SOBOLEV HOMEOMORPHISM WITH ZERO JACOBIAN ALMOST EVERYWHERE

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Abstract. Let $1 \leq p < N$. We construct a homeomorphism f in the Sobolev space $W^{1,p}((0,1)^N,(0,1)^N)$ such that $J_f = 0$ almost everywhere.

1. Introduction

In this paper we address the following issue. Suppose that $\Omega \subset \mathbb{R}^N$ is an open set and $f : \Omega \to \mathbb{R}^N$ is a homeomorphism of the Sobolev class $W^{1,p}(\Omega, \mathbb{R}^N)$, $p \geq 1$. Here $W^{1,p}(\Omega, \mathbb{R}^N)$ consists of all p-integrable mappings of Ω into $\mathbb{R}^N$ whose coordinate functions have p-integrable distributional derivatives. How big can be the zero set of the jacobian J_f (the determinant of the matrix of derivatives)? Is it possible that $J_f = 0$ almost everywhere?

Let us mention some strange consequences of the existence of a mapping such that $J_f = 0$ a.e. The area formula for Sobolev mappings (see e.g. [2]) holds up to a set of measure zero Z, i.e.

$$0 = \int_{\Omega\setminus Z} J_f(x) = \int_{f(\Omega\setminus Z)} 1 = \mathcal{L}_n\bigl(f(\Omega \setminus Z)\bigr),$$

but $\mathcal{L}_n(\Omega \setminus Z) = \mathcal{L}_n(\Omega)$. It also follows that

$$\mathcal{L}_n(Z) = 0 \quad \text{but} \quad \mathcal{L}_n\bigl(f(Z)\bigr) = \mathcal{L}_n\bigl(f(\Omega)\bigr).$$

It means that such a mapping would simultaneously send a null set to a set of full measure and a set of full measure to a null set.

Let us recall that it is possible to construct a Lipschitz homeomorphism which maps a set of positive measure to a null set (see e.g. [7], [5]) and thus $J_f = 0$ on a set of positive measure. However the simple iteration of this construction does not work because the Sobolev norm of such a mapping would grow too fast. Indeed, the standard counterexamples ([10], [5]) are mappings of finite distortion (see [1] or [4] for basic properties and applications), i.e. $J_f \geq 0$ and $J_f(x) = 0 \Rightarrow |Df(x)| = 0$ a.e. It is easy to see that if a homeomorphism is a mapping of finite distortion, then it cannot satisfy $J_f = 0$ a.e. Otherwise $|Df| = 0$ a.e. and the absolute continuity on almost all lines easily give us a contradiction. It means that for such a homeomorphism we would need to invent the novel construction.

For each $1 \leq p < N$ it is also possible to construct a homeomorphism $f \in W^{1,p}((0,1)^N, \mathbb{R}^N)$ such that f maps a null set to a set of positive measure (see [10] and [6]). On the other hand each homeomorphism in the Sobolev space $W^{1,N}((0,1)^N, \mathbb{R}^N)$

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satisfies the Lusin (N) condition [8] and therefore the image of each null set is a null set, in particular there is no homeomorphism in $W^{1,N}$ such that $J_f = 0$ a.e. We show that surprisingly such a strange mapping exists for any $1 \leq p < N$.

Theorem 1.1. *Let $1 \leq p < N$. There is a homeomorphism $f \in W^{1,p}((0,1)^N, (0,1)^N)$ such that $J_f(x) = 0$ almost everywhere.*

Our research is motivated by our interest in geometric function theory, where the nonnegativity (or positivity) of the jacobian is a standing assumption. Our result implies that far from the natural space $W^{1,N}$ we can have serious difficulties with the development of the reasonable theory. For an overview of the field, discussion of interdisciplinary links and further references see [4].

It was shown by Müller [9] that there is a mapping $f \in W^{1,p}((0,1)^N, \mathbb{R}^N)$, $1 \leq p < N$, such that the distributional jacobian is a singular measure supported on some set of prescribed Hausdorff dimension $\alpha \in (0, N)$. If we take any $N - 1 < p < N$, then the distributional jacobian (see e.g. [4] or [9] for the definition and basic properties) is well-defined for continuous mappings. It follows from our result and its construction that there is even a homeomorphism such that the distributional jacobian is a singular measure (see Remark 7.1).

Our research was also partially motivated by the following recent result [3]: Let $\Omega \subset \mathbb{R}^3$ and let $f \in W^{1,1}(\Omega, \mathbb{R}^3)$ be a sense-preserving homeomorphism. Then $J_f \geq 0$ almost everywhere. It follows from Theorem 1.1 that the converse implication is not valid, because now we can construct a sense-reversing Sobolev homeomorphism such that $J_f = 0$ almost everywhere.

It will be essential for us to construct a mapping which is not a mapping of finite distortion. We will construct a sequence of homeomorphisms F_j which will eventually converge to f and disjoint Cantor type sets C_j of positive measure such that $J_{F_j} = 0$ a.e. on C_j. For $N = 2$ the mapping F_j for j odd will squeeze the sets C_j in the horizontal direction and the derivative in the vertical direction will be non-zero. Analogously F_j for j even will squeeze the sets C_j in the vertical direction and the derivative in the horizontal direction will be non-zero.

At the end we will need to estimate the derivatives of our functions F_j and since they will be composition of finitely many functions on some properly chosen sets, we will use the chain rule to compute the derivative as a product of finitely many matrices. Another key ingredient of our construction will be a fact that all the matrices in the construction will be almost diagonal. That means that the stretching in the horizontal and vertical direction do not multiply and thus the derivative is not big and the norm is finite.

For simplicity we will give the details of the whole construction only for $N = 2$ and in the last section we will briefly outline how to proceed with general $N > 2$. From some technical reasons we construct a mapping from some rhomboid onto the same rhomboid and not from the unit cube onto the unit cube. This difference is of course immaterial.

We will use the usual convention that c denotes a generic constant whose value may change at each occurrence but for fixed N and $1 \leq p < N$ it is an absolute constant. We write $c(j)$ if the value may also depend on some additional parameter j.

2. Basic building block

We begin by defining "building blocks". For $0 < w$ and $s \in (0,1)$, we denote the diamond of width w by

$$Q(w) = \{(x,y) \in \mathbb{R}^2 : |x| < w(1-|y|)\}.$$

We will often work with the inner smaller diamond and the outer annular diamond defined as

$$I(w,s) = Q(ws) \text{ and } O(w,s) = Q(w) \setminus Q(ws).$$

Given parameters $s, s' \in (0,1)$, we will repeatedly employ the mapping $\varphi_{w,s,s'} : Q(w) \to Q(w)$ defined by

$$\varphi_{w,s,s'}(x,y) = \begin{cases} \left(\left(\frac{1-s'}{1-s}\right)x + \operatorname{sgn}(x)(1-|y|)w\left(1-\frac{1-s'}{1-s}\right), y\right) & (x,y) \in O(w,s), \\ \left(\left(\frac{s'}{s}\right)x, y\right) & (x,y) \in I(w,s). \end{cases}$$

If $s' < s$, then this linear homeomorphism horizontally compresses $I(w,s)$ onto $I(w,s')$, while stretching $O(w,s)$ onto $O(w,s')$. Note that $\varphi_{w,s,s'}$ is the identity on the boundary of $Q(w)$.

