Stability and Interaction of Vortices in two-dimensional viscous flows

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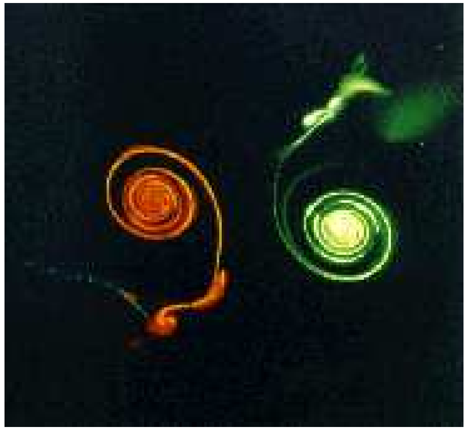


(Merging of a pair of co-rotating vortices : pictures by P. Meunier, IRPHE, Marseille)

Overview

The goal of these lectures is to present a few mathematical results which illustrate the role of vortices in the dynamics of two-dimensional incompressible viscous flows. They will be organized as follows :

1) The Cauchy problem for the 2D vorticity equation :
   General properties of the vorticity equation; nonsmooth initial data; local existence in critical spaces, obstructions to uniqueness.

2) Self-similar variables, Lyapunov functions, and long-time behavior :
   Oseen vortices, similarity variables, compactness, Liouville's theorem.

3) Asymptotic stability of Oseen vortices :
   Structure of the linearized operator at Oseen's vortex; spectral gap, pseudospectral estimates, spectral asymptotics.

4) Interaction of vortices in weakly viscous flows :
   Phenomenology of vortex interactions; the inviscid limit in presence of point vortices, the viscous N-vortex solution.

Headlines of Lecture 1

The Cauchy problem for the 2D vorticity equation

- The two-dimensional Navier-Stokes and vorticity equations
- Classical estimates for the Biot-Savart Kernel
- General properties of the 2D vorticity equation : conservation laws, Lyapunov functions, scaling invariance
- The Cauchy problem in $L^1(\mathbb{R}^2)$
- Finite measures, canonical decompositions
- The Cauchy problem in $\mathcal{M}(\mathbb{R}^2)$ (small atomic part)
- Heat kernel estimates, control of the nonlinearity
- The Cauchy problem in $\mathcal{M}(\mathbb{R}^2)$ (general case)

The Two-Dimensional Navier-Stokes Equations

We consider the incompressible Navier-Stokes equations :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + (u(x,t)\cdot\nabla)u(x,t) \;=\; \nu\Delta u(x,t) - \dfrac{1}{\rho}\nabla p(x,t)\;, \\ \operatorname{div} u(x,t) \;=\; 0\;, \end{cases} \qquad \text{(NS)}$$

where $x \in \mathbb{R}^2$ is the space variable, $t \geq 0$ is the time, and

- $u(x,t) = (u_1(x,t), u_2(x,t)) \in \mathbb{R}^2$ is the velocity field;
- $p(x,t) \in \mathbb{R}$ is the pressure field;
- $\nu > 0$ is the kinematic viscosity;
- $\rho > 0$ is the fluid density.

Eq. (NS) is an idealized model for real 3D flows, which is appropriate in some limiting situations (flows in thin domains, geophysical flows, stratified flows).

The Two-Dimensional Vorticity Equation

In our simple setting, the Navier-Stokes equation is most conveniently written in terms of the vorticity field :

$$\omega(x,t) \;=\; \partial_1 u_2(x,t) - \partial_2 u_1(x,t) \in \mathbb{R}\ ,$$

which satisfies the following advection-diffusion equation :

$$\frac{\partial \omega}{\partial t}(x,t) + u(x,t)\cdot\nabla\omega(x,t) \;=\; \nu\Delta\omega(x,t)\ . \tag{V}$$

The velocity field can be reconstructed from the vorticity by solving the elliptic system $\partial_1 u_1 + \partial_2 u_2 = 0$, $\partial_1 u_2 - \partial_2 u_1 = \omega$. Under mild assumptions†, the unique solution is given by the Biot-Savart formula

$$u(x,t) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y,t)\,\mathrm{d}y\ . \tag{BS}$$

† For instance, if $\omega/(1+|x|) \in L^1(\mathbb{R}^2)$ or $\omega \in L^p(\mathbb{R}^2)$ for some $p < 2$.

Classical Estimates for the Biot-Savart Kernel

Lemma 1 Assume that $\omega \in L^p(\mathbb{R}^2)$ for some $p \in (1,2)$. Then the velocity field u given by (BS) satisfies:

i) (Hardy-Littlewood-Sobolev bound)

$$\|u\|_{L^q(\mathbb{R}^2)} \le C\|\omega\|_{L^p(\mathbb{R}^2)}\,, \quad \text{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{2}\,.$$

ii) (Calderón-Zygmund bound)

$$\|\nabla u\|_{L^p(\mathbb{R}^2)} \le C\|\omega\|_{L^p(\mathbb{R}^2)}\,.$$

Estimate i) follows from the HLS (or weak Young) inequality, since

$$K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} \qquad \text{satisfies} \qquad K \in L^{2,\infty}(\mathbb{R}^2)\,.$$

Estimate ii) follows from CZ theory since ∇K is homogeneous of degree -2.

General Properties of the 2D Vorticity Equation

I. Conservation laws

Let $\omega(x,t)$ be a solution of the vorticity equation (V) with initial data ω_0.

- Total circulation: If $\omega_0 \in L^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} \omega(x,t)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} \omega_0\,\mathrm{d}x\;, \quad \text{for all} \quad t \ge 0\;.$$

- First order moments: If $(1+|x|)\omega_0 \in L^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} x_i\,\omega(x,t)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} x_i\,\omega_0\,\mathrm{d}x\;, \quad \text{for all} \quad t \ge 0\;, \quad i = 1,2\;.$$

- Symmetric second order moment: If $(1+|x|^2)\omega_0 \in L^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} |x|^2\omega(x,t)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} |x|^2\omega_0\,\mathrm{d}x + 4\nu t\int_{\mathbb{R}^2} \omega_0\,\mathrm{d}x\;, \quad \text{for all} \quad t \ge 0\;.$$

II. Lyapunov functions

- L^p norms : If $\omega_0 \in L^p(\mathbb{R}^2)$ for some $p \in [1,\infty]$, then

$$\|\omega(t)\|_{L^p} \le \|\omega_0\|_{L^p} , \quad \text{for all} \quad t \ge 0 .$$

- Pseudo-energy : Let

$$\mathcal{E}_d(t) = \frac{1}{4\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{d}{|x-y|}\, \omega(x,t)\omega(y,t)\,\mathrm{d}x\,\mathrm{d}y ,$$

where $d > 0$ is an arbitrary length scale. Then

a) $\mathcal{E}_d'(t) = -\nu \int_{\mathbb{R}^2} \omega(x,t)^2\,\mathrm{d}x \le 0$

b) $\mathcal{E}_d(t) = \frac{1}{2} \int_{\mathbb{R}^2} |u(x,t)|^2\,\mathrm{d}x = E(t)$ if $u(\cdot,t) \in L^2(\mathbb{R}^2)$.

Remark : If $u \in L^2(\mathbb{R}^2)$ and $\omega \in L^1(\mathbb{R}^2)$, then necessarily $\int_{\mathbb{R}^2} \omega\,\mathrm{d}x = 0$.

III. Scaling invariance

Solutions of the Navier-Stokes equations are invariant under the rescaling

$$u(x,t) \mapsto \lambda u(\lambda x, \lambda^2 t)\ , \qquad \text{or} \qquad \omega(x,t) \mapsto \lambda^2 \omega(\lambda x, \lambda^2 t)\ ,$$

for any $\lambda > 0$. Possible scale invariant or critical function spaces are:

a) $u \in C_b^0(\mathbb{R}_+, L^2(\mathbb{R}^2))$, with $\|u\| = \sup_{t\ge 0} \|u(t)\|_{L^2}$ (energy space);

b) $\omega \in C_b^0(\mathbb{R}_+, L^1(\mathbb{R}^2))$, with $\|\omega\| = \sup_{t\ge 0} \|\omega(t)\|_{L^1}$.

General principle: "For a scale invariant nonlinear PDE, critical spaces are the largest spaces, in terms of local regularity of the solutions, in which we can hope that the Cauchy problem is locally well-posed".

In the rest of this first lecture, we discuss the Cauchy problem for the 2D vorticity equation in two different critical spaces: $L^1(\mathbb{R}^2)$ and $\mathcal{M}(\mathbb{R}^2)$.

The Cauchy Problem for the Vorticity Equation (1)

Theorem 1 (Giga, Miyakawa & Osada 1988, Ben-Artzi 1994)
For all initial data $\omega_0 \in L^1(\mathbb{R}^2)$, the vorticity equation (V) has a unique global solution

$$\omega \in C^0([0,\infty), L^1(\mathbb{R}^2)) \cap C^0((0,\infty), L^\infty(\mathbb{R}^2)) .$$

Moreover $\|\omega(t)|_{L^1} \le \|\omega_0\|_{L^1}$ for all $t \ge 0$, and

- $\displaystyle\int_{\mathbb{R}^2} \omega(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^2} \omega_0(x)\,\mathrm{d}x$ for all $t \ge 0$;
- $\displaystyle\|\omega(t)\|_{L^p} \le \frac{C_p}{t^{1-1/p}}\|\omega_0\|_{L^1}$, for all $t > 0$ and all $p \in [1,\infty]$.

The proof uses classical ideas which go back to Fujita & Kato (1964). The mild solution $\omega(x,t)$ is smooth for $t > 0$ and depends continuously on the initial data ω_0, uniformly in time on compact intervals.
Theorem 1 is in fact subsumed by Theorem 3 below.

The Space of Finite Measures

Let $\mathcal{M}(\mathbb{R}^2)$ be the space of all real-valued Radon measures on $\mathbb{R}^2$, equipped with the total variation norm

$$\|\mu\|_{\mathrm{tv}} = \sup\left\{ \int_{\mathbb{R}^2} \varphi \,\mathrm{d}\mu \;\middle|\; \varphi \in C_0(\mathbb{R}^2)\,,\ \|\varphi\|_{L^\infty} \le 1 \right\}\,.$$

- $\mathcal{M}(\mathbb{R}^2)$ is a Banach space, containing $L^1(\mathbb{R}^2)$ as a closed subspace; if $\omega \in L^1(\mathbb{R}^2)$, then $\|\omega\|_{\mathrm{tv}} = \|\omega\|_{L^1}$.
- The total variation norm is scale invariant.
- $\mathcal{M}(\mathbb{R}^2) = C_0(\mathbb{R}^2)'$ is the tolopogical dual of the space of all continuous functions vanishing at infinity.
- The unit ball in $\mathcal{M}(\mathbb{R}^2)$ is compact for the weak convergence defined by:

$$\mu_n \xrightarrow[n\to\infty]{} \mu \quad \text{if} \quad \int_{\mathbb{R}^2} \varphi \,\mathrm{d}\mu_n \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^2} \varphi \,\mathrm{d}\mu \quad \text{for all} \quad \varphi \in C_0(\mathbb{R}^2)\,.$$

Decomposition of a Finite Measure

Any finite measure $\mu \in \mathcal{M}(\mathbb{R}^2)$ can be decomposed as follows :

1) Lebesgue decomposition : $\mu = \mu_{ac} + \mu_s$, where

- μ_{ac} is absolutely continuous with respect to Lebesgue's measure;
- μ_s is singular with respect to Lebesgue's measure.

Furthermore, $\mu_{ac}(E) = \int_E \omega \, \mathrm{d}x$ for some $\omega \in L^1(\mathbb{R}^2)$ (Radon-Nikodym).

