Real functions

M. Zelený

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Part I

Winter semester

Chapter 1

Differentiation of measures

1.1 Covering theorems

Covering theorems provide a tool which enables us to infer global properties from local ones in the context of measure theory.

Vitali theorem

Definition. Let $A \subset \mathbf{R}^n$. We say that a system $\mathcal{V}$ consisting of closed balls from $\mathbf{R}^n$ forms **Vitali cover of** A, if

$$\forall x \in A\, \forall \varepsilon > 0\, \exists B \in \mathcal{V} \colon x \in B \ \wedge\ \operatorname{diam} B < \varepsilon.$$

Notation.

- $\lambda_n \ldots$ Lebesgue measure on $\mathbf{R}^n$
- $\lambda_n^* \ldots$ outer Lebesgue measure on $\mathbf{R}^n$
- If $B \subset \mathbf{R}^n$ is a ball and $\alpha > 0$, then $\alpha \star B$ denotes the ball, which is concentric with B and with α-times greater radius than B.

Theorem 1.1 (Vitali)**.** *Let $A \subset \mathbf{R}^n$ and $\mathcal{V}$ be a system of closed balls forming a Vitali cover of A. Then there exists a countable disjoint subsystem $\mathcal{A} \subset \mathcal{V}$ such that $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$.*

Proof. First assume that A is bounded. Take an open bounded set $G \subset \mathbf{R}^n$ with $A \subset G$. Set

$$\mathcal{V}^* = \{B \in \mathcal{V};\ B \subset G\}.$$

The system $\mathcal{V}^*$ is a Vitali cover of A again. If there exists a finite disjoint subsystem $\mathcal{V}^*$ covering A, we are done. So assume

($\star$) there is no finite disjoint subsystem of $\mathcal{V}^*$ covering A.

1st step. We set

$$s_1 = \sup\{\operatorname{diam} B; B \in \mathcal{V}^*\}$$

and choose a ball $B_1 \in \mathcal{V}^*$ such that $\operatorname{diam} B_1 > s_1/2$. We know that $\mathcal{V}^* \neq \emptyset$ and $s_1 \leq \operatorname{diam} G < \infty$.

k**-th step.** Suppose that we have already chosen balls $B_1, \dots, B_{k-1}$. We set

$$s_k = \sup\bigl\{\operatorname{diam} B;\ B \in \mathcal{V}^* \ \wedge\ B \cap \bigcup_{i=1}^{k-1} B_i = \emptyset\bigr\}.$$

The supremum is considered for a nonempty set since the set $\bigcup_{i=1}^{k-1} B_i$ is closed, which by ($\star$) does not cover A, and $\mathcal{V}^*$ is a Vitali cover of A. We choose a ball $B_k \in \mathcal{V}^*$ such that $B_k \cap \bigcup_{i=1}^{k-1} B_i = \emptyset$ and $\operatorname{diam} B_k > s_k/2$.

This finishes the construction of the sequence $(B_k)_{k=1}^{\infty}$. Set $\mathcal{A} = \{B_k;\ k \in \mathbf{N}\}$. We verify that $\mathcal{A}$ is the desired system.

- $\mathcal{A}$ *is countable.* This follows immediately from the construction.

- $\mathcal{A}$ *is disjoint.* This follows from the construction.

- *It holds* $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$. We have

$$\sum_{i=1}^{\infty} \lambda_n(B_i) = \lambda_n\Bigl(\bigcup_{i=1}^{\infty} B_i\Bigr) \leq \lambda_n(G) < \infty.$$

Thus the series $\sum_{i=1}^{\infty} \lambda_n(B_i)$ is convergent, therefore $\lim_i \lambda_n(B_i) = 0$. Using the fact that B_i, $i \in \mathbf{N}$, are balls we also have $\lim_i \operatorname{diam} B_i = 0$. We know that $2 \operatorname{diam} B_i > s_i$, consequently $\lim_i s_i = 0$.