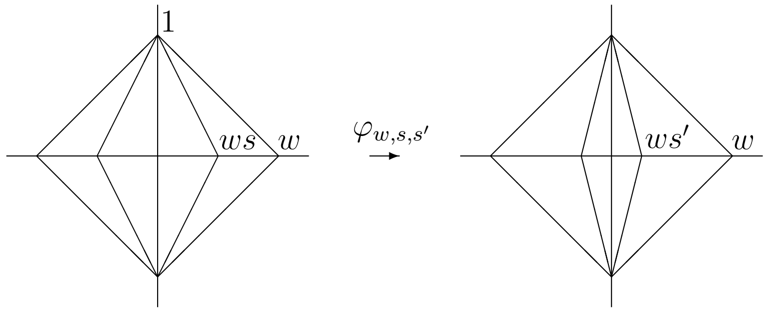

Fig. 1. The mapping $\varphi_{w,s,s'}$

If (x_0,y_0) is an interior point of $I(w,s)$, then

$$D\varphi_{w,s,s'}(x_0,y_0) = \begin{pmatrix} \frac{s'}{s} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

and if (x_0,y_0) is an interior point of $O(w,s)$ and $y_0 \neq 0$, then

$$D\varphi_{w,s,s'}(x_0,y_0) = \begin{pmatrix} \left(\frac{1-s'}{1-s}\right) & -\operatorname{sgn}(x_0y_0)w\left(1-\frac{1-s'}{1-s}\right) \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

Note that by choosing w sufficiently small we can make the matrix arbitrarily close to diagonal matrix.

Suppose that Q is a scaled and translated version of $Q(w)$. We define $\varphi^Q_{w,s,s'}$ to be the corresponding scaled and translated version of $\varphi_{w,s,s'}$. By I^s_Q and O^s_Q we will denote the corresponding inner diamond and outer annular diamond.

Suppose that P is a scaled and translated copy of a rotated diamond

$$P(w) = \{(x,y) \in \mathbb{R}^2 : |y| < w(1-|x|)\}.$$

We define $\varphi^P_{w,s,s'}$ to be the corresponding rotated, scaled and translated version of $\varphi_{w,s,s'}$. That is $\varphi^P_{w,s,s'}$ maps P onto P and is the identity on the boundary. We will also use a notation I_P^s and O_P^s for the corresponding inner diamond and outer annular diamond.

3. Choice of parameters

Let $1 \leq p < 2$. We can clearly fix $t > 1$ such that

$$10^p \frac{\pi^4}{6^2} t^{p-2} < \frac{1}{2} \tag{3.1}$$

For $k \in \mathbb{N}$, we set

$$w_k = \frac{k+1}{tk^2 - 1},\ s_k = 1 - \frac{1}{tk^2} \text{ and } s'_k = s_k \frac{k}{k+1}. \tag{3.2}$$

In this case,

$$\frac{1-s'_k}{1-s_k} = \frac{tk^2+k}{k+1} \text{ and } \Big(\frac{1-s'_k}{1-s_k} - 1\Big) w_k = \frac{tk^2-1}{k+1} w_k = 1. \tag{3.3}$$

It is also easy to check that $0 < s_k < 1$ and

$$\prod_{i=1}^{\infty} s_i > 0.$$

4. Construction of F_1

Let us denote $Q_0 := Q(w_1)$. We will construct a sequence of bi-Lipschitz mappings $f_{k,1} : Q_0 \to Q_0$ and our mapping $F_1 \in W^{1,p}(Q_0, \mathbb{R}^2)$ will be later defined as $F_1(x) = \lim_{k\to\infty} f_{k,1}(x)$. We will also construct a Cantor-type set C_1 of positive measure such that $J_{F_1} = 0$ almost everywhere on C_1.

We define a sequence of families $\{\mathcal{Q}_{k,1}\}$ of building blocks, and a sequence of homeomorphisms $f_{k,1} \colon Q_0 \to Q_0$. Let $\mathcal{Q}_{1,1} = \{Q(w_1)\} = \{Q_0\}$, and define $f_{1,1} \colon Q_0 \to Q_0$ by

$$f_{1,1}(x, y) = \varphi_{w_1, s_1, s'_1}(x, y).$$

Clearly $f_{1,1}$ is a bi-Lipschitz homeomorphism. Now each $f_{k,1}$ will equal to $f_{1,1}$ on the set $G_{1,1} := O(w_1, s_1)$ and it remains to define it on $R_{1,1} := I(w_1, s_1)$. Clearly

$$\mathcal{L}_2(G_{1,1}) = (1 - s_1)\mathcal{L}_2(Q_0) \text{ and } \mathcal{L}_2(R_{1,1}) = s_1 \mathcal{L}_2(Q_0).$$

Let $\mathcal{Q}_{2,1}$ be any collection of disjoint, scaled and translated copies of $Q(w_2)$ which covers $f_{1,1}(R_{1,1}) = I(w_1, s'_1)$ up to a set of measure zero. That is any two elements of $\mathcal{Q}_{2,1}$ have disjoint interiors, and there is a set $E_{2,1} \subseteq I(w_1, s'_1)$ of measure 0 such that

$$I(w_1, s'_1) \setminus E_{2,1} \subseteq \bigcup_{Q \in \mathcal{Q}_{2,1}} Q \subseteq I(w_1, s'_1).$$

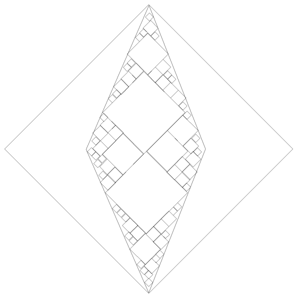

Fig. 2. Sketch of family $\mathcal{Q}_{2,1}$

Clearly such a collection exists. Note that if $Q \in \mathcal{Q}_{2,1}$, then the inverse image of Q under $f_{1,1}$ is a scaled and translated copy of $Q(\frac{s_1}{s_1'}w_2) = Q(2w_2)$ and

$$I(w_1, s_1) \setminus (f_{1,1})^{-1}(E_{2,1}) \subseteq \bigcup_{Q\in\mathcal{Q}_{2,1}} (f_{1,1})^{-1}(Q) \subseteq I(w_1, s_1).$$

Note that $J_{f_{1,1}} \neq 0$ a.e. and hence the inverse image of a null set $E_{2,1}$ has measure zero.