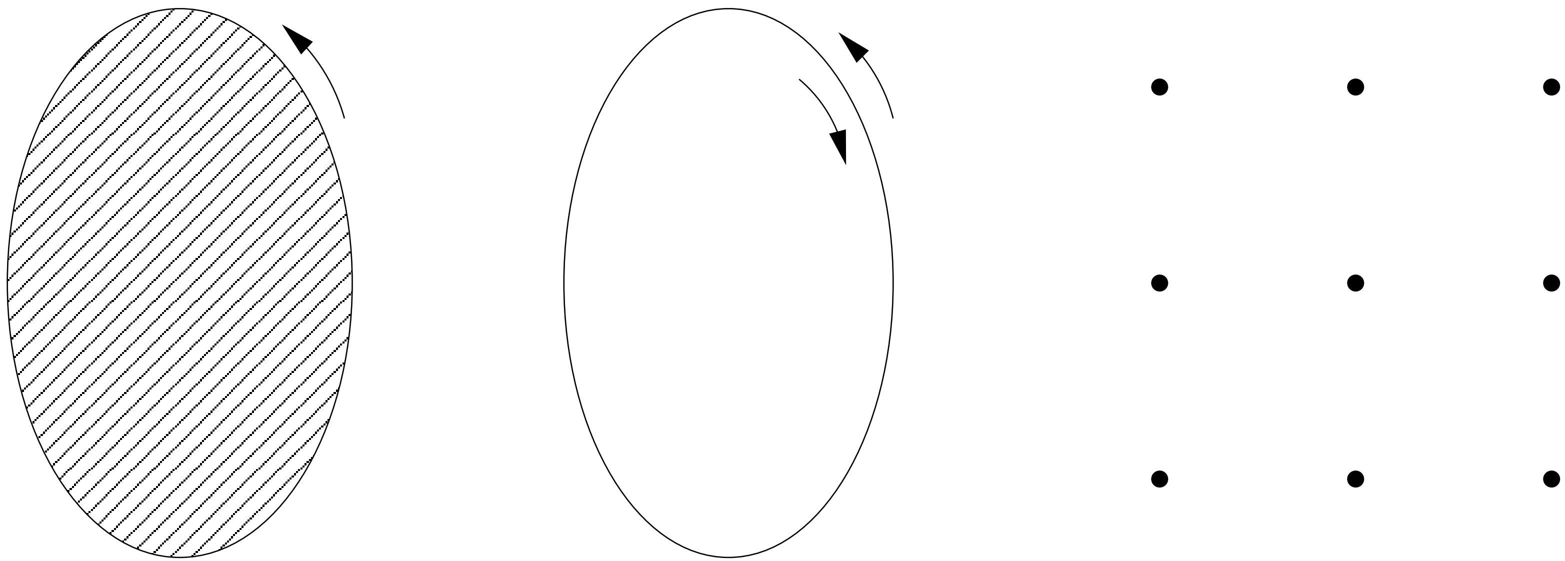
2) Atomic decomposition : $\mu_s = \mu_{sc} + \mu_{pp}$, where

$$\mu_{pp} \;=\; \mu|_{\Sigma} \;=\; \sum_{i=1}^{\infty} \alpha_i \, \delta_{x_i} \;, \quad \text{and} \quad \Sigma \;=\; \left\{ x \in \mathbb{R}^2 \,\middle|\, \mu(\{x\}) \neq 0 \right\} .$$

Finally $\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$, with $\mu_{ac} \perp \mu_{sc} \perp \mu_{pp}$. In particular,

$$\|\mu\|_{\mathrm{tv}} \;=\; \|\mu_{ac}\|_{\mathrm{tv}} + \|\mu_{sc}\|_{\mathrm{tv}} + \|\mu_{pp}\|_{\mathrm{tv}} \;=\; \|\omega\|_{L^1} + \|\mu_{sc}\|_{\mathrm{tv}} + \sum_{i=1}^{\infty} |\alpha_i| \; .$$

Typical Examples of Nonsmooth Flows



Vortex patch | Vortex sheet | Point vortices

The Cauchy Problem for the Vorticity Equation (2)

We start with a preliminary result that is relatively easy to prove.

Theorem 2 (Giga, Miyakawa & Osada 1988, Kato 1994)
There exists a universal constant $C_0 > 0$ such that, if the initial vorticity $\mu \in \mathcal{M}(\mathbb{R}^2)$ satisfies $\|\mu_{pp}\|_{\mathrm{tv}} \le C_0\,\nu$, then the vorticity equation (V) has a unique global solution

$$\omega \in C^0((0,\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$$

such that $\|\omega(\cdot,t)\|_{L^1} \le \|\mu\|_{\mathrm{tv}}$ for all $t > 0$, and $\omega(\cdot,t) \rightharpoonup \mu$ as $t \to 0$.

The smallness condition $\|\mu_{pp}\|_{\mathrm{tv}} \le C_0\,\nu$ inevitably arises if one tries to prove uniqueness of the solution by a standard application of Gronwall's lemma. This restriction is technical, however, and can be completely relaxed (see below). It is automatically fulfilled if the initial measure is non-atomic, so that Theorem 2 implies Theorem 1.

Sketch of the proof of Theorem 2

We assume that $\nu = 1$ without loss of generality. Given $\mu \in \mathcal{M}(\mathbb{R}^2)$, we consider the integral equation associated to (V) :

$$\omega(t) = e^{t\Delta}\mu - \int_0^t \operatorname{div}\Big(e^{(t-s)\Delta}\, u(s)\omega(s)\Big)\,\mathrm{d}s\;, \quad t > 0\;, \qquad \text{(IE)}$$

where $e^{t\Delta}$ denotes the heat semigroup defined by

$$(e^{t\Delta}\mu)(x) = \frac{1}{4\pi t}\int_{\mathbb{R}^2} e^{-|x-y|^2/(4t)}\,\mathrm{d}\mu_y\;, \quad t > 0\;, \quad x \in \mathbb{R}^2\;. \qquad \text{(HS)}$$

Our goal is to solve the integral equation (IE) by a fixed point argument in an appropriate function space. This is a classical idea which goes back, in the Navier-Stokes context, to Fujita & Kato (1964). Once such a mild solution is obtained, standard regularity arguments imply that the solution $\omega(x,t)$ is smooth for $t > 0$ and satisfies (V) in a classical sense. Similarly, the velocity field $u(x,t)$ given by (BS) satisfies (NS).

Heat kernel estimates

Lemma 2 Let $\mu \in \mathcal{M}(\mathbb{R}^2)$.

a) For $1 \le p \le \infty$ and $t > 0$, we have

$$\|e^{t\Delta}\mu\|_{L^p} \le \frac{1}{(4\pi t)^{1-\frac{1}{p}}}\,\|\mu\|_{\rm tv}\,, \qquad \|\nabla e^{t\Delta}\mu\|_{L^p} \le \frac{C}{t^{\frac{3}{2}-\frac{1}{p}}}\,\|\mu\|_{\rm tv}\,.$$

b) For $1 < p \le \infty$, we have

$$L_p(\mu) := \limsup_{t\to 0}\,(4\pi t)^{1-\frac{1}{p}}\|e^{t\Delta}\mu\|_{L^p} \le \|\mu_{pp}\|_{\rm tv}\,.$$

Estimates a) follow easily from (HS) and Young's inequality. Estimate b) is due to Giga, Miyakawa & Osada, and was strengthened by Kato in this way :

$$\lim_{t\to 0}\,(4\pi t)^{1-\frac{1}{p}}\|e^{t\Delta}\mu\|_{L^p} = p^{-1/p}\|\{\alpha_i\}_{i=1}^{\infty}\|_{\ell^p} \le \sum_{i=1}^{\infty}|\alpha_i|\,,$$

where $\mu_{pp} = \sum_{i=1}^{\infty}\alpha_i\,\delta_{x_i}$. The assumption $p > 1$ is of course crucial.

Digression 1 : Proof of Lemma 2.b

- In view of Lemma 2.a, since $\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$, it is sufficient to show that $L_p(\mu) = 0$ if $p > 1$ and $\mu_{pp} = 0$.
- As $L_p(\mu) \le L_1(\mu)^{1/p} L_\infty(\mu)^{1-1/p} \le \|\mu\|_{\rm tv}^{1/p} L_\infty(\mu)^{1-1/p}$, we only need to consider the case where $p = \infty$.
- Assume that $\mu \in \mathcal{M}(\mathbb{R}^2)$ satisfies $\mu_{pp} = 0$, and fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\sup_{x \in \mathbb{R}^2} |\mu|(B(x,\delta)) \le \varepsilon\,, \quad \text{where} \quad B(x,\delta) = \{y \in \mathbb{R}^2 \,|\, |y-x| \le \delta\}\,.$$

- For any $t > 0$, take $\bar{x}(t) \in \mathbb{R}^2$ such that $|(e^{t\Delta}\mu)(\bar{x}(t))| = \|e^{t\Delta}\mu\|_{L^\infty}$. Then

$$4\pi t\,\|e^{t\Delta}\mu\|_{L^\infty} \le \int_{B(\bar{x}(t),\delta)} e^{-\frac{|\bar{x}(t)-y|^2}{4t}}\,{\rm d}|\mu|_y + \int_{B(\bar{x}(t),\delta)^c} e^{-\frac{|\bar{x}(t)-y|^2}{4t}}\,{\rm d}|\mu|_y\,.$$

The first term is bounded by ε for all $t > 0$; the second one vanishes as $t \to 0$.

Sketch of the proof of Theorem 2 (continued)

The easiest existence result is obtained using the function space

$$X_T = \Big\{ \omega \in C^0((0,T], L^{4/3}(\mathbb{R}^2)) \,\Big|\, \|\omega\|_{X_T} < \infty \Big\}\,,$$

where $T > 0$ will be fixed later and

$$\|\omega\|_{X_T} = \sup_{0<t\le T} t^{1/4} \|\omega(t)\|_{L^{4/3}}\,.$$

A. Estimates for the linear term in (IE) :

By Lemma 2, there exist positive constants C_1, C_2 such that, for any measure $\mu \in \mathcal{M}(\mathbb{R}^2)$, the linear solution $\omega_0(t) = e^{t\Delta}\mu$ satisfies :

- $\|\omega_0\|_{X_T} \le C_1 \|\mu\|_{\mathrm{tv}}$ for any $T > 0$;
- $\|\omega_0\|_{X_T} \le C_2 \|\mu_{pp}\|_{\mathrm{tv}} + \varepsilon$ if $T > 0$ is small enough, depending on μ.

Here $\varepsilon > 0$ is an arbitrary positive number.

B. Estimate for the integral term in (IE):

Given $\omega \in X$, we define $F\omega \in X$ by

$$(F\omega)(t) \;=\; \int_0^t \mathrm{div}\Big(e^{(t-s)\Delta}\,u(s)\omega(s)\Big)\,\mathrm{d}s\;, \qquad 0 < t \le T\;.$$

Then

$$t^{1/4}\|(F\omega)(t)\|_{L^{4/3}}$$

$$\le\; t^{1/4}\int_0^t \frac{C}{(t-s)^{\frac{1}{2}+\frac{1}{4}}}\,\|u(s)\omega(s)\|_{L^1}\,\mathrm{d}s$$ (Heat kernel estimate with derivative)

$$\le\; t^{1/4}\int_0^t \frac{C}{(t-s)^{\frac{3}{4}}}\,\|u(s)\|_{L^4}\|\omega(s)\|_{L^{4/3}}\,\mathrm{d}s$$ (Hölder's inequality)

$$\le\; t^{1/4}\int_0^t \frac{C}{(t-s)^{\frac{3}{4}}}\,\|\omega(s)\|_{L^{4/3}}^2\,\mathrm{d}s$$ (HLS bound for the BS law)

$$\le\; C\|\omega\|_{X_T}^2\,t^{1/4}\int_0^t \frac{C}{(t-s)^{\frac{3}{4}}s^{\frac{1}{2}}}\,\mathrm{d}s \;\le\; C\|\omega\|_{X_T}^2\;.$$

Summarizing, there exists a positive constant C_3 such that

$$\begin{aligned} \|F\omega\|_{X_T} &\leq C_3 \|\omega\|_{X_T}^2 \,, \\ \|F\omega - F\tilde{\omega}\|_{X_T} &\leq C_3 (\|\omega\|_{X_T} + \|\tilde{\omega}\|_{X_T}) \|\omega - \tilde{\omega}\|_{X_T} \,. \end{aligned} \tag{NL}$$

C. The fixed point argument:

Fix $R > 0$ such that $2C_3 R < 1$, and consider the closed ball

$$B = \{\omega \in X_T \,|\, \|\omega\|_{X_T} \leq R\} \,.$$

If $\|\omega_0\|_{X_T} \leq R/2$, the map $\omega \mapsto \omega_0 - F\omega$ is a strict contraction in B, hence has a unique fixed point there. Three situations can occur:

1) If $2C_1 \|\mu\|_{\mathrm{tv}} \leq R$, then $T > 0$ can be chosen arbitrarily large: global well-posedness for small data.

2) If $2C_2 \|\mu_{pp}\|_{\mathrm{tv}} < R$, then $T > 0$ must be small enough, depending on μ: local well-posedness for large data with small atomic part.

3) If $\|\mu_{pp}\|_{\mathrm{tv}}$ is large, the argument breaks down.

D. Concluding remarks :

- A more appropriate space for continuing the solutions is

$$Y_T = \Big\{ \omega \in C^0((0,T], L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \,\Big|\, \|\omega\|_{Y_T} < \infty \Big\}\,,$$

  equipped with the norm $\|\omega\|_{Y_T} = \sup_{0<t\le T} \|\omega(t)\|_{L^1} + \sup_{0<t\le T} t\|\omega(t)\|_{L^\infty}$.

