j,\psi_h^i)\big)_{i,j=1}^{N_v}, \quad B = \big(b_h(\chi_h^j,\psi_h^i)\big)_{i,j=1}^{N_v,N_p}$$

$$N(x) = \big(n_h(v_h,\psi_h^j,\psi_h^i) + n_h(\psi_h^j,v_h^{in},\psi_h^i)\big)_{i,j=1}^{N_v}$$

$$S(x) = \big((-\nu\Delta\psi_h^j + v_h\cdot\nabla\psi_h^j, -\nu\Delta\psi_h^i + v_h\cdot\nabla\psi_h^i)_h + (\nabla\cdot\psi_h^j,\nabla\cdot\psi_h^i)_h\big)_{i,j=1}^{N_v}$$

$$T(x) = \big((\nabla\chi_h^j, -\nu\Delta\psi_h^i + v_h\cdot\nabla\psi_h^i)_h\big)_{i,j=1}^{N_v,N_p}, \quad C = \big((\nabla\chi_h^j,\nabla\chi_h^i)_h\big)_{i,j=1}^{N_p}$$

$$b = \big((f,\psi_h^i) - a(v_h^{in},\psi_h^i) - n_h(v_h^{in},v_h^{in},\psi_h^i) + (f,v_h\cdot\nabla\psi_h^i)_h\big)_{i=1}^{N_v},$$

$$c = \big((f,\nabla\chi_h^i)_h\big)_{i=1}^{N_p}$$

Notice that the nonhomogeneous inflow data $v_{h|\Gamma_{\rm in}} = v_h^{\rm in}$ is implicitly incorporated into the system, and that the stabilization only acts on velocity basis functions corresponding to interior nodes. Further, we introduce the velocity and pressure 'mass matrices':

$$M_v = \big((\psi_h^i, \psi_h^j)\big)_{i,j=1}^{N_v}, \quad M_p = \big((\chi_h^i, \chi_h^j)\big)_{i,j=1}^{N_p}$$

With this notation the variational problem (**??**), can equivalently be written in form of an algebraic system for the vectors $x \in \mathbb{R}^{N_v}$ and $y \in \mathbb{R}^{N_p}$ of expansion coefficients:

$$\begin{bmatrix} A + N(x) + S(x) & B + T(x) \\ -B^T + T(x)^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \tag{23}$$

Notice that this system has essentially the features of a saddle-point problem (since C small) and is generically nonsymmetric. This poses a series of problems for its iterative solution.

1.5 Discretization in time

We now consider the *nonstationary* Navier-Stokes problem: *Find* $v \in v^{in} + \mathbf{H}$ *and* $p \in L$, *such that* $v(0) = v^0$ *and, for* $t > 0$,

$$(\partial_t v, \varphi) + a(v, \varphi) + n(v, v, \varphi) + b(p, \varphi) = (f, \varphi)\,, \quad \forall\, \varphi \in \mathbf{H} \tag{24}$$

$$b(\chi, v) = 0\,, \quad \forall\, \chi \in L \tag{25}$$

where it is implicitly assumed that v as a function of time is continuous on $[0, \infty)$ and continuously differentiable on $(0, \infty)$. The choice of the function spaces $\mathbf{H} \subset \mathbf{H}^1(\Omega)^d$ and $L \subset L^2(\Omega)$ depends again on the specific boundary conditions chosen for the problem to be solved. Due to the incompressibility constraint the nonstationary Navier-Stokes system has the character of a “differential-algebraic equation” (in short “DAE”) of “index” two, in the language of ODE theory. This enforces an implicit treatment of the pressure within the time-stepping process, while the other flow quantities may, in principle, be treated explicitly.

1.5.1 The Rothe Method

In the "Rothe Method", at first, the time variable is discrete by one of the common time differencing schemes. For example, the backward Euler scheme leads to a sequence of stationary Navier-Stokes-like problems

$$k_n^{-1}(v^n - v^{n-1}, \varphi) + a(v^n, \varphi) + n(v^n, v^n, \varphi) + b(p^n, \varphi) = (f^n, \varphi) \tag{26}$$

$$b(\chi, v^n) = 0 \tag{27}$$

for all $\{\varphi, \chi\} \in \mathbf{H} \times L$, where $k_n = t_n - t_{n-1}$ is the time step. Each of these problems is then solved by some spatial discretization method as described in the preceding section. This provides the flexibility to vary the spatial discretization, i.e. the mesh or the type of trial functions in the finite element method, during the time stepping process. In the classical Rothe method the time discretization scheme is kept fixed and only the size of the time step may change. The transfer between different spatial meshes is accomplished by some interpolation or projection strategy.