We define $f_{2,1}\colon Q_0 \to Q_0$ by

$$f_{2,1}(x,y) = \begin{cases} \varphi^Q_{w_2,s_2,s_2'} \circ f_{1,1}(x,y) & f_{1,1}(x,y) \in Q \in \mathcal{Q}_{2,1}, \\ f_{1,1}(x,y) & \text{otherwise.} \end{cases}$$

It is not difficult to check that $f_{2,1}$ is a bi-Lipschitz homeomorphism. From now on each $f_{k,1}$ will equal to $f_{2,1}$ on

$$G_{1,1} \cup G_{2,1}, \text{ where } G_{2,1} := f_{1,1}^{-1}\Big(\bigcup_{Q\in\mathcal{Q}_{2,1}} O_Q^{s_2}\Big)$$

and it remains to define it on

$$R_{2,1} := f_{1,1}^{-1}\Big(\bigcup_{Q\in\mathcal{Q}_{2,1}} I_Q^{s_2}\Big).$$

Since each $f_{1,1}^{-1}(Q)$ is a scaled and translated copy of our basic building block and the ratio s_2 is fixed, we obtain

$$\begin{aligned}\mathcal{L}_2(G_{2,1}) &= \sum_{Q\in\mathcal{Q}_{2,1}} \mathcal{L}_2\big(f_{1,1}^{-1}(O_Q^{s_2})\big) = \sum_{Q\in\mathcal{Q}_{2,1}} \mathcal{L}_2\big(O^{s_2}_{f_{1,1}^{-1}(Q)}\big) \\ &= \sum_{Q\in\mathcal{Q}_{2,1}} (1-s_2)\mathcal{L}_2\big(f_{1,1}^{-1}(Q)\big) = (1-s_2)\mathcal{L}_2(R_{1,1}).\end{aligned}$$

It is also easy to see that

$$\mathcal{L}_2(R_{2,1}) = s_2\mathcal{L}_2(R_{1,1}).$$

We continue inductively. Assume that $\mathcal{Q}_{k,1}$, $f_{k,1}$, $G_{k,1}$ and $R_{k,1}$ have already been defined. We find a family of disjoint scaled and translated copies of $Q(w_{k+1})$ that cover $f_{k,1}(R_{k,1})$ up to a set of measure zero $E_{k+1,1}$. Define $\varphi_{k+1,1}\colon Q_0 \to Q_0$ by

$$\varphi_{k+1,1}(x,y) = \begin{cases} \varphi^Q_{w_{k+1},s_{k+1},s_{k+1}'}(x,y) & (x,y) \in Q \in \mathcal{Q}_{k+1,1}, \\ (x,y) & \text{otherwise.} \end{cases}$$

The mapping $f_{k+1,1}\colon Q_0\to Q_0$ is now defined by $\varphi_{k+1,1}\circ f_{k,1}$. Clearly each mapping $f_{k+1,1}$ is a bi-Lipschitz homeomorphism. We further define the sets

$$G_{k+1,1}:=f_{k,1}^{-1}\Bigl(\bigcup_{Q\in\mathcal{Q}_{k+1,1}}O_Q^{s_{k+1}}\Bigr)\text{ and }R_{k+1,1}:=f_{k,1}^{-1}\Bigl(\bigcup_{Q\in\mathcal{Q}_{k+1,1}}I_Q^{s_{k+1}}\Bigr).$$

Again it is not difficult to check that

$$\mathcal{L}_2(G_{k+1,1})=(1-s_{k+1})\mathcal{L}_2(R_{k,1})\text{ and }\mathcal{L}_2(R_{k+1,1})=s_{k+1}\mathcal{L}_2(R_{k,1}).$$

Using $\mathcal{L}_2(G_{1,1})=(1-s_1)\mathcal{L}_2(Q_0)$ and $\mathcal{L}_2(R_{1,1})=s_1\mathcal{L}_2(Q_0)$ we easily obtain

$$\mathcal{L}_2(R_{k,1})=s_1s_2\cdots s_k\mathcal{L}_2(Q_0)\text{ and }\mathcal{L}_2(G_{k,1})=s_1s_2\cdots s_{k-1}(1-s_k)\mathcal{L}_2(Q_0).\tag{4.1}$$

It follows that the resulting Cantor type set

$$C_1:=\bigcap_{k=1}^{\infty}R_{k,1}$$

satisfies

$$\mathcal{L}_2(C_1)=\mathcal{L}_2(Q_0)\prod_{i=1}^{\infty}s_i>0.$$

It is clear from the construction that $f_{k,1}$ converge uniformly and hence the limiting map $F_1(x):=\lim_{k\to\infty}f_{k,1}(x)$ exists and is continuous. It is not difficult to check that F_1 is a one-to-one mapping of Q_0 onto Q_0. Since Q_0 is compact and F_1 is continuous we obtain that F_1 is a homeomorphism. It remains to verify that $f_{k,1}$ form a Cauchy sequence in $W^{1,p}$ and thus $F_1\in W^{1,p}(Q_0,\mathbb{R}^2)$.

Let us estimate the derivative of our functions $f_{m,1}$. Let us fix $m,k\in\mathbb{N}$ such that $m\geq k$. If $Q\in\mathcal{Q}_{k,1}$ and $(x,y)\in\operatorname{int}(f_{k,1})^{-1}(I_Q^{s_k'})$, then we have squeezed our diamond k-times. Using (2.1), (3.2) and the chain rule we obtain

$$Df_{k,1}(x,y)=\prod_{i=1}^{k}\begin{pmatrix}\frac{i}{i+1}&0\\0&1\end{pmatrix}=\begin{pmatrix}\frac{1}{k+1}&0\\0&1\end{pmatrix}.\tag{4.2}$$

Moreover, if $(x,y)\in\operatorname{int}(f_{m,1})^{-1}(O_Q^{s_k'})$, then we have squeezed our diamond $k-1$ times and then we have stretched it once. It follows from (2.1), (3.2), (2.2), (3.3) and the chain rule that

$$\begin{aligned}Df_{m,1}(x,y)&=\begin{pmatrix}\frac{tk^2+k}{k+1}&\pm\left(\frac{tk^2-1}{k+1}\right)w_k\\0&1\end{pmatrix}\left(\prod_{i=1}^{k-1}\begin{pmatrix}\frac{i}{i+1}&0\\0&1\end{pmatrix}\right)\\&=\begin{pmatrix}\frac{tk^2+k}{k(k+1)}&\pm1\\0&1\end{pmatrix}.\end{aligned}\tag{4.3}$$

Now let us fix $m,n\in\mathbb{N}$, $m>n$. Since $f_{n,1}=f_{m,1}$ outside of $R_{n,1}$ we obtain

$$\begin{aligned}\int_{Q_0}|D(f_{m,1}-f_{n,1})|^p&=\int_{R_{n,1}}|D(f_{m,1}-f_{n,1})|^p\\&\leq\int_{R_{n,1}\setminus R_{m,1}}|Df_{n,1}|^p+\int_{R_{m,1}}|Df_{m,1}-Df_{n,1}|^p+\sum_{k=n+1}^{m}\int_{G_{k,1}}|Df_{m,1}|^p.\end{aligned}$$

From (4.2) and (4.1) we obtain

$$\int_{R_{n,1}\setminus R_{m,1}} |Df_{n,1}|^p \leq c\mathcal{L}_2(R_{n,1}\setminus R_{m,1}) \overset{n\to\infty}{\to} 0$$

and

$$\int_{R_{m,1}} |Df_{m,1}-Df_{n,1}|^p \leq c\Big(\frac{1}{n+1}-\frac{1}{m+1}\Big)^p \leq \frac{c}{(n+1)^p} \overset{n\to\infty}{\to} 0.$$

From (4.3) and (4.1) we obtain

$$\begin{aligned}\sum_{k=n+1}^{m}\int_{G_{k,1}} |Df_{m,1}|^p &\leq c\sum_{k=n+1}^{m}\mathcal{L}_2(G_{k,1})\Big(\frac{tk^2+k}{k(k+1)}\Big)^p\\ &\leq c\sum_{k=n+1}^{m}(1-s_k)\Big(\frac{tk^2+k}{k(k+1)}\Big)^p\\ &\leq c\sum_{k=n+1}^{m}\frac{1}{tk^2}t^p \overset{n\to\infty}{\to} 0.\end{aligned}$$

It follows that the sequence $Df_{k,1}$ is Cauchy in L^p and thus we can easily obtain that $f_{k,1}$ is Cauchy in $W^{1,p}$. Since $f_{k,1}$ converge to F_1 uniformly we obtain that $F_1\in W^{1,p}$.