- As before, one has local existence and uniqueness in Y_T if $\|\mu_{pp}\|_{\mathrm{tv}} \le C_0$. The local solution satisfies

$$\lim_{t\to 0}\Big(\|\omega(t) - e^{t\Delta}\mu\|_{L^1} + t\|\omega(t) - e^{t\Delta}\mu\|_{L^\infty}\Big) = 0\,.$$

  In particular $\|\omega(t)\|_{L^1} \le \|\mu\|_{\mathrm{tv}}$ for all $t > 0$, and $\omega(t) \rightharpoonup \mu$ as $t \to 0$.

- If $\|\mu\|_{L^p} \le R$ for some $p > 1$, the local existence time $T = T(\mu)$ is bounded from below by a positive constant depending only on p and R.

- Since the L^p norm of any solution is nonincreasing with time, we conclude that any local solution can be extended to a global solution.

Digression 2 : Short Time Behavior of Mild Solutions

Given $\mu \in \mathcal{M}(\mathbb{R}^2)$ with $\|\mu_{pp}\|_{\rm tv} \le C_0$, let $\omega_0(t) = e^{t\Delta}\mu$ and let $\omega \in X_T$ be the unique local solution of the integral equation $\omega = \omega_0 - F\omega$. We define

$$\ell \;=\; \limsup_{t\to 0}\, t^{1/4}\|\omega(t)-\omega_0(t)\|_{L^{4/3}} \;=\; \limsup_{T\to 0}\,\|\omega-\omega_0\|_{X_T}\;.$$

Since $\omega - \omega_0 = (F\omega_0 - F\omega) - F\omega_0$ and F satisfies (NL), we easily obtain

$$\ell \;\le\; (2C_3R)\ell + \ell_0\;, \quad \text{where} \quad \ell_0 \;=\; \limsup_{T\to 0}\,\|F\omega_0\|_{X_T}\;.$$

To prove that $\ell = 0$, it is therefore sufficient to show that $\ell_0 = 0$. This is done in two steps :

- As in the proof of Lemma 2.b, one proves that $\ell_0 = 0$ if μ is non-atomic.
- If μ is a finite sum of Dirac masses, an explicit calculation (taking into account the fact that self-interaction terms vanish) shows that $\ell_0 = 0$ too.

Finally, $\ell = 0$ implies that $\|\omega(t)-\omega_0(t)\|_{L^1} \to 0$ as $t \to 0$.

The Cauchy Problem for the Vorticity Equation (3)

The restriction $\|\mu_{pp}\|_{\mathrm{tv}} \le C_0\,\nu$ in Theorem 2 is technical and can be removed :

Theorem 3 Given any finite measure $\mu \in \mathcal{M}(\mathbb{R}^2)$, the vorticity equation (V) has a unique global solution

$$\omega \in C^0((0,\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$$

such that $\|\omega(\cdot,t)\|_{L^1} \le \|\mu\|_{\mathrm{tv}}$ for all $t > 0$, and $\omega(\cdot,t) \rightharpoonup \mu$ as $t \to 0$.

Global existence for any $\mu \in \mathcal{M}(\mathbb{R}^2)$ can be proved by approximation :

- G.-H. Cottet (1986)
- Y. Giga, T. Miyakawa & H. Osada (1988)
- T. Kato (1994)

Uniqueness without smallness assumption was obtained in two steps :

- ThG & C.E. Wayne (2005) : the case of a single Dirac mass
- I. Gallagher & ThG (2005) : the general case

Headlines of Lecture 2

Self-similar variables, Lyapunov functions, and long-time behavior

- Radially symmetric solutions of the vorticity equation
- The Lamb-Oseen vortices, elementary properties
- A gobal convergence result
- The vorticity equation in self-similar variables
- Compactness properties
- Liouville's theorem
- Sketch of the proof of Theorems 3 and 4
- Open questions

Radially Symmetric Solutions of the Vorticity Equation

We consider again the two-dimensional vorticity equation

$$\partial_t \omega(x,t) + u(x,t) \cdot \nabla \omega(x,t) \;=\; \nu \Delta \omega(x,t) \;, \tag{V}$$

where the velocity field $u(x,t)$ is given by the Biot-Savart formula

$$u(x,t) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, \omega(y,t)\, \mathrm{d}y \;. \tag{BS}$$

If the vorticity $\omega(x,t)$ is radially symmetric , it follows from (BS) that the velocity field $u(x,t)$ is azimuthal :

$$x^\perp \cdot \nabla \omega \;=\; 0 \quad \Rightarrow \quad x \cdot u \;=\; 0 \;.$$

In that case $u \cdot \nabla \omega = 0$, hence the vorticity equation (V) reduces to the linear heat equation $\partial_t \omega = \nu \Delta \omega$. In particular, radial symmetry is preserved under the evolution defined by (V).

The Lamb-Oseen Vortices

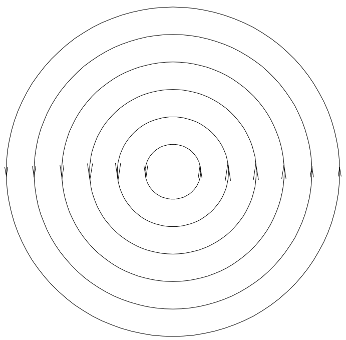
If $\mu = \alpha\delta_0$, the unique solution of (V) is the Lamb-Oseen vortex :

$$\omega(x,t) \;=\; \frac{\alpha}{\nu t}\, G\Big(\frac{x}{\sqrt{\nu t}}\Big)\,, \qquad u(x,t) \;=\; \frac{\alpha}{\sqrt{\nu t}}\, v^G\Big(\frac{x}{\sqrt{\nu t}}\Big)\,,$$

where the vorticity and velocity profiles are given by

$$G(\xi) \;=\; \frac{1}{4\pi}\, e^{-|\xi|^2/4}\,, \qquad v^G(\xi) \;=\; \frac{1}{2\pi}\, \frac{\xi^\perp}{|\xi|^2}\Big(1 - e^{-|\xi|^2/4}\Big)\,.$$

The parameter $\alpha \in \mathbb{R}$ is called the total circulation of the vortex.



Streamlines of an Oseen vortex with positive circulation number α

Elementary Properties of Oseen Vortices

- Oseen vortices are self-similar solutions of the vorticity equation (V).
- The vorticity profile $G(\xi)$ is radially symmetric, positive, and has Gaussian decay at infinity.
- Oseen's vortex with $\alpha = 1$ is the fundamental solution of the heat equation.
- The velocity profile $v^G(\xi)$ is azimuthal and satisfies

$$v^G(0) = 0 , \qquad |v^G(\xi)| \sim \frac{1}{2\pi|\xi|} \quad \text{as} \quad |\xi| \to \infty .$$

  In particular $v^G \notin L^2(\mathbb{R}^2)$, hence Oseen vortices have infinite energy for all $\alpha \neq 0$.

- By Theorem 3, Oseen vortices are the only self-similar solutions of the Navier-Stokes equation in $\mathbb{R}^2$ whose vorticity profile is integrable.

  Remark: Following Cannone and Planchon (1996), one can construct many (small) self-similar solutions for which $u \in L^{2,\infty}$, but $\omega \notin L^1(\mathbb{R}^2)$.

A Global Convergence Result

Theorem 4 (ThG & C.E. Wayne, 2005)
For all initial data $\mu \in \mathcal{M}(\mathbb{R}^2)$, the solution $\omega(x,t)$ of (V) satisfies

$$\lim_{t\to\infty} \Big\|\omega(x,t) - \frac{\alpha}{\nu t} G\Big(\frac{x}{\sqrt{\nu t}}\Big)\Big\|_{L^1} = 0\,, \quad \text{where} \quad \alpha = \int_{\mathbb{R}^2} \mathrm{d}\mu\,.$$

Some important consequences:

- Oseen vortices are the only self-similar solutions of the Navier-Stokes equation in $\mathbb{R}^2$ for which the vorticity profile is integrable.
- Oseen vortices are (globally) stable for all values of the circulation Reynolds number α/ν. No hydrodynamic instabilities appear for large α.

Further developments:

- Explicit convergence estimates: ThG & L.M. Rodrigues (2007)
- Intermediate asymptotics: Caglioti, Pulvirenti & Rousset (2009)

The Self-Similar Variables

Given $x_0 \in \mathbb{R}^2$, $t_0 > 0$, we introduce the self-similar variables

$$\xi = \frac{x - x_0}{\sqrt{\nu(t+t_0)}} , \qquad \tau = \log\Big(1 + \frac{t}{t_0}\Big) .$$

The vorticity and velocity fields are transformed as follows :

$$\begin{aligned} \omega(x,t) &= \frac{1}{t+t_0}\, w\left(\frac{x-x_0}{\sqrt{\nu(t+t_0)}}, \log\Big(1+\frac{t}{t_0}\Big)\right) , \\ u(x,t) &= \sqrt{\frac{\nu}{t+t_0}}\, v\left(\frac{x-x_0}{\sqrt{\nu(t+t_0)}}, \log\Big(1+\frac{t}{t_0}\Big)\right) . \end{aligned} \tag{SSV}$$

The rescaled vorticity $w(\xi,\tau)$ and velocity $v(\xi,\tau)$ are now dimensionless quantities, as are the space variable ξ and the time variable τ. Moreover, the velocity $v(\xi,\tau)$ is still obtained from the vorticity $w(\xi,\tau)$ by the Biot-Savart law (BS). If $w(\xi) = \bar\alpha G(\xi)$, then $\omega(x,t)$ is Oseen's vortex with circulation $\alpha = \bar\alpha\nu$.

The Vorticity Equation in Self-Similar Variables

If $\omega(x,t)$ is a solution of (V), the rescaled vorticity $w(\xi,\tau)$ defined by (SSV) satisfies the rescaled vorticity equation :

$$\frac{\partial w}{\partial \tau} + v \cdot \nabla_\xi w \;=\; \Delta_\xi w + \frac{1}{2}\xi \cdot \nabla_\xi w + w\,. \tag{RV}$$

The initial data of both systems are related through

$$w(\xi,0) \;=\; t_0\,\omega(x_0 + \xi\sqrt{\nu t_0}, 0)\,, \quad \xi \in \mathbb{R}^2\,.$$

Given any $w_0 \in L^1(\mathbb{R}^2)$, Theorem 1 shows that (RV) has a unique global solution $w \in C^0([0,\infty), L^1(\mathbb{R}^2))$ with initial data w_0. The L^1 norm $\|w(\tau)\|_{L^1}$ is nonincreasing with time, and the circulation number is conserved :

$$\bar{\alpha} \;=\; \int_{\mathbb{R}^2} w(\xi,\tau)\,\mathrm{d}\xi \;=\; \frac{1}{\nu}\int_{\mathbb{R}^2} \omega(x,t)\,\mathrm{d}x\,, \quad t,\tau \geq 0\,.$$

For any $\bar{\alpha} \in \mathbb{R}$, Oseen's vortex $w = \bar{\alpha}G$ is a stationary solution of (RV).

Compactness Properties

Positive trajectories of (RV) in $L^1(\mathbb{R}^2)$ are not only bounded, but also compact :

Lemma 3 For any $w_0 \in L^1(\mathbb{R}^2)$, the solution $\{w(\tau)\}_{\tau \geq 0}$ of (RV) with initial data w_0 is relatively compact in $L^1(\mathbb{R}^2)$.

As is clear from Theorem 4, this is not true for the original equation (V). The essential difference is that, in the rescaled system (RV), the Laplacian in the right-hand side has been replaced by the Fokker-Planck operator

$$\mathcal{L} = \Delta + \frac{1}{2}\xi \cdot \nabla + 1 \,.$$

The explicit formula for the associated semigroup

$$(e^{\tau\mathcal{L}}w_0)(\xi) = \frac{1}{4\pi a(\tau)}\int_{\mathbb{R}^2} \exp\Bigl(-\frac{|\xi - \eta\, e^{-\tau/2}|^2}{4a(\tau)}\Bigr) w_0(\eta)\,\mathrm{d}\eta\,, \quad \xi \in \mathbb{R}^2,\ \tau > 0\,,$$

where $a(\tau) = 1 - e^{-\tau}$, shows that $e^{\tau\mathcal{L}}$ is asymptotically confining. Compactness results from confinement and parabolic regularity.

Liouville's Theorem

In contrast, negative trajectories of (RV) in $L^1(\mathbb{R}^2)$ are usually not compact, but those which are compact have a simple characterization :

Proposition 1 If $\{w(\tau)\}_{\tau\in\mathbb{R}}$ is a complete trajectory of (RV) which is relatively compact in $L^1(\mathbb{R}^2)$, then there exists $\bar{\alpha} \in \mathbb{R}$ such that $w(\tau) = \bar{\alpha} G$ for all $\tau \in \mathbb{R}$.