We show that

$$\forall x \in A \setminus \bigcup \mathcal{A}\ \forall i \in \mathbf{N}\ \exists j \in \mathbf{N}, j > i :\ x \in 5 \star B_j.$$

Take $x \in A \setminus \bigcup \mathcal{A}$ and $i \in \mathbf{N}$. Denote $\delta = \operatorname{dist}(x, \bigcup_{k=1}^{i} B_k)$. It holds $\delta > 0$ and there exists $B \in \mathcal{V}^*$ such that $x \in B$ and $\operatorname{diam} B < \delta$. Then we have $B \cap \bigcup_{k=1}^{i} B_k = \emptyset$. Thus we have $\operatorname{diam} B > s_p$ for some $p \in \mathbf{N}$ since $\lim_i s_i = 0$. Therefore there exists $j > i$ with $B_j \cap B \neq \emptyset$. Let j be the smallest number with this property. Then we have $s_j \geq \operatorname{diam} B$ since $B \cap \bigcup_{l=1}^{j-1} B_l = \emptyset$. Further we have $\operatorname{diam} B_j > s_j/2 \geq \frac{1}{2} \operatorname{diam} B$. Together we have $2 \operatorname{diam} B_j \geq \operatorname{diam} B$. This implies $x \in B \subset 5 \star B_j$.

For any $i \in \mathbf{N}$ we have

$$\lambda_n^*(A \setminus \bigcup \mathcal{A}) \leq \lambda_n\bigl(\bigcup_{j=i}^{\infty} 5 \star B_j\bigr) \leq \sum_{j=i}^{\infty} \lambda_n(5 \star B_j) = 5^n \sum_{j=i}^{\infty} \lambda_n(B_j).$$

Using $\lim_{i\to\infty}\sum_{j=i}^{\infty}\lambda_n(B_j)=0$ we get $\lambda_n^*(A\setminus\bigcup\mathcal{A})=0$, and therefore $\lambda_n(A\setminus\bigcup\mathcal{A})=0$.

Now we assume that the set A is a general subset of $\mathbf{R}^n$. Let $(G_j)_{j=1}^{\infty}$ be a sequence of bounded disjoint open sets such that $\lambda_n(\mathbf{R}^n\setminus\bigcup_{j=1}^{\infty}G_j)=0$. Denote

$$\mathcal{V}_j^*=\{B\in\mathcal{V};\ B\subset G_j\}.$$

The system $\mathcal{V}_j^*$ forms a Vitali cover of the bounded set $G_j\cap A$. Using the previous part of the construction we find a countable disjoint system $\mathcal{A}_j\subset\mathcal{V}_j^*$ with $\lambda_n\big((G_j\cap A)\setminus\bigcup\mathcal{A}_j\big)=0$. Now we set $\mathcal{A}=\bigcup_j\mathcal{A}_j$. □

Definition. We say that a measure μ on $\mathbf{R}^n$ satisfies **Vitali theorem**, if for every $M\subset\mathbf{R}^n$ and every Vitali cover $\mathcal{V}$ of M there exists countable disjoint cover $\mathcal{A}\subset\mathcal{V}$ such that $\mu\big(M\setminus\bigcup\mathcal{A}\big)=0$.

Remark. (1) By Theorem 1.1 λ_n satisfies Vitali theorem.

(2) If μ satisfies Vitali theorem and $\nu\ll\mu$, then ν satisfies Vitali theorem.

Remark. If μ is the Borel measure on $\mathbf{R}^2$ such that $\mu(A)=\lambda_1\big(A\cap(\mathbf{R}\times\{0\})\big)$ for any $A\subset\mathbf{R}^2$ Borel, then Vitali theorem does not hold for μ.

——— The end of the lecture no. 1, 1. 10. 2024 ———

Theorem 1.2. *Let $E\subset\mathbf{R}^n$ be measurable and $\mathcal{S}$ be a finite system of closed balls covering E. Then there exists a disjoint system $\mathcal{L}\subset\mathcal{S}$ such that $\lambda_n(E)\le 3^n\sum_{B\in\mathcal{L}}\lambda_n(B)$.*

Proof. Without any loss of generality we may assume that $\mathcal{S}$ is nonempty. Choose $B_1\in\mathcal{S}$ with maximal radius among balls in $\mathcal{S}$. Suppose that we have already constructed $B_1,\dots,B_{k-1}$. If possible, choose $B_k\in\mathcal{S}$ disjoint with $\bigcup_{i<k}B_i$ and with maximal radius among balls in $\mathcal{S}$ satisfying this property. We construct a finite sequence of closed balls $B_1,\dots,B_N$ and set $\mathcal{L}=\{B_1,\dots,B_N\}$. We have $E\subset\bigcup_{B\in\mathcal{L}}3\star B$. To this end consider $x\in E$. Then there exists $B\in\mathcal{S}$ with $x\in B$. We find minimal k such that $B\cap B_k\neq\emptyset$. Then we have $\mathrm{radius}(B)\le\mathrm{radius}(B_k)$. This implies that $x\in B\subset 3\star B_k$.