1.5.2 The Method of Lines

The more traditional approach to solving time-dependent problems is the "Method of Lines". At first, the spatial variable is discrete, e.g. by a finite element method as described above, leading to a DAE system of the form

$$\begin{bmatrix} M \\ 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} A + N(x(t)) & B \\ -B^T & C \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} b(t) \\ c(t) \end{bmatrix} \tag{28}$$

for $t \geq 0$, with the initial value $x(0) = x^0$. This DAE system is now discretized with respect to time. Let k denote the time-step size. Frequently used schemes are the simple "one-step-θ schemes":

One-step θ-scheme: Step $t_{n-1} \to t_n$:

$$\begin{bmatrix} M + \theta k A^n & \theta k B \\ -B^T & C \end{bmatrix} \begin{bmatrix} x^n \\ y^n \end{bmatrix} = \begin{bmatrix} [M - (1-\theta) k A^{n-1}] x^{n-1} + \theta k b^n + (1-\theta) k b^{n-1} \\ c^n \end{bmatrix}$$

where $x^n \approx x(t_n)$ and $A^n := A(x^n)$.

Special cases are the "forward Euler scheme" for $\theta = 0$ (first-order explicit), the most popular "Crank-Nicolson scheme" for $\theta = 1/2$ (second-order implicit, A-stable), and the the "backward Euler scheme" for $\theta = 1$ (first-order implicit, strongly A-stable). The most robust implicit Euler scheme is very dissipative and therefore not suitable for the time-accurate computation of nonstationary flow. In contrast to that, the Crank-Nicolson scheme has only very little dissipation but occasionally suffers from instabilities caused by the possible occurrence of rough perturbations in the data which are not damped out due to the only weak stability properties of this scheme (not *strongly* A-stable). This defect can in principle be cured by an adaptive step size selection but this may enforce the use of an unreasonably small time step.

Alternative schemes of higher order are based on the (diagonally) implicit Runge-Kutta formulas or the backward differencing multi-step formulas, both being well known from the ODE literature. These schemes, however, have not yet found wide applications in practical flow computations.

Another alternative is the so-called "fractional-step-θ scheme":

Fractional-step-θ-scheme $(t_{n-1} \to t_{n-1+\theta} \to t_{n-\theta} \to t_n)$:

$$\begin{bmatrix} M + \alpha\theta k A^{n-1+\theta} & \theta k B \\ -B^T & C \end{bmatrix} \begin{bmatrix} x^{n-1+\theta} \\ y^{n-1+\theta} \end{bmatrix} = \begin{bmatrix} [M - \beta\theta k A^{n-1}]x^{n-1} + \theta k b^{n-1} \\ c^{n-1+\theta} \end{bmatrix}$$

$$\begin{bmatrix} M + \beta\theta' k A^{n-\theta} & \theta' k B y^{n-\theta} \\ -B^T & C \end{bmatrix} \begin{bmatrix} x^{n-\theta} \\ y^{n-\theta} \end{bmatrix} = \begin{bmatrix} [M - \alpha\theta' k A^{n-1+\theta}]x^{n-1+\theta} + \theta' k b^{n-\theta} \\ c^{n-\theta} \end{bmatrix}$$

$$\begin{bmatrix} M + \alpha\theta k A^{n} & \theta k B \\ -B^T & C \end{bmatrix} \begin{bmatrix} x^{n} \\ y^{n} \end{bmatrix} = \begin{bmatrix} [M - \beta\theta k A^{n-\theta}]x^{n-\theta} + \theta k b_h^{n-\theta} \\ c^{n} \end{bmatrix}$$

Here $\theta = 1 - \sqrt{2}/2 = 0.292893...$, $\theta' = 1 - 2\theta$, $\alpha \in (1/2, 1]$, and $\beta = 1 - \alpha$, in order to ensure second-order accuracy, and strong A-stability. Being *strongly* A-stable, for any choice of $\alpha \in (1/2, 1]$, it possesses the full smoothing property in the case of rough initial data, in contrast to the Crank–Nicolson scheme (case $\alpha = 1/2$).