Proposition 1 can be proved using two Lyapunov functions :

- The L^1 norm $\Phi(w) = \|w\|_{L^1}$, which is strictly decreasing except along constant-sign solutions;
- The relative entropy $H(w) = \int_{\mathbb{R}^2} w(\xi) \log\left(\frac{w(\xi)}{G(\xi)}\right) \mathrm{d}\xi$,

  which is defined for positive solutions, and stationary along Oseen vortices :

$$\frac{\mathrm{d}}{\mathrm{d}\tau} H(w) = -\int_{\mathbb{R}^2} w \left|\nabla \log\left(\frac{w}{G}\right)\right|^2 \mathrm{d}\xi \,.$$

By assumption, the solution $w(\tau)$ in Proposition 1 satisfies

$$\mathcal{A} \quad \xleftarrow[\tau\to-\infty]{L^1} \quad w(\tau) \quad \xrightarrow[\tau\to+\infty]{L^1} \quad \Omega\,,$$

where $\mathcal{A} \subset L^1(\mathbb{R}^2)$ is the α-limit set and $\Omega \subset L^1(\mathbb{R}^2)$ the ω-limit set of w.

1. Using the first Lyapunov function

By LaSalle's principle, $\mathcal{A}$ and Ω consist of constant-sign functions. Since the total circulation $\bar{\alpha}$ is conserved, we infer that $\Phi = |\bar{\alpha}|$ on both $\mathcal{A}$ and Ω. As Φ is a Lyapunov function, we must have $\Phi(\tau) = |\bar{\alpha}|$ for all $\tau \in \mathbb{R}$, which in turn implies that $w(\tau)$ has constant sign for $\tau \in \mathbb{R}$.

2. Intermediate step

If $\bar{\alpha} = 0$, we are done. Otherwise, replacing $w(\xi_1, \xi_2, \tau)$ by $-w(\xi_2, \xi_1, \tau)$ if needed, we can assume that $\bar{\alpha} > 0$, hence the solution w is strictly positive.

3. Using the second Lyapunov function

From LaSalle's principle and the conservation of the total circulation, we infer that $\mathcal{A} = \Omega = \{\bar{\alpha}G\}$. It follows that $H(\tau) = \bar{\alpha}\log(\bar{\alpha})$ for all $\tau \in \mathbb{R}$, which implies that $w(\tau) = \bar{\alpha}G$ for all $\tau \in \mathbb{R}$.

Sketch of the Proof of Theorems 3 and 4

Applying Proposition 1 to the ω-limit set of any trajectory of (RV), we find

Corollary 1 For any initial data $w_0 \in L^1(\mathbb{R}^2)$, the solution of (RV) satisfies

$$\|w(\tau) - \bar{\alpha} G\|_{L^1} \xrightarrow[\tau \to \infty]{} 0\,, \quad \text{where} \quad \bar{\alpha} = \int_{\mathbb{R}^2} w_0 \, \mathrm{d}\xi\,.$$

Returning to the original variables, we obtain Theorem 4. On the other hand, using Proposition 1 and classical estimates on the fundamental solution of advection-diffusion equations, due to H. Osada, we arrive at

Corollary 2 Assume that $\omega \in C^0((0,\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ is a solution of (V) satisfying $\|\omega(t)\|_{L^1} \le C$ for all $t > 0$ and $\omega(t) \rightharpoonup \alpha\delta_0$ as $t \to 0$. Then

$$\omega(x,t) = \frac{\alpha}{\nu t} G\Big(\frac{x}{\sqrt{\nu t}}\Big)\,, \quad x \in \mathbb{R}^2\,, \quad t > 0\,.$$

This proves uniqueness in Theorem 3 in the important case where $\mu = \alpha\delta_0$.

Open questions (Lectures 1 and 2)

1. Assume that $\omega \in C^0((0,T), L^1(\mathbb{R}^2))$ is a weak solution of (V) which is uniformly bounded in $L^1(\mathbb{R}^2)$. In the L^1 framework, the nonlinear term can be interpreted as follows :

$$\int_{\mathbb{R}^2} \varphi(u \cdot \nabla\omega)\,\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\cdot(\nabla\varphi(y)-\nabla\varphi(x))\,\omega(x)\omega(y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Then $\omega(\cdot,t)$ converges weakly to some measure $\mu \in \mathcal{M}(\mathbb{R}^2)$ as $t \to 0$. Is the solution $\omega(x,t)$ uniquely determined by its trace μ at $t = 0$?

2. Can one prove the analog of Theorem 3 in bounded domains, with nonslip boundary conditions ?

3. Can one prove the analog of Theorem 4 in exterior domains, with nonslip boundary conditions ?

Headlines of Lecture 3

Asymptotic Stability of Oseen vortices

- The linearized operator at Oseen's vortex
- Spectral stability of Oseen vortices
- Local stability of Oseen vortices
- Characterization of the kernel
- Spectral asymptotics for large circulation numbers, numerical results
- A semiclassical model problem, spectral and pseudospectral estimates
- The stabilizing effect of fast rotation
- Formal asymptotic expansions
- Open questions

Linearization at Oseen's Vortex

Setting $w = \bar{\alpha} G + \tilde{w}$, $v = \bar{\alpha} v^G + \tilde{v}$ in (RV), we obtain the perturbation equation

$$\partial_\tau \tilde{w} + \tilde{v} \cdot \nabla \tilde{w} \;=\; (\mathcal{L} - \bar{\alpha}\Lambda)\tilde{w} \;, \tag{PE}$$

where

$$\mathcal{L}\tilde{w} \;=\; \Delta \tilde{w} + \frac{1}{2}\xi \cdot \nabla \tilde{w} + \tilde{w} \;, \qquad \Lambda \tilde{w} \;=\; v^G \cdot \nabla \tilde{w} + \tilde{v} \cdot \nabla G \;.$$

Here $\tilde{v} = K * \tilde{w}$ is the velocity field obtained from $\tilde{w}$ via the Biot-Savart law (BS). From now on, we write w, v, α instead of $\tilde{w}, \tilde{v}, \bar{\alpha}$, and we consider the semigroup generated by the linearized operator $\mathcal{L} - \alpha\Lambda$.

Function space: We introduce the Hilbert space $X = L^2(\mathbb{R}^2, G^{-1}\,\mathrm{d}\xi)$ with scalar product

$$\langle w_1 \,,\, w_2 \rangle \;=\; \int_{\mathbb{R}^2} G(\xi)^{-1}\, w_1(\xi)\, w_2(\xi)\,\mathrm{d}\xi \;.$$

Functions in X have Gaussian decay at infinity, and $X \hookrightarrow L^p(\mathbb{R}^2)$ for $p \in [1,2]$.

Structure of the Linearized Operator (1)

Observation 1 : The operator $\mathcal{L}$ is selfadjoint in $X = L^2(\mathbb{R}^2, G^{-1}\,\mathrm{d}\xi)$ with compact resolvent and purely discrete spectrum

$$\sigma(\mathcal{L}) = \left\{ -\frac{n}{2} \,\middle|\, n = 0, 1, 2, \dots \right\} .$$

Indeed, if we conjugate $\mathcal{L}$ with the Gaussian weight $G^{1/2}$, we obtain the two-dimensional harmonic oscillator

$$L = G^{-1/2}\,\mathcal{L}\,G^{1/2} = \Delta - \frac{|\xi|^2}{16} + \frac{1}{2} .$$

In particular $\mathcal{L}G = 0$, and $\mathcal{L}\partial_i G = -\frac{1}{2}\partial_i G$ for $i = 1, 2$.

Observation 2 : The operator Λ is skew-symmetric in the same space :

$$\langle \Lambda w_1 \,,\, w_2 \rangle + \langle w_1 \,,\, \Lambda w_2 \rangle = 0 \,, \quad \text{for all} \quad w_1, w_2 \in D(\Lambda) \subset X \,.$$

(ThG & C.E. Wayne 2005, Y. Maekawa 2007).

Digression 3 : Proof of Observation 2

Let $\Lambda = \Lambda_1 + \Lambda_2$, where $\Lambda_1 w = v^G \cdot \nabla w$ and $\Lambda_2 w = v \cdot \nabla G = (K * w) \cdot \nabla G$. If $w_1, w_2 \in D(\Lambda) \subset X$, then

$$\begin{aligned}\langle \Lambda_1 w_1, w_2 \rangle + \langle w_1, \Lambda_1 w_2 \rangle \;&=\; \int_{\mathbb{R}^2} G^{-1} \Big(w_2\, v^G \cdot \nabla w_1 + w_1\, v^G \cdot \nabla w_2 \Big)\, \mathrm{d}\xi \\ &=\; \int_{\mathbb{R}^2} G^{-1}\, v^G \cdot \nabla (w_1 w_2)\, \mathrm{d}\xi \;=\; 0\,,\end{aligned}$$

because $G^{-1} v^G$ is divergence-free. Moreover, since $\nabla G = -\frac{1}{2}\xi G$, we have

$$\langle \Lambda_2 w_1, w_2 \rangle + \langle w_1, \Lambda_2 w_2 \rangle \;=\; -\frac{1}{2} \int_{\mathbb{R}^2} \Big((\xi \cdot v_1) w_2 + (\xi \cdot v_2) w_1 \Big)\, \mathrm{d}\xi$$

$$= \; -\frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left\{ \xi \cdot \frac{(\xi - \eta)^\perp}{|\xi - \eta|^2} + \eta \cdot \frac{(\eta - \xi)^\perp}{|\xi - \eta|^2} \right\} w_1(\eta) w_2(\xi)\, \mathrm{d}\eta\, \mathrm{d}\xi \;=\; 0\,.$$

Thus $\langle \Lambda w_1, w_2 \rangle + \langle w_1, \Lambda w_2 \rangle = 0$ for all $w_1, w_2 \in D(\Lambda) \subset X$.

Structure of the Linearized Operator (2)

Observation 3 : The operator Λ is relatively compact with respect to $\mathcal{L}$, in the space X. For any $\alpha \in \mathbb{R}$, the spectrum of $\mathcal{L} - \alpha\Lambda$ is thus a sequence of eigenvalues $\{\lambda_k(\alpha) \,|\, k \in \mathbb{N}\}$ with

$$\mathrm{Re}\,(\lambda_k(\alpha)) \to -\infty \quad \text{as} \quad k \to \infty\,.$$

Observation 4 : The following subspaces of X are left invariant by both operators $\mathcal{L}$ and Λ :

$$Y_0 = \Big\{ w \in X \,\Big|\, \int_{\mathbb{R}^2} w \,\mathrm{d}\xi = 0 \Big\} = \{G\}^\perp\,,$$

$$Y_1 = \Big\{ w \in Y_0 \,\Big|\, \int_{\mathbb{R}^2} \xi_i w \,\mathrm{d}\xi = 0 \text{ for } i = 1,2 \Big\} = \{G; \partial_1 G; \partial_2 G\}^\perp\,,$$

$$Y_2 = \Big\{ w \in Y_1 \,\Big|\, \int_{\mathbb{R}^2} |\xi|^2 w \,\mathrm{d}\xi = 0 \Big\} = \{G; \partial_1 G; \partial_2 G; \Delta G\}^\perp\,.$$

Spectral Stability of Oseen Vortices

Proposition 2 (ThG & C.E. Wayne 2005)
For any $\alpha \in \mathbb{R}$, Oseen's vortex $w = \alpha G$ is spectrally stable in X :

$$\sigma(\mathcal{L} - \alpha\Lambda) \subset \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{Re}\,(z) \le 0 \right\} .$$

Moreover,

$$\sigma(\mathcal{L} - \alpha\Lambda) \subset \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{Re}\,(z) \le -\frac{1}{2} \right\} \quad \text{in} \quad Y_0 \,,$$

$$\sigma(\mathcal{L} - \alpha\Lambda) \subset \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{Re}\,(z) \le -1 \right\} \quad \text{in} \quad Y_1 \,.$$

Proof : If $(\mathcal{L} - \alpha\Lambda)w = \lambda w$ for some normalized vector $w \in D(\mathcal{L}) \subset X$, then

$$\mathrm{Re}\,(\lambda) \;=\; \mathrm{Re}\,\langle(\mathcal{L} - \alpha\Lambda)w, w\rangle \;=\; \langle \mathcal{L}w, w\rangle \;\le\; 0 \,.$$

Moreover $\langle \mathcal{L}w, w\rangle \le -1/2$ if $w \in Y_0$, and $\langle \mathcal{L}w, w\rangle \le -1$ if $w \in Y_1$.