Then we have

$$\lambda_n(E)\le\lambda_n\Big(\bigcup_{B\in\mathcal{L}}3\star B\Big)\le\sum_{B\in\mathcal{L}}\lambda_n(3\star B)=3^n\sum_{B\in\mathcal{L}}\lambda_n(B).$$

□

Besicovitch theorem

Theorem 1.3 (Besicovitch [**?**]). *For each $n\in\mathbf{N}$ there exists $N\in\mathbf{N}$ with the following property. If $A\subset\mathbf{R}^n$ and $\Delta\colon A\to(0,\infty)$ is a bounded function, then there exist sets $A_1,\dots,A_N$ such that*

- $\{\overline{B}(x, \Delta(x));\ x \in A_i\}$ *is disjoint for every* $i \in \{1, \dots, N\}$,
- $A \subset \bigcup\{\overline{B}(x, \Delta(x));\ x \in \bigcup_{i=1}^{N} A_i\}$.

Proof. The case of a bounded set A. Let $R = \sup_A \Delta$. Choose $B_1 := \overline{B}(a_1, r_1)$ such that $a_1 \in A$ and $r_1 := \Delta(a_1) > \frac{3}{4}R$. Assume that we have already chosen balls $B_1, \dots, B_{j-1}$ where $j \geq 2$. If

$$F_j := A \setminus \bigcup_{i=1}^{j-1} \overline{B}(a_i, r_i) = \emptyset,$$

then the process stops and we set $J = j$. If $F_j \neq \emptyset$, we continue by choosing $B_j := \overline{B}(a_j, r_j)$ such that $a_j \in F_j$ and

$$r_j := \Delta(a_j) > \tfrac{3}{4} \sup_{F_j} \Delta. \tag{1.1}$$

If $F_j \neq \emptyset$ for all j, then we set $J = \infty$. In this case $\lim_{j\to\infty} r_j = 0$ because A is bounded and the inequalities

$$\|a_i - a_j\| \geq r_i = \frac{1}{3}r_i + \frac{2}{3}r_i > \frac{1}{3}r_i + \frac{1}{2}r_j > \frac{1}{3}r_i + \frac{1}{3}r_j$$

for $i < j < J$ imply that

$$\left\{\tfrac{1}{3} \star B_j;\ j < J\right\} \text{ is a disjoint family.} \tag{1.2}$$

In case $J < \infty$, we have $A \subset \bigcup_{j<J} B_j$. This is also true in the case $J = \infty$. Otherwise there exist $a \in \bigcap_{j=1}^{\infty} F_j$ and $j_0 \in \mathbf{N}$ with $r_{j_0} \leq \frac{3}{4}\Delta(a)$, contradicting the choice of r_{j_0}.

Fix $k < J$. We set $I = \{i < k;\ B_i \cap B_k \neq \emptyset\}$. We now prove that there exists $M \in \mathbf{N}$ depending only on n which estimates $|I|$. To this end we split I into I_1 and I_2 and we estimate their cardinality separately.

$$I_1 = \{i < k;\ B_i \cap B_k \neq \emptyset, r_i < 10r_k\},$$
$$I_2 = \{i < k;\ B_i \cap B_k \neq \emptyset, r_i \geq 10r_k\}.$$

The estimate of $|I_1|$. We have $\frac{1}{3} \star B_i \subset 15 \star B_k$ for every $i \in I_1$. Indeed, if $x \in \frac{1}{3} \star B_i$, then

$$\|x - a_k\| \leq \|x - a_i\| + \|a_i - a_k\| \leq \frac{10}{3}r_k + r_i + r_k \leq \frac{43}{3}r_k < 15r_k.$$