1.5.3 Splitting schemes

The fractional-step-θ scheme was originally introduced as an operator splitting in order to separate the two main difficulties in solving problem (24) namely the nonlinearity causing nonsymmetry and the incompressibility constraint causing indefiniteness. Using the notation from above, but suppressing here the terms stemming from pressure stabilization, the operator splitting scheme reads as follows:

Splitting-fractional-step-θ scheme $(t_{n-1} \to t_{n-1+\theta} \to t_{n-\theta} \to t_n)$:

$$\begin{bmatrix} M+\alpha\theta kA & \theta kB \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x^{n-1+\theta} \\ y^{n-1+\theta} \end{bmatrix} = \begin{bmatrix} [M-\beta\theta kA]x^{n-1} + \theta kb^{n-1} - \theta kN^{n-1}x^{n-1} \\ 0 \end{bmatrix}$$

$$[M+\beta\theta' kA^{n-\theta}]x^{n-\theta} = [M-\alpha\theta' kA^{n-1+\theta}]x^{n-1+\theta} - \theta' kBy^{n-\theta} + \theta' kb^{n-\theta}$$

$$\begin{bmatrix} M+\alpha\theta kA & \theta kB \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x^{n} \\ y^{n} \end{bmatrix} = \begin{bmatrix} [M-\beta\theta kA]x^{n-\theta} - \theta kN^{n-\theta}x^{n-\theta} + \theta kb^{n-\theta} \\ 0 \end{bmatrix}$$

The first and last step solve *linear* Stokes problems treating the nonlinearity explicitly, while in the middle step a nonlinear Burgers-type problem (without incompressibility constraint) is solved. The symmetric form of this splitting scheme guarantees second-order accuracy of the approximation.

Nowadays, the efficient solution of the *nonlinear* incompressible Navier-Stokes equations is standard by the use of new multigrid techniques. Hence, the splitting of nonlinearity and incompressibility is no longer an important issue.

1.5.4 Projection schemes

The Chorin projection scheme has originally been designed in order to overcome the problem with the incompressibility constraint $\nabla \cdot v = 0$. The continuity equation is decoupled from the momentum equation through an iterative process (again "operator splitting"). There are various schemes of this kind in the literature referred to as "projection method", "quasi-compressibility method", "SIMPLE method", etc. All these methods are based on the same principle idea. The continuity equation $\nabla \cdot \mathbf{v} = 0$ is supplemented by certain stabilizing terms involving the pressure, e.g.,

$$\nabla \cdot v + \epsilon p = 0, \tag{29}$$

$$\nabla \cdot v - \epsilon \Delta p = 0, \qquad \partial_n p_{|\partial\Omega} = 0 \tag{30}$$

$$\nabla \cdot v + \epsilon \partial_t p = 0, \qquad p_{|t=0} = 0 \tag{31}$$

$$\nabla \cdot v - \epsilon \partial_t \Delta p = 0, \quad \partial_n p_{|\partial\Omega} = 0, \ p_{|t=0} = 0 \tag{32}$$

where the small parameter ϵ is usually taken as $\epsilon \approx h^\alpha$, or $\epsilon \approx k^\beta$.

These approaches are closely related to the classical "Chorin projection method". Since this method used to be particularly attractive for computing nonstationary incompressible flow, we will discuss it in some more detail. For simplicity consider the case of pure homogeneous Dirichlet boundary conditions, $v_{|\partial\Omega} = 0$. Then, the projection method reads as follows:

For an admissible initial value v^0, solve for $n \geq 1$:

(i) Implicit 'Burgers step' for $\tilde{v}^n \in \mathbf{H}$:

$$k^{-1}(\tilde{v}^n - v^{n-1}) - \nu\Delta\tilde{v}^n + \tilde{v}^n \cdot \nabla\tilde{v}^n = f^n \tag{33}$$

(ii) "Projection step" for $v^n := \tilde{v}^n + k\nabla\tilde{p}^n$, where $\tilde{p}^n \in H^1(\Omega)$ is determined by

$$\Delta\tilde{p}^n = k^{-1}\nabla\cdot\tilde{v}^n, \quad \partial_n\tilde{p}^n_{|\partial\Omega} = 0 \tag{34}$$

This time stepping scheme can be combined with any spatial discretization method, e.g., the finite element methods described above.

Equation (34) amounts to a Poisson equation for $\tilde{p}^n$ with homogeneous Neumann boundary conditions. It is this non-physical boundary condition, $\partial_n \tilde{p}^n_{|\partial\Omega} = 0$, which has caused a lot of controversial discussion about the value of the projection method. Nevertheless, the method has proven to work well for representing the velocity field in many flow problems of physical interest. It is very economical as it requires in each time step only the solution of a (nonlinear) advection-diffusion system for v^n (of Burgers equation type) and a scalar Neumann problem for $\tilde{p}^n$.