Local Stability of Oseen Vortices

Corollary 3 (Linear stability) For all $\alpha \in \mathbb{R}$, we have

$$\|e^{\tau(\mathcal{L}-\alpha\Lambda)}\|_{Z\to Z} \le e^{-\mu\tau}\,, \quad \tau \ge 0\,,$$

where $\mu = 0$ if $Z = X$, $\mu = 1/2$ if $Z = Y_0$, and $\mu = 1$ if $Z = Y_1$.

Returning to the perturbation equation (PE), we obtain :

Corollary 4 (Local stability) For any $\mu \in (0, 1/2)$, there exists $\varepsilon > 0$ such that, if $w_0 \in X$ satisfies $w_0 - \alpha G \in Y_0$ and $\|w_0 - \alpha G\| \le \varepsilon$ for some $\alpha \in \mathbb{R}$, then the unique solution of (RV) with initial data w_0 satisfies

$$\|w(\tau) - \alpha G\| \le \|w_0 - \alpha G\|\, e^{-\mu\tau}\,, \quad \tau \ge 0\,.$$

If moreover $w_0 - \alpha G \in Y_1$, then $\|w(\tau) - \alpha G\| \le \|w_0 - \alpha G\|\, e^{-(\mu+\frac{1}{2})\tau}$, $\tau \ge 0$.

Remarkably, the size of the (immediate) basin of attraction of Oseen's vortex αG is uniform in $\alpha \in \mathbb{R}$.

The Kernel of the Skew-Symmetric Operator

Observation 5 : For any $m \in \mathbb{N}$ we define the subspace $X_m \subset X$ by

$$X_m = \left\{ w \in X \,\middle|\, w(\xi) = a_m(r)\cos(m\theta) + b_m(r)\sin(m\theta) \right\},$$

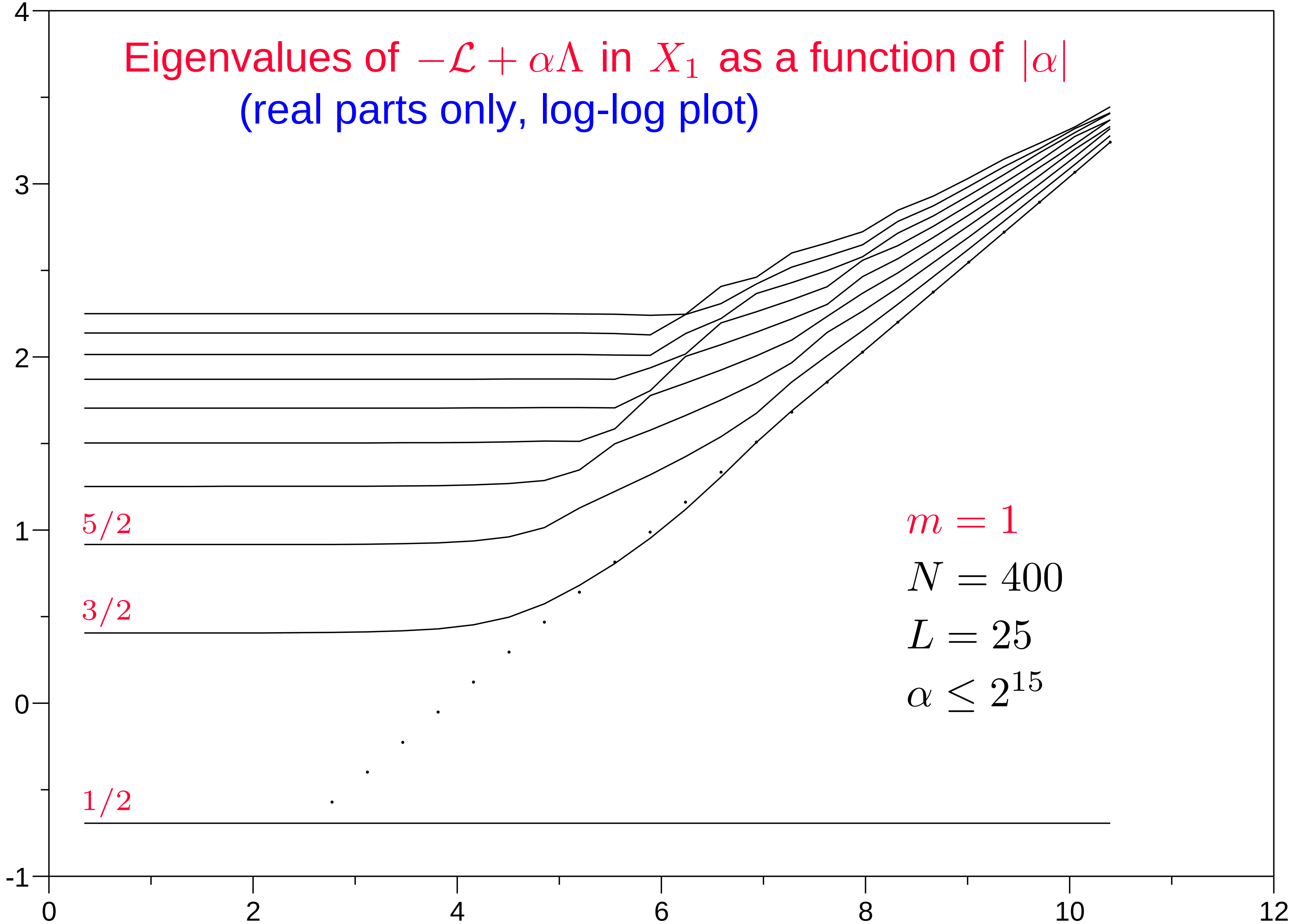
where $\xi = (r\cos\theta, r\sin\theta)$. Then X_m is left invariant by both $\mathcal{L}$ and Λ, so that

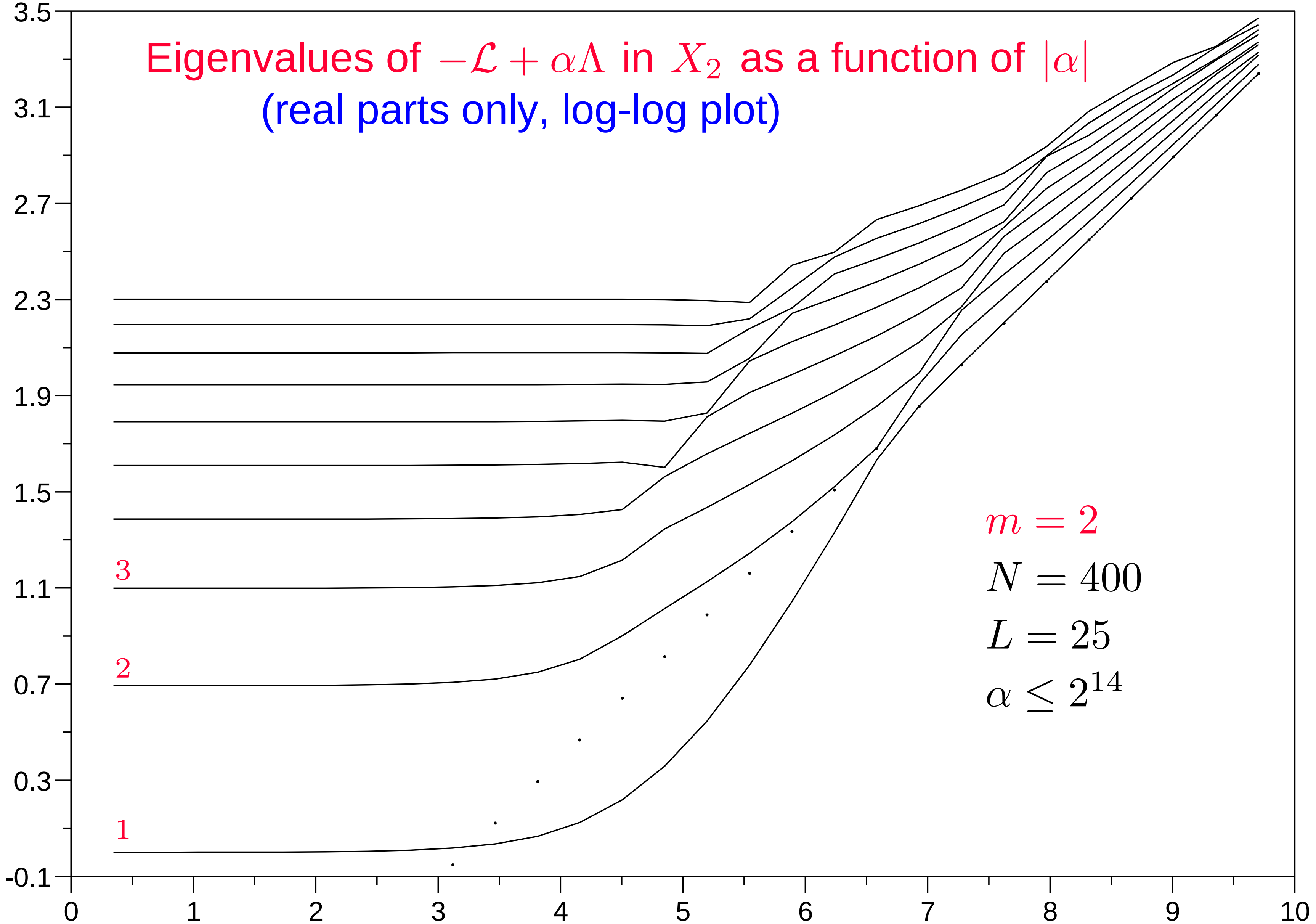
$$X = \bigoplus_{m\in\mathbb{N}} X_m\,, \qquad \mathcal{L} = \bigoplus_{m\in\mathbb{N}} \mathcal{L}_m\,, \qquad \Lambda = \bigoplus_{m\in\mathbb{N}} \Lambda_m\,.$$

Observation 6 : (Y. Maekawa 2007)

$$\ker(\Lambda) = X_0 \oplus \{\beta_1\partial_1 G + \beta_2\partial_2 G \,|\, \beta_1, \beta_2 \in \mathbb{R}\}\,.$$

Numerical observation : (A. Prochazka & D. Pullin, 1995)
In the invariant subspace $\ker(\Lambda)^\perp$, the real parts of all eigenvalues of $\mathcal{L} - \alpha\Lambda$ behave like $-C|\alpha|^{1/2}$ as $|\alpha| \to \infty$.


Eigenvalues of $-\mathcal{L} + \alpha\Lambda$ in X_1 as a function of $|\alpha|$
(real parts only, log-log plot)
$m = 1$
$N = 400$
$L = 25$
$\alpha \leq 2^{15}$
5/2
3/2
1/2
4
3
2
1
0
-1
0
2
4
6
8
10
12



Eigenvalues of $-\mathcal{L} + \alpha\Lambda$ in X_2 as a function of $|\alpha|$
(real parts only, log-log plot)
$m = 2$
$N = 400$
$L = 25$
$\alpha \leq 2^{14}$
1
2
3
3.5
3.1
2.7
2.3
1.9
1.5
1.1
0.7
0.3
-0.1
0
1
2
3
4
5
6
7
8
9
10

The Stabilizing Effect of Fast Rotation

Let $X_\perp$ denote the orthogonal complement of $\ker(\Lambda)$ in X.

Proposition 3 (Y. Maekawa 2007) Let

$$\sigma_\perp(\alpha) = \sigma\Big((\mathcal{L} - \alpha\Lambda)\Big|_{X_\perp}\Big)\,, \quad \text{and} \quad \Sigma(\alpha) = \sup\Big\{\mathrm{Re}\,(z)\,\Big|\, z \in \sigma_\perp(\alpha)\Big\}\,.$$

Then $\Sigma(\alpha) \to -\infty$ as $|\alpha| \to \infty$.

The proof is done by contradiction: assuming that $\Sigma(\alpha_n)$ stays bounded for some sequence $|\alpha_n| \to \infty$, and using compactness arguments, one constructs a normalized vector $w \in X_\perp$ such that $\Lambda w = i\mu w$ for some $\mu \in \mathbb{R}$. This is impossible, because it can be proved that

$$\sigma(\Lambda) = i\mathbb{R}\,, \qquad \text{and} \quad \sigma_p(\Lambda) = \{0\}\,.$$

This approach cannot give any precise estimate of $\Sigma(\alpha)$ for large $|\alpha|$.

A Semiclassical Model Problem

We consider the differential operator

$$H_\varepsilon = -\partial_x^2 + x^2 + \frac{i}{\varepsilon} f(x) , \quad x \in \mathbb{R} , \qquad (*)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth Morse function satisfying, for some $k > 0$,

$$f(x) \sim \frac{1}{|x|^k} \quad \text{as} \quad |x| \to \infty .$$

Relation to the Navier-Stokes problem: if $\tilde{\Lambda} = v^G \cdot \nabla$ denotes the local part of the skew-symmetric operator Λ, the restriction of $\mathcal{L} - \alpha\tilde{\Lambda}$ to the subspace $X_m \subset X$ is

$$\mathcal{L}_m - im\alpha\varphi(r) , \quad \text{where} \quad \varphi(r) = \frac{1}{2\pi r^2}(1 - e^{-r^2/4}) , \quad r > 0 .$$

This is of the form $(*)$ with $\varepsilon = \alpha^{-1}$ and $f = \varphi$ (hence $k = 2$).