Hence, there are at most 60^n elements of I_1, because for any $i \in I_1$ we have

$$\lambda_n(\tfrac{1}{3} \star B_i) = \lambda_n(\overline{B}(0,1)) \cdot \left(\tfrac{1}{3}r_i\right)^n > \lambda_n(\overline{B}(0,1)) \cdot \left(\tfrac{1}{4}r_k\right)^n = \frac{1}{60^n}\lambda_n(15 \star B_k).$$

——— The end of the lecture no. 2, 8. 10. 2024 ———

The estimate of $|I_2|$. Denote $b_i = a_i - a_k$. An elementary mesh-like construction gives a family $\{Q_m;\ 1 \le m \le (22n)^n\}$ of closed cubes with edge length $1/(11n)$ (so that $\operatorname{diam} Q_m \le 1/11$), which cover $[-1,1]^n$ and thus in particular the unit sphere. We claim that for each $1 \le m \le (22n)^n$ there is at most one $i \in I_2$ such that $b_i/\|b_i\| \in Q_m$, which estimates the cardinality of I_2.

If the claim were not valid, then there would exist $i, j \in I_2, i < j$, such that

$$\left\| \frac{b_i}{\|b_i\|} - \frac{b_j}{\|b_j\|} \right\| \le \frac{1}{11}.$$

Notice that

$$r_i < \|b_i\| < r_i + r_k \quad \text{and} \quad r_j < \|b_j\| < r_j + r_k, \tag{1.3}$$

as the balls B_i, B_j intersect B_k but does not contain a_k. Hence

$$\big| \|b_i\| - \|b_j\| \big| \le |r_i - r_j| + r_k \le |r_i - r_j| + \frac{1}{10} r_j.$$

and

$$\|b_j\| \le r_j + r_k \le r_j + \frac{1}{10} r_j = \frac{11}{10} r_j. \tag{1.4}$$

We have

$$\begin{aligned}
\|a_i - a_j\| = \|b_i - b_j\| &\le \left\| b_i - \frac{\|b_j\|}{\|b_i\|} b_i \right\| + \left\| \frac{\|b_j\|}{\|b_i\|} b_i - b_j \right\| \\
&= \left\| \frac{\|b_i\| b_i}{\|b_i\|} - \frac{\|b_j\|}{\|b_i\|} b_i \right\| + \left\| \frac{\|b_j\|}{\|b_i\|} b_i - \frac{\|b_j\|}{\|b_j\|} b_j \right\| \\
&\le \big| \|b_i\| - \|b_j\| \big| + \frac{1}{11} \|b_j\| \\
&\le |r_i - r_j| + \frac{1}{10} r_j + \frac{1}{10} r_j \qquad \text{(using (1.3) and (1.4))} \\
&\le \begin{cases} r_i - \frac{4}{5} r_j < r_i & \text{if } r_i > r_j, \\ -r_i + \frac{6}{5} r_j \le -r_i + \frac{8}{5} r_i < r_i & \text{if } r_i \le r_j. \end{cases}
\end{aligned}$$

In the last inequality we have used that $i < j$ and thus $r_j < \frac{4}{3} r_i$ by (1.1). We arrived at a contradiction as $i < j$ and thus $a_j \notin B_i$. Hence $|I_2| \le (22n)^n$.

Thus it is sufficient to choose $M > 60^n + (22n)^n$.

Choice of $A_1, \dots, A_M$. For each $k \in \mathbf{N}$ we define $\lambda_k \in \{1, 2, \dots, M\}$ such that $\lambda_k = k$ whenever $k \le M$ and for $k > M$ we define λ_k inductively as follows. There is $\lambda_k \in \{1, \dots, M\}$ such that

$$B_k \cap \bigcup \{B_i;\ i < k, \lambda_i = \lambda_k\} = \emptyset.$$

Now we set $A_j = \{a_i;\ \lambda_i = j\}$, $j = 1, \dots, M$.