From (4.2) we obtain that the derivative of $f_{k,1}$ on $R_{k,1}$ and especially on C_1 equals to

$$Df_{k,1}(x,y)=\begin{pmatrix}\frac{1}{k+1} & 0\\ 0 & 1\end{pmatrix}.$$

Since $Df_{k,1}$ converge to DF_1 in L^p we obtain that for almost every $(x,y)\in C_1$ we have

$$DF_1(x,y)=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

and therefore $J_{F_1}(x,y)=0$. From now on each F_k will equal to F_1 on C_1 and we need to define it only on $Q_0\setminus C_1$. Moreover it is easy to see from the construction that $J_{F_1}\neq 0$ a.e. on $Q_0\setminus C_1$. It follows that the preimage of each null set in $F_1(Q_0\setminus C_1)$ has zero measure.

5. Construction of F_2

We will construct a sequence of homeomorphisms $f_{k,2}:Q_0\to Q_0$ and our mapping $F_2\in W^{1,p}(Q_0,\mathbb{R}^2)$ will be later defined as $F_2(x)=\lim_{k\to\infty}f_{k,2}(x)$. We will also construct a Cantor-type set $C_2\subset Q_0\setminus C_1$ of positive measure such that $J_{F_2}=0$ almost everywhere on C_2.

The set C_1 is closed and thus we can find $\mathcal{Q}_{1,2}$, a collection of disjoint, scaled and translated copies of $P(w_1)$ which cover $F_1(Q_0\setminus C_1)$ up to a set of measure zero $E_{1,2}$. We will moreover require two additional properties. We know that $Q_0\setminus C_1$ is equal up to a set of measure zero to $\bigcup_{l=1}^{\infty}G_{l,1}$. Hence we will also require that

(5.1) $\qquad$ for each $P\in\mathcal{Q}_{1,2}$ there is $l\in\mathbb{N}$ such that $F_1^{-1}(P)\subset G_{l,1}$.

Secondly, we know that J_{F_1} is constant in each diamond from $G_{l,1}$ (see (2.2)) and thus we may assume that $F_1^{-1}(P)$ is a subset of one diamond and thus

(5.2) $\qquad J_{F_1}(x_1,y_1)=J_{F_1}(x_2,y_2)$ for every $(x_1,y_1),\ (x_2,y_2)\in F_1^{-1}(P)$.

This fact, $\mathcal{L}_2(I_P^{s_1}) = s_1\mathcal{L}_2(P)$ and $\mathcal{L}_2(O_P^{s_1}) = (1-s_1)\mathcal{L}_2(P)$ imply that

$$\mathcal{L}_2\big(F_1^{-1}(I_P^{s_1})\big) = s_1\mathcal{L}_2\big(F_1^{-1}(P)\big) \text{ and } \mathcal{L}_2\big(F_1^{-1}(O_P^{s_1})\big) = (1-s_1)\mathcal{L}_2\big(F_1^{-1}(P)\big).$$

We define $f_{1,2}\colon Q_0 \to Q_0$ by

$$f_{1,2}(x,y) = \begin{cases} \varphi^P_{w_1,s_1,s_1'} \circ F_1(x,y) & F_1(x,y) \in P \in \mathcal{Q}_{1,2}, \\ F_1(x,y) & \text{otherwise}. \end{cases}$$

It is not difficult to check that $f_{1,2}$ is a homeomorphism. Moreover it is a $W^{1,p}$ mapping since it is a composition of a Sobolev and bi-Lipschitz mapping. From now on each $f_{k,2}$ will equal to $f_{1,2}$ on

$$C_1 \cup G_{1,2}, \text{ where } G_{1,2} := F_1^{-1}\Big(\bigcup_{P\in\mathcal{Q}_{1,2}} O_P^{s_1}\Big)$$

and it remains to define it on

$$R_{1,2} := F_1^{-1}\Big(\bigcup_{P\in\mathcal{Q}_{1,2}} I_P^{s_1}\Big).$$

Let us note that $J_{F_1} \neq 0$ on $Q_0 \setminus C_1$ and thus the preimage of the null set $E_{1,2}$ under F_1 is a null set. Clearly

$$\mathcal{L}_2(F_1(R_{1,2})) = s_1\mathcal{L}_2\big(F_1(Q_0\setminus C_1)\big) \text{ and } \mathcal{L}_2(F_1(G_{1,2})) = (1-s_1)\mathcal{L}_2\big(F_1(Q_0\setminus C_1)\big).$$

We continue inductively. Assume that $\mathcal{Q}_{k,2}$, $f_{k,2}$, $G_{k,2}$ and $R_{k,2}$ have already been defined. We find a family of disjoint scaled and translated copies of $P(w_{k+1})$ that cover $f_{k,2}(R_{k,2})$ up to a set of measure zero $E_{k+1,2}$. Define $\varphi_{k+1,2}\colon Q_0 \to Q_0$ by

$$\varphi_{k+1,2}(x,y) = \begin{cases} \varphi^P_{w_{k+1},s_{k+1},s_{k+1}'}(x,y) & (x,y) \in P \in \mathcal{Q}_{k+1,2}, \\ (x,y) & \text{otherwise}. \end{cases}$$

The mapping $f_{k+1,2}\colon Q_0 \to Q_0$ is now defined by $\varphi_{k+1,2} \circ f_{k,2}$. Clearly each mapping $f_{k+1,2}$ is a homeomorphism. Moreover it is a $W^{1,p}$ mapping since it is a composition of a Sobolev and bi-Lipschitz mapping. We further define the sets

$$G_{k+1,2} := f_{k,2}^{-1}\Big(\bigcup_{P\in\mathcal{Q}_{k+1,2}} O_P^{s_{k+1}}\Big) \text{ and } R_{k+1,2} := f_{k,2}^{-1}\Big(\bigcup_{P\in\mathcal{Q}_{k+1,2}} I_P^{s_{k+1}}\Big).$$