Spectral and Pseudospectral Estimates

For the operator $H_\varepsilon = -\partial_x^2 + x^2 + i\varepsilon^{-1} f(x)$ in $L^2(\mathbb{R})$ we define

- The spectral lower bound : $\Sigma(\varepsilon) = \inf \operatorname{Re}(\sigma(H_\varepsilon))$,
- The pseudospectral lower bound : $\Psi(\varepsilon) = \Big(\sup_{\lambda\in\mathbb{R}} \|(H_\varepsilon - i\lambda)^{-1}\|\Big)^{-1}$.

It is easy to verify that $\Sigma(\varepsilon) \geq \Psi(\varepsilon) \geq 1$.

Theorem 5 (I. Gallagher, ThG & F. Nier 2009)
If $f(x) \sim |x|^{-k}$ as $|x| \to \infty$, the following estimate holds as $\varepsilon \to 0$:

$$\Psi(\varepsilon) = \mathcal{O}(\varepsilon^{-\gamma}), \quad \text{where} \quad \gamma = \frac{2}{k+4}.$$

If moreover $f(x) = (1+x^2)^{-k/2}$, then

$$\Sigma(\varepsilon) \geq \mathcal{O}(\varepsilon^{-\kappa}), \quad \text{where} \quad \kappa = \min\Big\{\frac{1}{2}, \frac{2}{k+2}\Big\} > \gamma.$$

The proof is based on semiclassical subelliptic estimates.

Constructive Estimates for the Vortex Problem ?

For the linearized operator $\mathcal{L} - \alpha\Lambda$, we define as before

- The spectral bound in $X_\perp$: $\Sigma(\alpha) = \sup\Big\{\mathrm{Re}\,(z)\,\Big|\, z \in \sigma_\perp(\alpha)\Big\}$,
- The pseudospectral bound in $X_\perp$: $\Psi(\alpha) \,=\, \Big(\sup_{\lambda\in\mathbb{R}} \|(\mathcal{L}_\perp - \alpha\Lambda_\perp - i\lambda)^{-1}\|\Big)^{-1}$.

"Proposition" (work in progress with I. Gallagher)
There exist constants $\kappa > \gamma > 0$ such that

$$\Psi(\alpha) \,=\, \mathcal{O}(\alpha^\gamma) \quad \text{and} \quad |\Sigma(\alpha)| \,=\, \mathcal{O}(\alpha^\kappa)\,, \quad \text{as} \quad |\alpha| \to \infty\,.$$

We conjecture that $\gamma = 1/3$, $\kappa = 1/2$ as in the model problem with $k = 2$.

The pseudospectral exponent γ determines the size of the local basin of attraction of Oseen's vortex. The spectral exponent κ gives the asymptotic decay rate of the perturbations as $\tau \to \infty$.

Formal Asymptotic Expansions

Using a saddle-point analysis in the complex plane and formal semiclassical arguments, one is led to the following conjecture :

The eigenvalue of $\mathcal{L} - \alpha\Lambda$ in $X_\perp$ with largest real part satisfies

$$\lambda_0(\alpha) \approx -\Big(\frac{|\alpha|}{16\pi}\Big)^{1/2}(1+i)\,, \quad \text{as} \quad |\alpha| \to +\infty\,,$$

and the corresponding eigenfunction has the following expression :

$$\varphi_0(r,\theta) \approx e^{-\frac{1}{4}(r-z_\alpha)^2}\, e^{i\theta}\,, \quad \text{where} \quad z_\alpha \approx \Big(\frac{8i|\alpha|}{\pi}\Big)^{1/4}.$$

Observe that $\varphi_0 \in X_1$, and that φ_0 is concentrated in an annulus located at distance $\mathcal{O}(|\alpha|^{1/4})$ from the origin.

Similar asymptotic expansions can be derived for the principal eigenvalues in X_m, for each $m \geq 2$.

Open questions (Lecture 3)

1. Can one prove the optimal spectral and pseudospectral estimates for the linearized operator at Oseen's vortex (see the "Proposition" on page 49) ?

2. For the rescaled vorticity equation, can one show that the size of the (immediate) basin of attraction of Oseen's vortex αG grows unboundedly as $|\alpha| \to \infty$?

3. Can one justify the formal asymptotic expansion for the leading eigenvalue on page 50 ?

4. Can one extend the results above to larger function spaces, allowing algebraic decay of the perturbations at infinity ?

Headlines of Lecture 4

Interaction of vortices in weakly viscous flows

- Phenomenology of vortex interactions
- The viscous N-vortex solution
- The inviscid limit for rough solutions
- The Helmholtz-Kirchhoff system
- The weak convergence result
- Decomposition of the N-vortex solution, self-similar variables
- The strong convergence result
- Self-interaction effects and higher-order expansions
- Skectch of the proof of the main result
- Open questions

Interaction of two co-rotating vortices

Circulation : $\Gamma = \int \omega_i \, \mathrm{d}x > 0$

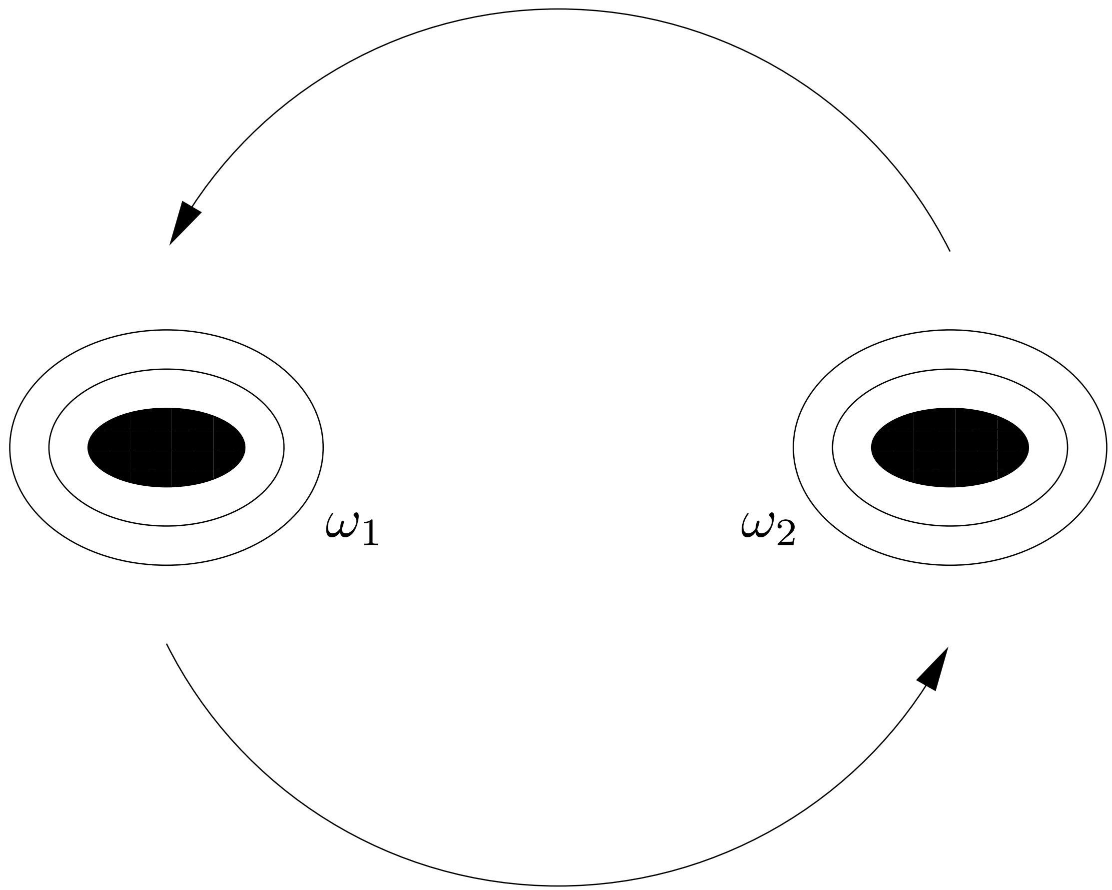
Separation distance : $d > 0$

Rotation period : $T_0 = \frac{2\pi^2 d^2}{\Gamma}$

Vortex size : $a(t)^2 = a(0)^2 + 4\nu t$

Reynolds number : $\mathrm{Re} = \frac{\Gamma}{\nu}$

Remark : $\mathrm{Re} \cdot \frac{\nu T_0}{d^2} = 2\pi^2$



Phenomenology of Vortex Interactions

- When two vortices start interacting, each vortex adapts its shape to the strain field generated by the other vortex. Depending on the initial conditions, oscillations of the vortex ellipticity may be observed during the adaptation stage.
- After oscillations have disappeared, the system reaches a metastable state which evolves slowly on a viscous time scale. This regime is characterized by a single parameter : the ratio a/d of the vortex size to the separation distance. When this parameter reaches the critical value ≈ 0.44, the vortices start merging.

Basic idea : The metastable regime describing the early stage of interaction of a pair of identical vortices can be computed by solving the two-dimensional vorticity equation with point vortices as initial data.

The Viscous N-Vortex Solution

Fix $N \in \mathbb{N}$, $N \geq 1$, and choose

$$x_1, \dots, x_N \in \mathbb{R}^2 \,, \quad \text{with} \quad x_i \neq x_j \quad \text{for } i \neq j \,,$$
$$\alpha_1, \dots, \alpha_N \in \mathbb{R} \,, \quad \text{with} \quad \alpha_i \neq 0 \quad \text{for all } i \,.$$

Given any $\nu > 0$, let $\omega^\nu(x,t)$ denote the unique solution of the vorticity equation (V) with initial data

$$\mu = \sum_{i=1}^{N} \alpha_i \, \delta(\cdot - x_i) \,.$$

In other words, μ is a superposition of N point vortices of circulations $\alpha_1, \dots, \alpha_N$ located at the points $x_1, \dots, x_N$ in $\mathbb{R}^2$. Note that μ does not depend on the viscosity ν.

Question : What is the behavior of $\omega^\nu(x,t)$ as $\nu \to 0$?

Remarks on the Inviscid Limit

Convergence of solutions of the Navier-Stokes equation to solutions of Euler's equation in the vanishing viscosity limit can be established at least for smooth solutions in domains without boundaries :

- D. Ebin & J. Marsden (1970)
- H. Swann (1971)
- T. Kato (1972)
- Th. Beale & A. Majda (1981) . . .

Some convergence results were also obtained for nonsmooth flows :

- Vortex patches : P. Constantin & J. Wu (1995, 1996), J.-Y. Chemin (1996), R. Danchin (1997, 1999), H. Abidi & R. Danchin (2004), T. Hmidi (2005, 2006), N. Masmoudi (2007), F. Sueur (2008)
- Vortex sheets : R. Caflisch & M. Sammartino (2006)
- Point vortices : L. Ting & C. Tung (1965), C. Marchioro (1990, 1998)

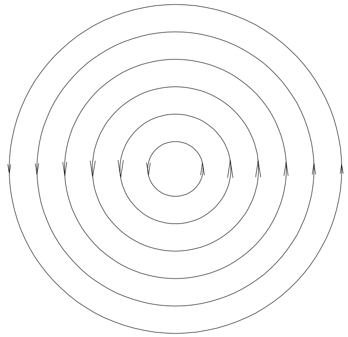
N = 1 : The Lamb-Oseen Vortex

When $\mu = \alpha\delta_0$, we have an explicit self-similar solution of the vorticity equation:

$$\omega(x,t) = \frac{\alpha}{\nu t}\, G\Big(\frac{x}{\sqrt{\nu t}}\Big)\,, \qquad u(x,t) = \frac{\alpha}{\sqrt{\nu t}}\, v^G\Big(\frac{x}{\sqrt{\nu t}}\Big)\,.$$

Here $\alpha \in \mathbb{R}$ is a free parameter (the total circulation of the vortex), and

$$G(\xi) = \frac{1}{4\pi}\, e^{-|\xi|^2/4}\,, \qquad v^G(\xi) = \frac{1}{2\pi}\, \frac{\xi^\perp}{|\xi|^2}\Big(1 - e^{-|\xi|^2/4}\Big)\,.$$



Streamlines of an Oseen vortex
with positive circulation number α

N > 1 : The Helmholtz-Kirchhoff System

Let $z_1(t), \ldots, z_N(t)$ be the solution of the point vortex system

$$z_i'(t) = \frac{1}{2\pi} \sum_{j \neq i} \alpha_j \frac{(z_i(t) - z_j(t))^\perp}{|z_i(t) - z_j(t)|^2}, \qquad z_i(0) = x_i . \qquad \text{(PV)}$$

We fix $T > 0$ such that (PV) is well-posed on $[0, T]$, and we define

– the minimal distance $d = \min_{t \in [0,T]} \min_{i \neq j} |z_i(t) - z_j(t)| > 0$,

– the turnover time $T_0 = \frac{d^2}{|\alpha|}$, where $|\alpha| = |\alpha_1| + \ldots + |\alpha_N|$.