The case of a general set A. For each $l \in \mathbf{N}$ apply the previously obtained result with A replaced by

$$A^l = A \cap \{x;\ 3(l-1)R \le \|x\| < 3lR\},$$

and denote resulting sets as $A_i^l,\ i = 1, \dots, M$. Then we set

$$A_i = \bigcup_{l \text{ is odd}} A_i^l, \qquad A_{M+i} = \bigcup_{l \text{ is even}} A_i^l, \qquad i = 1, \dots, M.$$

Then we constructed $N := 2M$ subsets which have the required properties. □

Definition. Let P be a locally compact space and $\mathcal{S}$ be a σ-algebra of subsets of P. We say that μ is a **Radon measure** on $(P, \mathcal{S})$ if

(a) $\mathcal{S}$ contains all Borel subsets of P,

(b) $\mu(K) < \infty$ for every compact set $K \subset P$,

(c) $\mu(G) = \sup\{\mu(K);\ K \subset G \text{ is compact}\}$ for every open set $G \subset P$,

(d) $\mu(A) = \inf\{\mu(G);\ A \subset G, G \text{ is open}\}$ for every $A \in \mathcal{S}$,

(e) μ is complete.

Definition. Let μ be a measure on X. **Outer measure corresponding** to μ is defined by

$$\mu^*(A) = \inf\{\mu(B);\ A \subset B, B \text{ is } \mu\text{-measurable}\}.$$

Remark. Let μ be a Radon measure on $(\mathbf{R}^n, \mathcal{S})$ and $A \in \mathcal{S}$. Then there exist a Borel set $B \subset \mathbf{R}^n$ such that $A \subset B$ and $\mu(B \setminus A) = 0$. If ν is a Radon measure on $(\mathbf{R}^n, \mathcal{S}')$ with $\nu \ll \mu$, then $\mathcal{S} \subset \mathcal{S}'$.

The end of the lecture no. 3, 15. 10. 2024

Lemma 1.4. *Let μ be a measure on X and $\{A_j\}_{j=1}^\infty$ be an increasing sequence of subset of X. Then $\lim \mu^*(A_j) = \mu^*\big(\bigcup_{j=1}^\infty A_j\big)$.*

Proof. For every $j \in \mathbf{N}$ find a μ-measurable set B_j with $A_j \subset B_j$ and $\mu^*(A_j) = \mu(B_j)$. We set $M_k = \bigcap_{j=1}^k A_j$. Then M_k is μ-measurable $A_k \subset M_k$, and $\mu(M_k) = \mu^*(A_k)$ for every $k \in \mathbf{N}$. Moreover, $\{M_k\}$ is nondecreasing sequence of sets. Then we have

$$\lim_{k\to\infty} \mu^*(A_k) = \lim_{k\to\infty} \mu(M_k) = \mu\Big(\bigcup_{k=1}^\infty M_k\Big) \geq \mu^*\Big(\bigcup_{k=1}^\infty A_k\Big) \geq \lim_{k\to\infty} \mu^*(A_k)$$

and we are done. □

Theorem 1.5. *Let μ be a Radon measure on $\mathbf{R}^n$ and $\mathcal{F}$ be a system of closed balls in $\mathbf{R}^n$. Let A denote the set of centers of the balls in $\mathcal{F}$. Assume $\inf\{r;\ B(a,r) \in \mathcal{F}\} = 0$ for each $a \in A$. Then there exists a countable disjoint system $\mathcal{G} \subset \mathcal{F}$ such that $\mu\big(A \setminus \bigcup \mathcal{G}\big) = 0$.*

Proof. The case $\mu^*(A) < \infty$. Let N be the natural number from Theorem 1.3. Fix θ such that $1 - \frac{1}{N} < \theta < 1$.

Claim. Let $U \subset \mathbf{R}^n$ be an open set. There exists a disjoint finite system $\mathcal{H} \subset \mathcal{F}$ such that $\bigcup \mathcal{H} \subset U$ and

$$\mu^*\big((A \cap U) \setminus \bigcup \mathcal{H}\big) \le \theta \mu^*(A \cap U). \tag{1.5}$$

Proof of Claim. We may assume that $\mu^*(A \cap U) > 0$. Let $\mathcal{F}_1 = \{B \in \mathcal{F};\ \operatorname{diam} B < 1, B \subset U\}$. By Theorem 1.3 there exist disjoint families $\mathcal{G}_1, \ldots, \mathcal{G}_N \subset \mathcal{F}_1$ such that

$$A \cap U \subset \bigcup_{i=1}^{N} \bigcup \mathcal{G}_i.$$

Thus

$$\mu^*(A \cap U) \le \sum_{i=1}^{N} \mu^*\big(A \cap U \cap \bigcup \mathcal{G}_i\big).$$