A rigorous convergence analysis shows that the quantities $\tilde{p}^n$ are indeed reasonable approximations to the pressure $p(t_n)$. This may be seen by re-interpreting the projection method in the context of "pressure stabilization". To this end the quantity $v^{n-1} = \tilde{v}^{n-1} - k\nabla\tilde{p}^{n-1}$ is inserted into the momentum equation yielding

$$k^{-1}(\tilde{v}^n - \tilde{v}^{n-1}) - \nu\Delta\tilde{v}^n + (\tilde{v}^n\cdot\nabla)\tilde{v}^n + \nabla\tilde{p}^{n-1} = f^n, \quad \tilde{v}^n_{|\partial\Omega} = 0 \qquad (35)$$

$$\nabla\cdot\tilde{v}^n - k\Delta\tilde{p}^n = 0, \quad \partial_n\tilde{p}^n_{|\partial\Omega} = 0 \qquad (36)$$

This appears like an approximation of the Navier-Stokes equations involving a first-order (in time) “pressure stabilization” term, i.e., the projection method can be viewed as a pressure stabilization method with a global stabilization parameter $\epsilon = k$, and an explicit treatment of the pressure term. This explains the success of the not inf-sup stable Q_1/Q_1 Stokes element in the context of nonstationary computations. The pressure error is actually confined to a small boundary strip of width $\delta \approx \sqrt{\nu k}$ and decays exponentially into the interior of Ω.

The projection approach can be extended to formally higher order. The most popular example is the method of Van Kan:
For starting values v^0 *and* p^0 *compute, for* $n \geq 1$ *and some* $\alpha \geq \frac{1}{2}$*:*

(i) Second order implicit “Burgers step” for $\tilde{v}^n \in \mathbf{H}$:

$$k^{-1}(\tilde{v}^n - v^{n-1}) - \tfrac{1}{2}\nu\Delta(\tilde{v}^n + v^{n-1}) + \tilde{v}^n \cdot \nabla \tilde{v}^n + \nabla p^{n-1} = f^{n-1/2} \qquad (37)$$

(ii) Pressure Poisson problem for $q^n \in H^1(\Omega)$:

$$\Delta q^n = \alpha^{-1} k^{-1} \nabla \cdot \tilde{v}^n, \quad q^n_{|\partial\Omega} = 0 \qquad (38)$$

(iii) Pressure and velocity update:

$$v^n = \tilde{v}^n - \alpha k \nabla q^n, \quad p^n = p^{n-1} + q^n \tag{39}$$

This scheme can also be interpreted in the context of pressure stabilization methods using a stabilization of the form

$$\nabla \cdot v - \alpha k^2 \partial_t \Delta p = 0, \quad \partial_n p_{|\partial\Omega} = 0 \tag{40}$$

i.e., this method appears like a quasi-compressibility method of the form (32) with $\epsilon \approx k^2$.

1.7 Convergence analysis and error estimates

Problem: Regularity loss due to lacking initial compatibility

Best known (possible?) convergence behavior:

a) Spatial discretization

- without "smoothing": $\|(u - u_h)(t)\| \leq ch^2 \quad (\text{or } \; ch^{5/2}?)$
- with "smoothing": $\|(u - u_h)(t)\| \leq ct^{-1}h^5 \quad (\text{or } \; ct^{-5/4}h^6?)$

b) Time discretization

- without "smoothing": $\|u_h(t_n) - U_h^n\| \leq ck \quad (\text{or } \; ck^{3/2}?)$
- with "smoothing": $\|u_h(t_n) - U_h^n\| \leq ct_n^{-1}k^2 \quad (\text{or } \; ct_n^{-5/4}k^{9/4}?)$

It is known that for the nonlinear heat equation there is an order bound for the smoothing property. But what is the best achievable order?

1.8 Application to turbulent flow

(i) Adaptive multiscale discretization (LES)

Claims (Braack/Burman):

- A Variational Multi Scale Method (VMS) à la Hughes et al., based on two-scale stabilization is proposed for numerical subgrid modeling within an LES.
- Following an idea of Guermond, the local stabilization, which is not residual-based, only applies to the smallest (resolved) scale and does not affect the macro-scale.
- An a priori error analysis is given for the smooth-solution case, with constants independent of the local Peclet numers.
- Numerical results are not available yet.

(ii) Turbulence "modeling" by adaptive discretization

Claims (Johnson/Hoffman):

- DNS with DWR-controled SUPG (Streamline Upwinding Petrov Galerkin) is a valid LES approach for computing average quantities (such as drag and lift) in turbulent flow.

- Adaptive SUPG provides the right damping of energy production of the unresolved fine scales, without spoiling the average accuracy on the macro scale. The letter is controled by an error estimate involving the computed "dual solution".

$$E_{\mathrm{disc}} \approx \sqrt{h}\,\|h\nabla_h^2 z_h\|, \qquad E_{\mathrm{mod}} \approx \sqrt{h}\|\nabla z_h\|.$$

- Computational evidence for flow around a wall-mounted cube (but conflicts with statements in Hughes et al.).