Remarks :

- The system (PV) can be rigorously derived from Euler's equation, through an approximation procedure (C. Marchioro & M. Pulvirenti).
- The system (PV) is not always globally well-posed : vortex collisions can occur in finite time for exceptional initial configurations.

The Weak Convergence Result

Theorem 6 Suppose that system (PV) is well-posed on the time interval $[0, T]$. Then the solution of (V) with initial data $\mu = \sum_{i=1}^{N} \alpha_i\, \delta(\cdot - x_i)$ satisfies

$$\omega^\nu(\cdot, t) \xrightharpoonup[\nu \to 0]{} \sum_{i=1}^{N} \alpha_i\, \delta(\cdot - z_i(t))\,, \quad \text{for all } t \in [0, T]\,.$$

A similar result was proved by Marchioro (1990, 1998), who considered initial data of the form $\mu = \sum_{i=1}^{N} \omega_i^\varepsilon(x)$, where ω_i^ε is a smooth vortex patch with definite sign, of size $\mathcal{O}(\varepsilon)$, centered at x_i, and such that

$$\int_{\mathbb{R}^2} \omega_i^\varepsilon(x)\, \mathrm{d}x \;=\; \alpha_i^\varepsilon \xrightarrow[\varepsilon \to 0]{} \alpha_i\,.$$

Convergence is obtained as $\varepsilon, \nu \to 0$ provided $\nu \le \nu_0\, \varepsilon^\beta$ for some $\beta > 0$.

Theorem 6 is the limiting case $\varepsilon = 0$, $\nu \to 0$, which is precisely excluded by Marchioro's condition.

Decomposition of the N-Vortex Solution

For any $t \in [0,T]$ we decompose the N-vortex solution as

$$\omega^\nu(x,t) \,=\, \sum_{i=1}^N \omega_i^\nu(x,t)\;, \qquad u^\nu(x,t) \,=\, \sum_{i=1}^N u_i^\nu(x,t)\;,$$

where $\omega_i^\nu(x,t)$ is the solution of the linear convection-diffusion equation

$$\partial_t \omega_i^\nu + (u^\nu \cdot \nabla)\omega_i^\nu \,=\, \nu\Delta\omega_i^\nu\;, \quad \text{with} \quad \omega_i^\nu(\cdot,t) \xrightarrow[t\to 0]{} \alpha_i\,\delta(\cdot - x_i)\;,$$

and $u_i^\nu(x,t)$ is obtained from $\omega_i^\nu(x,t)$ via the Biot-Savart law.

Then $\omega_i^\nu(x,t)$ has a definite sign (the sign of α_i), and satisfies Gaussian upper and lower bounds for any fixed ν (Osada 1988, Carlen & Loss 1996). Moreover,

$$\int_{\mathbb{R}^2} \omega_i^\nu(x,t)\,\mathrm{d}x \,=\, \alpha_i\;, \qquad \text{for} \quad i \in \{1,\dots,N\} \quad \text{and} \quad t \in [0,T]\;.$$

Self-Similar Variables

Motivated by the exact solution for $N = 1$ (Oseen's vortex), we define the rescaled vorticity $w_i^\nu(\xi, t)$ and the rescaled velocity $v_i^\nu(\xi, t)$ by setting

$$\begin{cases} \omega_i^\nu(x,t) \;=\; \dfrac{\alpha_i}{\nu t}\, w_i^\nu\Big(\dfrac{x - z_i(t)}{\sqrt{\nu t}}\,,\, t\Big)\,, \\ u_i^\nu(x,t) \;=\; \dfrac{\alpha_i}{\sqrt{\nu t}}\, v_i^\nu\Big(\dfrac{x - z_i(t)}{\sqrt{\nu t}}\,,\, t\Big)\,, \end{cases} \qquad i \in \{1, \dots, N\}\,.$$

Given any $i \in \{1, \dots, N\}$ we denote by ξ the self-similar variable

$$\xi \;=\; \frac{x - z_i(t)}{\sqrt{\nu t}}\,.$$

Our goal is to compute an asymptotic expansion of $w_i^\nu(\xi, t)$ as $\nu \to 0$. The first term in this expansion is the profile $G(\xi)$ of Oseen's vortex, but higher-order corrections will be needed to control the remainder terms.

The Strong Convergence Result

Theorem 7 Suppose that system (PV) is well-posed on the time interval $[0, T]$. Then the rescaled vortex patches of the N-vortex solution $\omega^\nu(x,t)$ satisfy, for $i \in \{1, \ldots, N\}$,

$$\|w_i^\nu(\cdot, t) - G\|_{X_\beta} = \mathcal{O}\Big(\frac{\nu t}{d^2}\Big)\,, \quad \text{as} \quad \nu \to 0\,,$$

uniformly for $t \in (0, T]$.

Here X_β is the weighted L^2 space defined by the norm

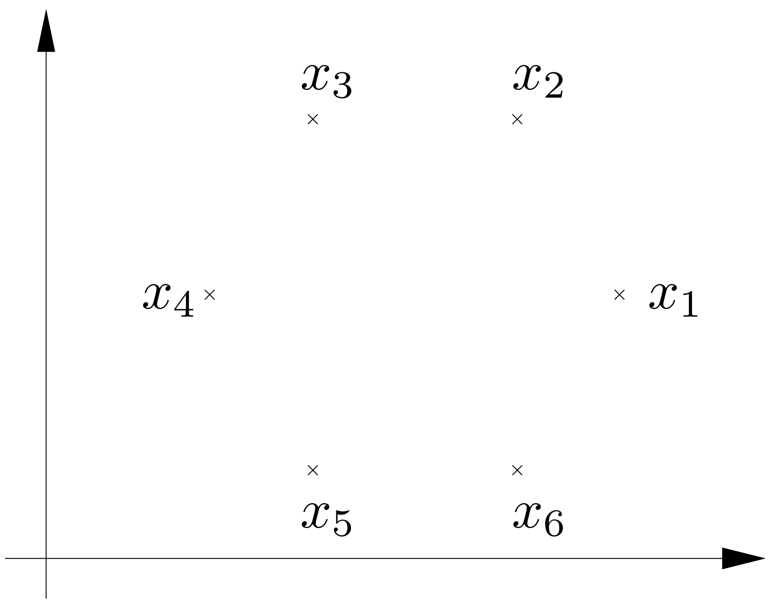
$$\|w\|_{X_\beta} = \Big(\int_{\mathbb{R}^2} |w(\xi)|^2 \, e^{\beta|\xi|/4} \, \mathrm{d}\xi\Big)^{1/2}\,,$$

for some small $\beta > 0$, and $d = \min_{t \in [0,T]} \min_{i \neq j} |z_i(t) - z_j(t)| > 0$.

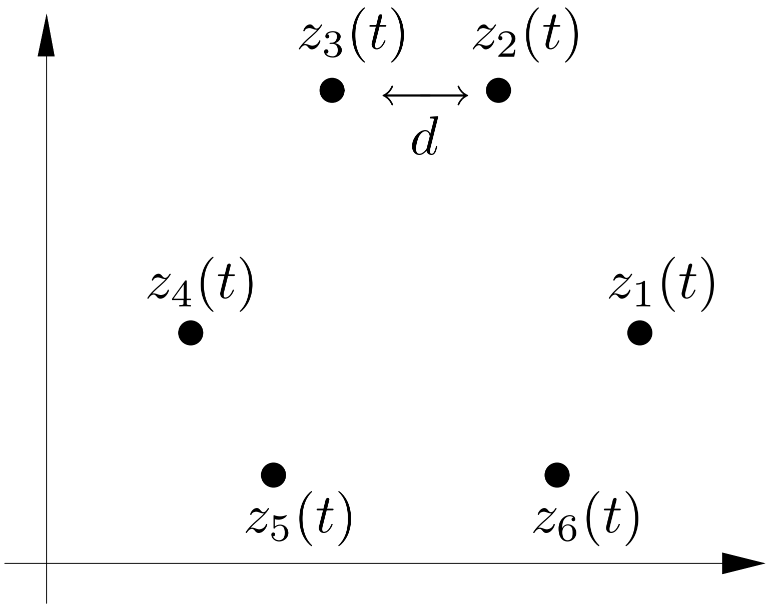
Note that $X_\beta \hookrightarrow L^1(\mathbb{R}^2)$, hence Theorem 7 implies Theorem 6.

Illustration of Theorem 7

$t = 0$: point vortices



$t > 0$: Oseen vortices of size $\mathcal{O}(\sqrt{\nu t})$



The expansion is valid as long as $\nu t \ll d^2$, where d is the minimal distance.

The Self-Interaction Effects

- An isolated Oseen vortex is radially symmetric and does not feel any self-interaction, no matter how large the Reynolds number is.
- When an external strain field is applied, the vortex becomes elliptical and is therefore advected by its own velocity field.
- If the Reynolds number is large, this self-interaction effect can be very strong even if the vortex is nearly symmetric.

General principle : A rapidly rotating Oseen vortex in an external field adapts its shape in such a way that the self-interaction counterbalances the strain of the external field (L. Ting & C. Tung, 1965).

This remarkable stability property explains why elliptical vortices can be advected like rigid bodies in an external field. It is an essential ingredient in the study of the N-vortex solution.

The Second-Order Approximation

For $\xi \in \mathbb{R}^2$, $t \in [0,T]$, and $i \in \{1,\dots,N\}$, we define

$$w_i^{\mathrm{app}}(\xi,t) \;=\; G(\xi) + F(\xi)\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\,\frac{\nu t}{|z_{ij}(t)|^2}\Bigl(2\frac{|\xi\cdot z_{ij}(t)|^2}{|\xi|^2|z_{ij}(t)|^2} - 1\Bigr) + \dots\;,$$

where $z_{ij}(t) = z_i(t) - z_j(t)$. Here $F : \mathbb{R}^2 \to \mathbb{R}_+$ is a smooth radially symmetric function satisfying

$$F(\xi) \;\sim\; \begin{cases} C_1|\xi|^2 & \text{as} \quad |\xi| \to 0\,, \\ C_2|\xi|^4\,e^{-|\xi|^2/4} & \text{as} \quad |\xi| \to \infty\,, \end{cases}$$

for some $C_1, C_2 > 0$. In polar coordinates $\xi = (r\cos\theta, r\sin\theta)$ we have

$$w_i^{\mathrm{app}}(\xi,t) \;=\; g(r) + f(r)\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\,\frac{\nu t}{|z_{ij}(t)|^2}\cos\Bigl(2(\theta - \theta_{ij}(t))\Bigr) + \dots\;,$$

where $\theta_{ij}(t)$ is the argument of $z_{ij}(t) = z_i(t) - z_j(t)$.

The Final Convergence Result

Theorem 8 Suppose that system (PV) is well-posed on the time interval $[0, T]$. Then the rescaled vortex patches of the N-vortex solution $\omega^\nu(x,t)$ satisfy, for $i \in \{1, \ldots, N\}$,

$$\|w_i^\nu(\cdot, t) - w_i^{\mathrm{app}}(\cdot, t)\|_{X_\beta} = \mathcal{O}\Big(\Big(\frac{\nu t}{d^2}\Big)^{3/2}\Big) , \quad \text{as} \quad \nu \to 0 ,$$

uniformly for $t \in (0, T]$.

The error term in Theorem 8 is smaller than the non-radially symmetric corrections to the Gaussian profile in the approximate solution $w_i^{\mathrm{app}}(\xi, t)$. These corrections depend on the instantaneous relative positions of the vortices $z_i(t) - z_j(t)$, without oscillations or inertia.

The approximate solution $w_i^{\mathrm{app}}(\xi, t)$ therefore describes, to leading order, the metastable regime observed in the early stage of vortex interaction.