Consequently, there exists an integer $1 \le j \le N$ for which

$$\mu^*\big(A \cap U \cap \bigcup \mathcal{G}_j\big) \ge \frac{1}{N} \mu^*(A \cap U) > (1-\theta)\mu^*(A \cap U).$$

Using Lemma 1.4 we find a finite system $\mathcal{H} \subset \mathcal{G}_j$ such that

$$\mu^*\big(A \cap U \cap \bigcup \mathcal{H}\big) > (1-\theta)\mu^*(A \cap U).$$

The set $\bigcup \mathcal{H}$ is μ-measurable and therefore

$$\begin{aligned} \mu^*(A \cap U) &= \mu^*\big(A \cap U \cap \bigcup \mathcal{H}\big) + \mu^*\big(A \cap U \setminus \bigcup \mathcal{H}\big) \\ &\ge (1-\theta)\mu^*(A \cap U) + \mu^*\big(A \cap U \setminus \bigcup \mathcal{H}\big). \end{aligned}$$

This gives (1.5). □

Set $U_1 = \mathbf{R}^n$. Using Claim we find a disjoint finite system $\mathcal{H}_1 \subset \mathcal{F}$ such that $\bigcup \mathcal{H}_1 \subset U_1$ and

$$\mu^*\big((A \cap U_1) \setminus \bigcup \mathcal{H}_1\big) \le \theta \mu^*(A \cap U_1).$$

Continuing by induction we obtain a sequence of open set (U_j) and finite disjoint finite systems $(\mathcal{H}_j)$ such that $U_{j+1} = U_j \setminus \bigcup \mathcal{H}_j$, $\mathcal{H}_j \subset \mathcal{F}$, $\bigcup \mathcal{H}_j \subset U_j$, and

$$\mu(A \cap U_{j+1}) = \mu^*\big((A \cap U_j) \setminus \bigcup \mathcal{H}_j\big) \le \theta \mu^*(A \cap U_j)$$

for every $j \in \mathbf{N}$. Together we have

$$\mu^*\big(A \cap U_{j+1}\big) \le \theta^j \mu^*(A)$$

for every $j \in \mathbf{N}$. Since $\mu^*(A) < \infty$ we get $\mu^*\big(A \setminus \bigcup_{j=1}^{\infty} \bigcup \mathcal{H}_j\big) = 0$. Thus we set $\mathcal{G} = \bigcup_{j=1}^{\infty} \mathcal{H}_j$ and we are done.

The general case. We find a sequence of bounded disjoint open sets $(G_j)_{j=1}^{\infty}$ such that $\mu\big(\mathbf{R}^n \setminus \bigcup_{j=1}^{\infty} G_j\big) = 0$. Then $\mu(G_j) < \infty$ for every $j \in \mathbf{N}$ and we proceed as in the proof of Theorem 1.1 □

1.2 Differentiation of measures

Notation. The symbol $\mathcal{B}$ stands for the family of all closed balls in $\mathbf{R}^n$.

Definition. Let ν and μ are measures on $\mathbf{R}^n$ and $x \in \mathbf{R}^n$. Then we define

- **upper derivative of ν with respect to μ at x** by
$$\overline{D}(\nu, \mu, x) = \lim_{r \to 0+} \left(\sup\{\nu(B)/\mu(B);\ x \in B,\ B \in \mathcal{B},\ \operatorname{diam} B < r\} \right),$$
if the term at the right side is defined,
- **lower derivative of ν with respect to μ at x** by
$$\underline{D}(\nu, \mu, x) = \lim_{r \to 0+} \left(\inf\{\nu(B)/\mu(B);\ x \in B,\ B \in \mathcal{B},\ \operatorname{diam} B < r\} \right),$$
if the term at the right side is defined,
- **derivative of ν with respect to μ at x** (denoting $D(\nu, \mu, x)$) as the common value of $\overline{D}(\nu, \mu, x)$ and $\underline{D}(\nu, \mu, x)$, if it is defined.