The linear maps $\varphi_{j,2}$, $1 \leq j \leq k$, on inner diamonds do not change the ratio of volumes of P and $O_P^{s_{k+1}}$. Therefore we obtain that

$$\mathcal{L}_2(F_1(G_{k+1,2})) = (1-s_{k+1})\mathcal{L}_2(F_1(R_{k,2})) \text{ and } \mathcal{L}_2(F_1(R_{k+1,2})) = s_{k+1}\mathcal{L}_2(F_1(R_{k,2})).$$

Analogously as before we obtain

$$\mathcal{L}_2(F_1(R_{k,2})) = s_1 s_2 \cdots s_k \mathcal{L}_2\big(F_1(Q_0\setminus C_1)\big)$$

and

$$\mathcal{L}_2(F_1(G_{k,2})) = s_1 s_2 \cdots s_{k-1}(1-s_k)\mathcal{L}_2\big(F_1(Q_0\setminus C_1)\big).$$

Therefore using (5.2) we obtain that

$$\mathcal{L}_2(R_{k,2}) = s_1 s_2 \cdots s_k \mathcal{L}_2\big(Q_0\setminus C_1\big) \tag{5.3}$$

and

$$\mathcal{L}_2(G_{k,2}) = s_1 s_2 \cdots s_{k-1}(1-s_k)\mathcal{L}_2\big(Q_0\setminus C_1\big).$$

Since the sets P are uniformly placed among $F_1(G_{l,1})$ (see (5.1)) we can moreover obtain the similar estimate on each $G_{l,1}$, $l \in \mathbb{N}$. Therefore

$$\mathcal{L}_2(G_{k,2} \cap G_{l,1}) = s_1 s_2 \cdots s_{k-1}(1 - s_k)\mathcal{L}_2(G_{l,1}). \tag{5.4}$$

It follows from (5.3) that the resulting Cantor type set

$$C_2 := \bigcap_{k=1}^{\infty} R_{k,2}$$

satisfies

$$\mathcal{L}_2(C_2) = \mathcal{L}_2(Q_0 \setminus C_1) \prod_{i=1}^{\infty} s_i > 0.$$

It is clear from the construction that $f_{k,2}$ converge uniformly and hence it is not difficult to check that the limiting map $F_2(x) := \lim_{k\to\infty} f_{k,2}(x)$ exists and is a homeomorphism. It remains to verify that $f_{k,2}$ form a Cauchy sequence in $W^{1,p}$ and thus $F_2 \in W^{1,p}(Q_0, \mathbb{R}^2)$.

Let us estimate the derivative of our functions $f_{m,2}$. Let us fix $m, k \in \mathbb{N}$ such that $m \geq k$. If $R \in \mathcal{Q}_{k,2}$ and $(x, y) \in \operatorname{int}(f_{k,2})^{-1}(I_R^{s'_k})$, then after applying F_1 we have squeezed our diamond k-times. Analogously to (4.2) we can use (2.1), (3.2) and the chain rule to obtain

$$Df_{k,2}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k+1} \end{pmatrix} DF_1(x, y). \tag{5.5}$$

Moreover, if $(x, y) \in \operatorname{int}(f_{m,2})^{-1}(O_R^{s'_k})$, then after applying F_1 we have squeezed our diamond $k-1$ times and then we have stretched it once. Analogously to (4.3) we can use (2.1), (3.2), (2.2), (3.3) and the chain rule to obtain that

$$\begin{aligned} Df_{m,2}(x, y) &= \begin{pmatrix} 1 & 0 \\ \pm\left(\frac{tk^2-1}{k+1}\right) w_k & \frac{tk^2+k}{k+1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix} DF_1(x, y) \\ &= \begin{pmatrix} 1 & 0 \\ \pm 1 & \frac{tk^2+k}{k(k+1)} \end{pmatrix} DF_1(x, y). \end{aligned} \tag{5.6}$$

Now let us fix $m, n \in \mathbb{N}$, $m > n$. Since $f_{n,2} = f_{m,2}$ outside of $R_{n,2}$ we obtain

$$\begin{aligned} \int_{Q_0} |D(f_{m,2} - f_{n,2})|^p &= \int_{R_{n,2}} |D(f_{m,2} - f_{n,2})|^p \\ &\leq \int_{R_{n,2}\setminus R_{m,2}} |Df_{n,2}|^p + \int_{R_{m,2}} |Df_{m,2} - Df_{n,2}|^p + \sum_{k=n+1}^{m} \int_{G_{k,2}} |Df_{m,2}|^p. \end{aligned}$$

By (5.5) we get

$$\int_{R_{n,2}\setminus R_{m,2}} |Df_{n,2}|^p \leq c \int_{R_{n,2}\setminus R_{m,2}} |DF_1|^p \overset{n\to\infty}{\to} 0$$

since $DF_1 \in L^p$ and $\mathcal{L}_2(R_{n,2} \setminus R_{m,2}) \to 0$. From (5.5) we obtain

$$\int_{R_{m,2}} |Df_{m,2} - Df_{n,2}|^p \leq \frac{c}{(n+1)^p} \int_{R_{m,2}} |DF_1|^p \overset{n\to\infty}{\to} 0.$$

Clearly

$$\begin{pmatrix} 1 & 0 \\ \pm 1 & \frac{tk^2+k}{k(k+1)} \end{pmatrix} \begin{pmatrix} \frac{tl^2+l}{l(l+1)} & \pm 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{tl^2+l}{l(l+1)} & \pm 1 \\ \pm \frac{tl^2+l}{l(l+1)} & \pm 1 + \frac{tk^2+k}{k(k+1)} \end{pmatrix}. \tag{5.7}$$

Thus we may use (4.3), (5.6), (4.1) and (5.4) to obtain

$$\begin{aligned} \sum_{k=n+1}^{m} \int_{G_{k,2}} |Df_{m,2}|^p &\leq \sum_{k=n+1}^{m} \sum_{l=1}^{\infty} \int_{G_{k,2} \cap G_{l,1}} |Df_{m,2}|^p \\ &\leq \sum_{k=n+1}^{m} \sum_{l=1}^{\infty} \mathcal{L}_2(G_{k,2} \cap G_{l,1}) \left\| \begin{pmatrix} \frac{tl^2+l}{l(l+1)} & \pm 1 \\ \pm \frac{tl^2+l}{l(l+1)} & \pm 1 + \frac{tk^2+k}{k(k+1)} \end{pmatrix} \right\|^p \\ &\leq \sum_{k=n+1}^{m} \sum_{l=1}^{\infty} (1-s_k)(1-s_l) 10^p \Big(\max\Big\{ \frac{tk^2+k}{k(k+1)}, \frac{tl^2+l}{l(l+1)} \Big\} \Big)^p \\ &\leq c \sum_{k=n+1}^{m} \sum_{l=1}^{\infty} \frac{1}{tl^2 tk^2} t^p \leq c \sum_{k=n+1}^{m} \frac{1}{k^2} t^{p-2} \overset{n\to\infty}{\to} 0. \end{aligned} \tag{5.8}$$

It follows that the sequence $Df_{k,2}$ is Cauchy in L^p and thus we can easily obtain that $f_{k,2}$ is Cauchy in $W^{1,p}$. Since $f_{k,2}$ converge to F_2 uniformly we obtain that $F_2 \in W^{1,p}$.