Evolution Equation for the Vorticity Profiles (1)

Setting $w_i(\xi,t) = w_i^\nu(\xi,t)$ and $v_i(\xi,t) = v_i^\nu(\xi,t)$, we have

$$t\partial_t w_i(\xi,t) + \left\{\sum_{j=1}^N \frac{\alpha_j}{\nu}\, v_j\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\,t\Big) - \sqrt{\frac{t}{\nu}}\, z_i'(t)\right\}\cdot \nabla w_i(\xi,t) \tag{1}$$

$$= (\mathcal{L} w_i)(\xi,t)\ ,$$

where $\mathcal{L}w = \Delta w + \frac{1}{2}\xi\cdot\nabla w + w$ and $z_{ij}(t) = z_i(t) - z_j(t)$.
To kill the most singular terms as $\nu \to 0$, we set

$$z_i'(t) \;=\; \sum_{j=1}^N \frac{\alpha_j}{\sqrt{\nu t}}\, v^G\Big(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)\,, \qquad i \in \{1,\dots,N\}\ . \tag{2}$$

This is a viscous regularization of the point vortex system (PV). In particular, system (2) is globally well-posed for all initial configurations.

Evolution Equation for the Vorticity Profiles (2)

Replacing (2) into (1) we obtain the evolution system

$$t\partial_t w_i(\xi,t) + \sum_{j=1}^{N} \frac{\alpha_j}{\nu}\left\{ v_j\left(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\, t\right) - v^G\left(\frac{z_{ij}(t)}{\sqrt{\nu t}}\right)\right\} \cdot \nabla w_i(\xi,t) \qquad (3)$$
$$= (\mathcal{L} w_i)(\xi,t)\;,$$

which is still singular in the limit $\nu \to 0$.

The Cauchy problem for (3) is not well-posed at $t = 0$, because of the singular term $t\partial_t$. A possible way to avoid this difficulty is to introduce a logarithmic time

$$\tau = \log\left(\frac{t}{T}\right) \in (-\infty, 0]\;,$$

so that $\partial_\tau = t\partial_t$. We then look for a solution of (3) satisfying $w_i(\xi,t) \to G(\xi)$ as $t \to 0$ (that is, as $\tau \to -\infty$). This is possible because $\mathcal{L}G = 0$.

Residuum of the First-Order Approximation

Replacing $w_i(\xi,t) = G(\xi)$, $v_i(\xi,t) = v^G(\xi)$ into (3) we obtain a residuum

$$R_i^{(1)}(\xi,t) \;=\; \sum_{j\neq i} \frac{\alpha_j}{\nu} \left\{ v^G\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\Big) - v^G\Big(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)\right\} \cdot \nabla G(\xi)\;.$$

Since $|z_{ij}(t)| = |z_i(t) - z_j(t)| \geq d$, we have the asymptotic expansion

$$R_i^{(1)}(\xi,t) \;=\; \frac{\alpha_i t}{d^2} \left\{ A_i(\xi,t) + \Big(\frac{\nu t}{d^2}\Big)^{1/2} B_i(\xi,t) + \mathcal{O}\Big(\frac{\nu t}{d^2}\Big) \right\}\;,$$

where

$$A_i(\xi,t) \;=\; \frac{d^2}{2\pi} \sum_{j\neq i} \frac{\alpha_j}{\alpha_i} \, \frac{(\xi\cdot z_{ij}(t))(\xi\cdot z_{ij}(t)^\perp)}{|z_{ij}(t)|^4}\, G(\xi)\;,$$

$$B_i(\xi,t) \;=\; \frac{d^3}{4\pi} \sum_{j\neq i} \frac{\alpha_j}{\alpha_i} \, \frac{(\xi\cdot z_{ij}(t)^\perp)}{|z_{ij}(t)|^6} \Big(|\xi|^2|z_{ij}(t)|^2 - 4(\xi\cdot z_{ij}(t))^2\Big)\, G(\xi)\;.$$

Higher-Order Approximation of the Solution

We look for an approximate solution of (3) in the form

$$w_i^{\rm app}(\xi,t) \;=\; G(\xi) \;+\left(\frac{\nu t}{d^2}\right)F_i(\xi,t) \;+\left(\frac{\nu t}{d^2}\right)^{3/2}H_i(\xi,t)\;,$$

$$v_i^{\rm app}(\xi,t) \;=\; v^G(\xi)+\left(\frac{\nu t}{d^2}\right)v^{F_i}(\xi,t)+\left(\frac{\nu t}{d^2}\right)^{3/2}v^{H_i}(\xi,t)\;,$$

where the profiles $F_i(\xi,t)$, $H_i(\xi,t)$ are determined so as to minimize the error terms.

To first order we have

$$R_i^{(2)}(\xi,t) = \frac{\alpha_i t}{d^2}\Big\{v^G(\xi)\cdot\nabla F_i(\xi,t) + v^{F_i}(\xi,t)\cdot\nabla G(\xi) + A_i(\xi,t) + \mathcal{O}\Big(\frac{\nu t}{d^2}\Big)^{\frac{1}{2}}\Big\}\;,$$

hence we would like to set $\Lambda F_i(\xi,t)+A_i(\xi,t)=0$, where Λ is the integro-differential operator

$$\Lambda w \;=\; v^G\cdot\nabla w + v\cdot\nabla G\;, \quad \text{with} \quad v = K*w\;.$$

Elliptic Equation for the First Correction Term

Since $A_i(\cdot,t)$ lies in the subspace $X_2 \subset \ker(\Lambda)^\perp = \overline{\mathrm{Im}\,(\Lambda)}$, one can show that the equation $\Lambda F_i(\xi,t) + A_i(\xi,t) = 0$ has a unique solution in X_2 :

$$F_i(\xi,t) \;=\; F(\xi)\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\,\frac{\nu t}{|z_{ij}(t)|^2}\Bigl(2\frac{|\xi\cdot z_{ij}(t)|^2}{|\xi|^2|z_{ij}(t)|^2}-1\Bigr)\,.$$

Here $F(\xi) = f(|\xi|)$ and the profile f is determined as follows.

Let $h(r) = (r^2/4)(e^{r^2/4}-1)^{-1}$ and let $\Omega : (0,\infty) \to \mathbb{R}$ be the unique solution of the second-order ODE

$$-\frac{1}{r}(r\Omega'(r))' + \Bigl(\frac{4}{r^2} - h(r)\Bigr)\Omega(r) \;=\; \frac{r^2 h(r)}{4\pi}\,, \quad r>0\,,$$

such that $\Omega(r) \approx C_1\, r^2$ as $r \to 0$, and $\Omega(r) \approx C_2\, r^{-2}$ as $r \to \infty$. Then

$$f(r) \;=\; -\frac{1}{r}(r\Omega'(r))' + \frac{4}{r^2}\Omega(r) \;\equiv\; h(r)\Bigl(\Omega(r) + \frac{r^2}{4\pi}\Bigr)\,, \quad r>0\,.$$

Residuum of the Higher-Order Approximations

First step : If $F_i(\xi,t)$ is chosen so that $\Lambda F_i(\xi,t) + A_i(\xi,t) = 0$, the error term satisfies

$$|R_i^{(2)}(\xi,t)| \le C\,\frac{|\alpha_i|t}{d^2}\Big(\frac{\nu t}{d^2}\Big)^{1/2} e^{-\beta|\xi|^2/4}\,, \qquad \xi\in\mathbb{R}^2\,, \quad t\in[0,T]\,.$$

Second step : If $H_i(\xi,t)$ is chosen so that $\Lambda H_i(\xi,t) + B_i(\xi,t) = 0$, the error term satisfies

$$|R_i^{(3)}(\xi,t)| \le C\,\frac{|\alpha_i|t}{d^2}\Big(\frac{\nu t}{d^2}\Big) e^{-\beta|\xi|^2/4}\,, \qquad \xi\in\mathbb{R}^2\,, \quad t\in[0,T]\,.$$

Third step : A similar, but more complicated procedure allows to obtain an error term satisfying

$$|R_i^{(4)}(\xi,t)| \le C\,\frac{|\alpha_i|t}{d^2}\Big(\frac{\nu t}{d^2}\Big)^{3/2} e^{-\beta|\xi|^2/4}\,, \qquad \xi\in\mathbb{R}^2\,, \quad t\in[0,T]\,.$$

Evolution Equation for the Remainder

Setting $w_i(\xi,t) = w_i^{\mathrm{app}}(\xi,t) + \tilde{w}_i(\xi,t)$, $v_i(\xi,t) = v_i^{\mathrm{app}}(\xi,t) + \tilde{v}_i(\xi,t)$, we obtain for the remainder $\tilde{w}_i$, $\tilde{v}_i$ the evolution system

$$\begin{aligned}
& t\partial_t \tilde{w}_i(\xi,t) - (\mathcal{L}\tilde{w}_i)(\xi,t) \\
& + \frac{\alpha_i}{\nu}\Big(v_i^{\mathrm{app}}(\xi,t)\cdot\nabla\tilde{w}_i(\xi,t) + \tilde{v}_i(\xi,t)\cdot\nabla w_i^{\mathrm{app}}(\xi,t)\Big) \\
& + \sum_{j\neq i}\frac{\alpha_j}{\nu}\left\{v_j^{\mathrm{app}}\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\,t\Big) - v^G\Big(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)\right\}\cdot\nabla\tilde{w}_i(\xi,t) \\
& + \sum_{j\neq i}\frac{\alpha_j}{\nu}\,\tilde{v}_j\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\,t\Big)\cdot\nabla w_i^{\mathrm{app}}(\xi,t) \\
& + \sum_{j=1}^{N}\frac{\alpha_j}{\nu}\,\tilde{v}_j\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\,t\Big)\cdot\nabla\tilde{w}_i(\xi,t) + R_i^{(4)}(\xi,t) \;=\; 0\,,
\end{aligned} \tag{4}$$

which is now “nonsingular” in the limit $\nu \to 0$.

Control of the Remainder (1)

To bound the remainder $\tilde{w}_i(\xi,t)$ we introduce a weighted energy :

$$E(t) \;=\; \sum_{i=1}^{N} \int_{\mathbb{R}^2} p_i(\xi,t)|\tilde{w}_i(\xi,t)|^2 \,\mathrm{d}\xi \;.$$

If $T > 0$ is small with respect to the turnover time

$$T_0 \;=\; \frac{d^2}{|\alpha|}\;, \qquad \text{where} \quad |\alpha| = |\alpha_1| + \ldots + |\alpha_N|\;,$$

we can take $p_i(\xi,t) = p_{a(t)}(\xi)$ for $i = 1,\ldots,N$, where $a(t) = d/(3\sqrt{\nu t})$ and

$$p_a(\xi) \;=\; \begin{cases} e^{|\xi|^2/4} & \text{if} \quad |\xi| \le a\;, \\ e^{a^2/4} & \text{if} \quad a \le |\xi| \le Ka\;, \\ e^{|\xi|^2/(4K^2)} & \text{if} \quad |\xi| \ge Ka\;, \end{cases}$$

for some $K \gg 1$. We then have $e^{|\xi|^2/(4K^2)} \le p_i(\xi,t) \le e^{|\xi|^2/4}$ for all x and t.

Control of the Remainder (2)

With this choice, we obtain from (4) a differential inequality for the weighted energy $E(t)$, which can be integrated using Gronwall's lemma and yields the bound :

$$\int_{\mathbb{R}^2} e^{\frac{\beta|\xi|^2}{4K^2}} \Big(|\tilde{w}_1(\xi,t)|^2 + \ldots + |\tilde{w}_N(\xi,t)|^2\Big)\,\mathrm{d}\xi \;\le\; E(t) \;\le\; C\Big(\frac{\nu t}{d^2}\Big)^3 .$$

This concludes the proof of Theorem 8 if $T \ll T_0$.

In the general case, one has to introduce more complicated weights, which can be constructed using the same procedure as the approximate solution itself. These weights satisfy $e^{\beta|\xi|/4} \le p_i(\xi,t) \le e^{|\xi|^2/4}$, for some small $\beta > 0$ depending only on T/T_0. We thus obtain the weaker estimate :

$$\int_{\mathbb{R}^2} e^{\frac{\beta|\xi|}{4}} \Big(|\tilde{w}_1(\xi,t)|^2 + \ldots + |\tilde{w}_N(\xi,t)|^2\Big)\,\mathrm{d}\xi \;\le\; E(t) \;\le\; C\Big(\frac{\nu t}{d^2}\Big)^3 ,$$

which implies the desired conclusion.

Open questions (Lecture 4)

1. Can one control the inviscid limit in the case where, in addition to point vortices, the initial measure contains a smooth component, or a vortex patch ?

2. Can one control the viscous N-vortex solution in a bounded domain (with nonslip boundary conditions) or on a manifold ?

3. Is it possible to carry on to arbitrarily high orders the large-Reynolds-number expansion used in the proof of Theorem 8 ?

4. Can one follow the interaction of a vortex pair closer to the point where merging occurs ?

5. In the exceptional case where the point vortex system is not globally well-posed, what is the vanishing viscosity limit of the N-vortex solution after the first collision time ?

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