From (5.5) we obtain that the derivative of $f_{k,2}$ on $R_{k,2}$ and especially on C_2 equals to

$$Df_{k,2}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k+1} \end{pmatrix} DF_1(x, y).$$

Since $Df_{k,2}$ converge to DF_2 in L^p we obtain that for almost every $(x, y) \in C_2$ we have

$$J_{F_2}(x, y) = \det\Big(\lim_{k\to\infty} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k+1} \end{pmatrix} DF_1(x, y) \Big) = 0.$$

From now on each F_k will equal to F_2 on $C_1 \cup C_2$ and we need to define it only on $Q_0 \setminus (C_1 \cup C_2)$. Analogously as before $J_{F_2} \neq 0$ a.e. on $Q_0 \setminus (C_1 \cup C_2)$ and thus the preimages of the exceptional null sets will be null sets.

6. Construction of general F_j

Assume that the mapping F_{j-1} and the Cantor type set C_{j-1} have already been defined. We will construct a sequence of homeomorphisms $f_{k,j} : Q_0 \to Q_0$ and our mapping $F_j \in W^{1,p}(Q_0, \mathbb{R}^2)$ will be later defined as $F_j(x) = \lim_{k\to\infty} f_{k,j}(x)$. We will also construct a Cantor-type set $C_j \subset Q_0 \setminus (\cup_{i=1}^{j-1} C_i)$ of positive measure such that $J_{F_j} = 0$ almost everywhere on C_j.

The set $C := \cup_{i=1}^{j-1} C_i$ is closed and thus we can find $\mathcal{Q}_{1,j}$, a collection of disjoint, scaled and translated copies of $P(w_1)$ for j even (or copies of $Q(w_1)$ for j odd) which cover $F_1(Q_0 \setminus C)$ up to a set of measure zero. From now on we will assume that j is even but it will be clear that the proof works with obvious minor modifications also for j odd. We will moreover require that

$$\text{for each } P \in \mathcal{Q}_{1,j} \text{ there are } k_1, \ldots, k_{j-1} \in \mathbb{N} \text{ such that } F_{j-1}^{-1}(P) \subset \bigcap_{i=1}^{j-1} G_{k_i,i} \tag{6.1}$$

and that $F_{j-1}^{-1}(P)$ is a subset of a single diamond from the previous construction and thus

$$J_{F_{j-1}}(x_1,y_1)=J_{F_{j-1}}(x_2,y_2) \text{ for every } (x_1,y_1),\ (x_2,y_2)\in F_{j-1}^{-1}(P). \tag{6.2}$$

We define $f_{1,j}\colon Q_0\to Q_0$ by

$$f_{1,j}(x,y)=\begin{cases}\varphi^P_{w_1,s_1,s_1'}\circ F_{j-1}(x,y) & F_{j-1}(x,y)\in P\in\mathcal{Q}_{1,j},\\ F_{j-1}(x,y) & \text{otherwise.}\end{cases}$$

It is not difficult to check that $f_{1,j}$ is a Sobolev homeomorphism since it is a composition of a Sobolev and bi-Lipschitz mapping. From now on each $f_{k,j}$ will equal to $f_{1,j}$ on

$$C\cup G_{1,j},\text{ where } G_{1,j}:=F_{j-1}^{-1}\Big(\bigcup_{P\in\mathcal{Q}_{1,j}}O_P^{s_1}\Big)$$

and it remains to define it on

$$R_{1,j}:=F_{j-1}^{-1}\Big(\bigcup_{P\in\mathcal{Q}_{1,j}}I_P^{s_1}\Big).$$

Clearly

$$\mathcal{L}_2(F_{j-1}(R_{1,j}))=s_1\mathcal{L}_2\big(F_{j-1}(Q_0\setminus C)\big)\text{ and}$$
$$\mathcal{L}_2(F_{j-1}(G_{1,j}))=(1-s_1)\mathcal{L}_2\big(F_{j-1}(Q_0\setminus C)\big).$$

We continue inductively. Assume that $\mathcal{Q}_{k,j}$, $f_{k,j}$, $G_{k,j}$ and $R_{k,j}$ have already been defined. We find a family of disjoint scaled and translated copies of $P(w_{k+1})$ that cover $f_{k,j}(R_{k,j})$ up to a set of measure zero $E_{k+1,j}$. Define $\varphi_{k+1,j}\colon Q_0\to Q_0$ by

$$\varphi_{k+1,j}(x,y)=\begin{cases}\varphi^P_{w_{k+1},s_{k+1},s'_{k+1}}(x,y) & (x,y)\in P\in\mathcal{Q}_{k+1,j},\\ (x,y) & \text{otherwise.}\end{cases}$$

The mapping $f_{k+1,j}\colon Q_0\to Q_0$ is now defined by $\varphi_{k+1,j}\circ f_{k,j}$. Clearly each mapping $f_{k+1,j}$ is a Sobolev homeomorphism since it is a composition of a Sobolev and bi-Lipschitz mapping. We further define the sets

$$G_{k+1,j}:=f_{k,j}^{-1}\Big(\bigcup_{P\in\mathcal{Q}_{k+1,j}}O_P^{s_{k+1}}\Big)\text{ and } R_{k+1,j}:=f_{k,2}^{-1}\Big(\bigcup_{P\in\mathcal{Q}_{k+1,j}}I_P^{s_{k+1}}\Big).$$

The linear maps $\varphi_{i,j}$, $1\le i\le k$, on inner diamonds do not change the ratio of volumes of P and $O_P^{s_{k+1}}$. Therefore we obtain that

$$\mathcal{L}_2(F_{j-1}(G_{k+1,j}))=(1-s_{k+1})\mathcal{L}_2(F_{j-1}(R_{k,j}))\text{ and }\mathcal{L}_2(F_{j-1}(R_{k+1,j}))=s_{k+1}\mathcal{L}_2(F_{j-1}(R_{k,j})).$$

Analogously as before we obtain using (6.2) that

$$\mathcal{L}_2(R_{k,j})=s_1s_2\cdots s_k\mathcal{L}_2\big(Q_0\setminus C\big)$$

and

$$\mathcal{L}_2(G_{k,j})=s_1s_2\cdots s_{k-1}(1-s_k)\mathcal{L}_2\big(Q_0\setminus C\big).$$

Since the sets P are uniformly placed among $F_{j-1}(G_{l,i})$ (see (6.1)) we moreover obtain that

$$\mathcal{L}_2\Big(\bigcap_{i=1}^{j}G_{k_i,i}\Big)=s_1s_2\cdots s_{k_j-1}(1-s_{k_j})\mathcal{L}_2\Big(\bigcap_{i=1}^{j-1}G_{k_i,i}\Big). \tag{6.3}$$

It follows that the resulting Cantor type set

$$C_j := \bigcap_{k=1}^{\infty} R_{k,j}$$

satisfies

$$\mathcal{L}_2(C_j) = \mathcal{L}_2(Q_0 \setminus C) \prod_{i=1}^{\infty} s_i > 0. \tag{6.4}$$

It is clear from the construction that $f_{k,j}$ converge uniformly and hence it is not difficult to check that the limiting map $F_j(x) := \lim_{k\to\infty} f_{k,j}(x)$ exists and is a homeomorphism. It remains to verify that $f_{k,j}$ form a Cauchy sequence in $W^{1,p}$ and thus $F_j \in W^{1,p}(Q_0, \mathbb{R}^2)$.

Let us estimate the derivative of our functions $f_{m,j}$. Let us fix $m, k \in \mathbb{N}$ such that $m \geq k$. If $R \in \mathcal{Q}_{k,j}$ and $(x, y) \in \operatorname{int}(f_{k,j})^{-1}(I_R^{s'_k})$, then after applying F_{j-1} we have squeezed our diamond k-times. Analogously to (4.2) we can use (2.1), (3.2) and the chain rule to obtain

$$Df_{k,j}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k+1} \end{pmatrix} DF_{j-1}(x, y). \tag{6.5}$$

Moreover, if $(x, y) \in \operatorname{int}(f_{m,j})^{-1}(O_R^{s'_k})$, then after applying F_{j-1} we have squeezed our diamond $k-1$ times and then we have stretched it once. Analogously to (4.3) we can use (2.1), (3.2), (2.2), (3.3) and the chain rule to obtain that

$$\begin{aligned} Df_{m,j}(x, y) &= \begin{pmatrix} 1 & 0 \\ \pm\left(\frac{tk^2-1}{k+1}\right) w_k & \frac{tk^2+k}{k+1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix} DF_{j-1}(x, y) \\ &= \begin{pmatrix} 1 & 0 \\ \pm 1 & \frac{tk^2+k}{k(k+1)} \end{pmatrix} DF_{j-1}(x, y). \end{aligned} \tag{6.6}$$

Now let us fix $m, n \in \mathbb{N}$, $m > n$. Since $f_{n,j} = f_{m,j}$ outside of $R_{n,j}$ we obtain

$$\begin{aligned} \int_{Q_0} |D(f_{m,j} - f_{n,j})|^p &= \int_{R_{n,j}} |D(f_{m,j} - f_{n,j})|^p \\ &\leq \int_{R_{n,j}\setminus R_{m,j}} |Df_{n,j}|^p + \int_{R_{m,j}} |Df_{m,j} - Df_{n,j}|^p + \sum_{k=n+1}^{m} \int_{G_{k,j}} |Df_{m,j}|^p. \end{aligned}$$

By (6.5) we get

$$\int_{R_{n,j}\setminus R_{m,j}} |Df_{n,j}|^p \leq c \int_{R_{n,j}\setminus R_{m,j}} |DF_{j-1}|^p \overset{n\to\infty}{\to} 0$$

since $DF_{j-1} \in L^p$ and $\mathcal{L}_2(R_{n,j} \setminus R_{m,j}) \to 0$. From (6.5) we obtain

$$\int_{R_{m,j}} |Df_{m,j} - Df_{n,j}|^p \leq \frac{c}{(n+1)^p} \int_{R_{m,j}} |DF_{j-1}|^p \overset{n\to\infty}{\to} 0.$$

Let us denote $d_i := \frac{ti^2+i}{i(i+1)}$. In the estimate of the norm of the derivative we use the chain rule and then we multiply couples of adjacent matrices as in (5.7). Later we estimate the norm of the product by the product of norms. Then we use (6.6), (6.3), $\sum \frac{1}{k^2} = \frac{\pi^2}{6}$ and we proceed similarly to (5.8)

(6.7)
$$\begin{aligned}
\sum_{k=n+1}^{m} \int_{G_{k,j}} |Df_{m,j}|^p &\leq \sum_{k_j=n+1}^{m} \sum_{k_1,\dots,k_{j-1}=1}^{\infty} \int_{\bigcap_{i=1}^{j} G_{k_i,i}} |Df_{m,j}|^p \\
&\leq c \sum_{k_j=n+1}^{m} \sum_{k_1,\dots,k_{j-1}=1}^{\infty} \mathcal{L}_2\Big(\bigcap_{i=1}^{j} G_{k_i,i}\Big) 10^p \max\{d_{k_1}, d_{k_2}\}^p \cdots 10^p \max\{d_{k_{j-1}}, d_{k_j}\}^p \\
&\leq c\Big(\sum_{k_1,k_2=1}^{\infty} 10^p \frac{t^p}{tk_1^2 tk_2^2}\Big)\Big(\sum_{k_3,k_4=1}^{\infty} 10^p \frac{t^p}{tk_3^2 tk_4^2}\Big) \cdots \Big(\sum_{k_j=n+1}^{m} \sum_{k_{j-1}=1}^{\infty} 10^p \frac{t^p}{tk_{j-1}^2 tk_j^2}\Big) \\
&\leq c\left(10^p \frac{\pi^4}{6^2} t^{p-2}\right)^{\frac{j}{2}-1} \cdot \Big(10^p \frac{\pi^2}{6} t^{p-2} \sum_{k_j=n+1}^{m} \frac{1}{k_j^2}\Big) \overset{n\to\infty}{\to} 0.
\end{aligned}$$

As before this implies that $F_j \in W^{1,p}$ and similarly we also obtain that $J_{F_j} = 0$ almost everywhere on C_j and that $J_{F_j} \neq 0$ almost everywhere on $Q_0 \setminus C$.

7. Properties of f

Now we define $f(x) = \lim_{j\to\infty} F_j(x)$. Since F_j converge uniformly it is easy to see that f is a homeomorphism. It remains to show that DF_j is Cauchy in L^p and thus $F \in W^{1,p}$.

Since $F_j = F_{j-1}$ on $\bigcup_{i=1}^{j-1} C_i$ we obtain

$$\int_{Q_0} |D(F_j - F_{j-1})|^p \leq \int_{C_j} \big(|DF_j|^p + |DF_{j-1}|^p\big) + \sum_{k=1}^{\infty} \int_{G_{k,j}} \big(|DF_j|^p + |DF_{j-1}|^p\big).$$

We will proceed analogously to (6.7) but we will estimate the multiplicative constant more carefully. Again we will suppose that j is even but everything works for j odd analogously. Analogously to (6.7) we can use (3.1) to obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \int_{G_{k,j}} \big(|DF_j|^p + |DF_{j-1}|^p\big) &\leq \sum_{k_1,\dots,k_j=1}^{\infty} \int_{\bigcap_{i=1}^{j} G_{k_i,i}} \big(|DF_j|^p + |DF_{j-1}|^p\big) \\
&\leq c \sum_{k_1,\dots,k_j=1}^{\infty} \mathcal{L}_2\Big(\bigcap_{i=1}^{j} G_{k_i,i}\Big) 10^p \max\{d_{k_1}, d_{k_2}\}^p \cdots 10^p \max\{d_{k_{j-1}}, d_{k_j}\}^p \\
&\leq c\Big(10^p \frac{\pi^4}{6^2} t^{p-2}\Big)^{\frac{j}{2}} \leq c\Big(\frac{1}{2}\Big)^{\frac{j}{2}}.
\end{aligned}$$

From (6.5) we know that

$$Df_{k,j}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k+1} \end{pmatrix} DF_{j-1}(x,y)$$

on C_j. Since the limit as $k\to\infty$ exists it is easy to see that $|DF_j|\leq C|DF_{j-1}|$ there. Hence

$$\begin{aligned}\int_{C_j}\big(|DF_j|^p+|DF_{j-1}|^p\big)&\leq c\sum_{k_1,\dots,k_{j-1}=1}^{\infty}\int_{C_j\cap\bigcap_{i=1}^{j-1}G_{k_i,i}}|DF_{j-1}|^p\\ &\leq c\sum_{k_1,\dots,k_{j-1}=1}^{\infty}\mathcal{L}_2\big(\bigcap_{i=1}^{j-1}G_{k_i,i}\big)10^p\max\{d_{k_1},d_{k_2}\}^p\cdots 10^p d_{k_j}^p\leq c\Big(\frac{1}{2}\Big)^{\frac{j}{2}-1}.\end{aligned}$$

It follows that

$$\sum_{j=1}^{\infty}\int_{Q_0}|D(F_j-F_{j-1})|^p<\infty$$

and thus DF_j forms a Cauchy sequence in L^p and $f\in W^{1,p}$.

From (6.4) we know that

$$\mathcal{L}_2(C_j)=\mathcal{L}_2\Big(Q_0\setminus\bigcup_{i=1}^{j-1}C_i\Big)\prod_{i=1}^{\infty}s_i$$

for each j. Since $\prod_{i=1}^{\infty}s_i>0$ we easily obtain

$$\mathcal{L}_2\Big(\bigcup_{j=1}^{\infty}C_j\Big)=\mathcal{L}_2(Q_0).$$

Together with $J_{F_j}=0$ on C_j and $F_k=F_j$ on C_j for each $k>j$ this implies that $J_f=0$ almost everywhere on Q_0.

Remark 7.1. *It follows from our construction that the mappings F_j are Lipschitz with constant $(Ct)^j$ (see (6.7)). Therefore the distributional jacobian of F_j can be represented by the usual jacobian and we get*

$$\begin{aligned}\mathcal{J}_{F_j}(\varphi)&=-\int_{Q_0}(F_j)_1(x)J\big(\varphi(x),(F_j)_2(x)\big)\,dx\\ &=\int_{Q_0}\varphi(x)J_{F_j}(x)\,dx=\int_{Q_0}\varphi(F_j^{-1}(y))\,dy\end{aligned}$$

for every test function $\varphi\in C_0^{\infty}(Q_0)$. Here $(F_j)_i$ denotes the i-th component of the function F_j and $J(\varphi,(F_j)_2)$ denotes the jacobian of a mappings with first component φ and second $(F_j)_2$. Since F_j converge to F uniformly and in $W^{1,p}$ we get that the left hand side converges to the distributional jacobian of f. It also follows that it is a nonnegative distribution and thus can be represented by some measure. Since $J_f=0$ a. e. we get that this measure is singular with respect to the Lebesgue measure. Similarly we obtain the same conclusion also in higher dimension if $p>N-1$.

8. Construction for $N>2$

The construction in higher dimension is similar and therefore we will only sketch it and point out the differences. Let $0<w$ and $s,s'\in(0,1)$. Our basic building block is a diamond of width w in the first coordinate

$$Q(w)=\big\{x\in\mathbb{R}^N:|x_1|<w(1-|x_2|-|x_3|-\ldots-|x_N|)\big\}$$

and again we denote

$$I(w,s) = Q(ws) \text{ and } O(w,s) = Q(w) \setminus Q(ws).$$

We define the mapping $\varphi_{s,s'}\colon Q(w) \to Q(w)$ by

$$\begin{aligned}&\Big(\Big(\tfrac{1-s'}{1-s}\Big)\, x_1 + \operatorname{sgn}(x_1)(1-|x_2|-\ldots-|x_N|)w\left(1-\tfrac{1-s'}{1-s}\right), x_2, \ldots, x_N\Big) \text{ for } x \in I(w,s),\\ &\Big(\Big(\tfrac{s'}{s}\Big)\, x_1, x_2, \ldots, x_N\Big) \text{ for } x \in O(w,s).\end{aligned}$$

If $s' < s$, then this linear homeomorphism horizontally compresses $I(w,s)$ onto $I(w,s')$, while stretching $O(w,s)$ onto $O(w,s')$.

If x is an interior point of $I(w,s)$, then

$$D\varphi_{w,s,s'}(x) = \begin{pmatrix} \frac{s'}{s} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

and if x is an interior point of $O(w,s)$ and $x_2, \ldots, x_N \neq 0$, then

$$D\varphi_{w,s,s'}(x) = \begin{pmatrix} \frac{1-s'}{1-s} & \pm w(1-\frac{1-s'}{1-s}) & \ldots & \pm w(1-\frac{1-s'}{1-s}) \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \tag{8.1}$$

and again this matrix is close to diagonal matrix for w small enough.

Let $1 \leq p < N$. We can clearly fix $t > 1$ such that

$$A^p \Big(\frac{\pi^2}{6}\Big)^N t^{p-N} < \frac{1}{2} \tag{8.2}$$

where A is a fixed constant to be chosen later. We define the sequences w_k, s_k and s'_k by the same formula as in (3.2).

The mapping $f_{k,j}$ and F_j are defined by the use of our N-dimensional building blocks similarly as in dimension $N = 2$. Given j we find $a \in \mathbb{N}_0$ and $b \in \{1, \ldots, N\}$ such that $j = aN + b$. Then we define the mapping F_j with the use of building blocks that are thin in the direction of the x_b-axis. That is in the key estimate of the derivative we multiply N adjacent matrices and we obtain a matrix that is almost diagonal and almost of the form

$$M_a := \begin{pmatrix} d_{k_{aN+1}} & 0 & \ldots & 0 \\ 0 & d_{k_{aN+2}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_{k_{aN+N-1}} \end{pmatrix}.$$

The actual matrix is non-diagonal because of the non-diagonal terms in (8.1). On the other hand the non-diagonal terms in (8.1) equal to ± 1 and hence it is not difficult to deduce that the norm of the actual matrix can be estimated by a constant A times the norm of the matrix M_a and thus by At.

Similarly to $N=2$ we may deduce that all the constructed sequences are Cauchy in L^p and thus $F_j \in W^{1,p}$ and $f \in W^{1,p}$. The key for the last is (8.2) and the estimate

$$\sum_{k_1,\dots,k_j=1}^{\infty} \mathcal{L}_2\Big(\bigcap_{i=1}^{j} G_{k_i,i}\Big) \prod_{a=0}^{j/N} A^p \,\|M_a\|^p \leq c \sum_{k_1,\dots,k_j=1}^{\infty} \frac{1}{tk_1^2\cdots tk_j^2} \prod_{a=0}^{j/N} A^p t^p$$
$$\leq c\left(A^p \Big(\frac{\pi^2}{6}\Big)^N t^{p-N}\right)^{\frac{j}{N}}.$$

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