The end of the lecture no. 4, 22. 10. 2024

Remark. The value $\overline{D}(\nu, \mu, x)$ ($\underline{D}(\nu, \mu, x)$) is well defined if and only if
$$\forall B \in \mathcal{B},\ x \in B \colon \mu(B) > 0.$$

Theorem 1.6. *Let ν and μ be Radon measures on $\mathbf{R}^n$ and μ satisfy Vitali theorem. Then $\overline{D}(\nu, \mu, x)$ and $\underline{D}(\nu, \mu, x)$ exist μ-a.e.*

Proof. Denote
$$\begin{aligned} M &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu, \mu, x) \text{ is not defined}\}, \\ \mathcal{V} &= \{B \in \mathcal{B};\ \mu(B) = 0\}. \end{aligned}$$

The family $\mathcal{V}$ is a Vitali cover of M. We find a countable disjoint system $\mathcal{A} \subset \mathcal{V}$ such that $\mu(M \setminus \bigcup \mathcal{A}) = 0$. The we have
$$\mu\Big(\bigcup \mathcal{A}\Big) = \sum_{B \in \mathcal{A}} \mu(B) = 0,$$
therefore $\mu(M) = 0$.

The proof for $\underline{D}(\nu, \mu, x)$ is analogous. □

Theorem 1.7. *Let ν and μ be Radon measures on $\mathbf{R}^n$, μ satisfy Vitali theorem, $c \in (0, \infty)$, and $M \subset \mathbf{R}^n$.*

(i) *If for every $x \in M$ we have $\overline{D}(\nu,\mu,x) > c$, then $\nu^*(M) \geq c\mu^*(M)$.*

(ii) *If for every $x \in M$ we have $\underline{D}(\nu,\mu,x) < c$, then there exists $H \subset M$ such that $\mu(M \setminus H) = 0$ and $\nu^*(H) \leq c\mu^*(M)$.*

Proof. (i) Choose $\varepsilon > 0$. There exists an open set $G \subset \mathbf{R}^n$ with $M \subset G$ and $\nu(G) \leq \nu^*(M) + \varepsilon$. Set

$$\mathcal{V} = \{B \in \mathcal{B};\ B \subset G, \nu(B) > c\mu(B)\}.$$

The family $\mathcal{V}$ is a Vitali cover of M. There exists a disjoint countable subfamily $\mathcal{A} \subset \mathcal{V}$ with $\mu(M \setminus \bigcup \mathcal{A}) = 0$. Then we have

$$\begin{aligned}\nu^*(M) + \varepsilon \geq \nu(G) \geq \nu\big(\bigcup \mathcal{A}\big) &= \sum_{B \in \mathcal{A}} \nu(B) \\ &\geq \sum_{B \in \mathcal{A}} c\mu(B) = c\mu(\bigcup \mathcal{A}) \geq c\mu^*(M).\end{aligned}$$

Taking $\varepsilon \to 0+$ we get the desired inequality.

(ii) Choose $k \in \mathbf{N}$. There exists an open set $G_k \subset \mathbf{R}^n$ such that $M \subset G_k$ and $\mu(G_k) \leq \mu^*(M) + 1/k$. Set

$$\mathcal{V}_k = \{B \in \mathcal{B};\ B \subset G_k, \nu(B) < c\mu(B)\}.$$

The system $\mathcal{V}_k$ is a Vitali cover of M. Thus there exists a countable disjoint subfamily $\mathcal{A}_k \subset \mathcal{V}_k$ such that $\mu(M \setminus \bigcup \mathcal{A}_k) = 0$. Set $H_k = M \cap \bigcup \mathcal{A}_k$. Then $\mu(M \setminus H_k) = 0$, $H_k \subset M$ and we have

$$\begin{aligned}\nu^*(H_k) \leq \nu\big(\bigcup \mathcal{A}_k\big) = \sum_{B \in \mathcal{A}} \nu(B) \leq c \sum_{B \in \mathcal{A}} \mu(B) = c\mu\big(\bigcup \mathcal{A}\big) \\ \leq c\mu(G_k) \leq c\big(\mu^*(M) + \tfrac{1}{k}\big).\end{aligned}$$

Now we set $H = \bigcap_{k=1}^{\infty} H_k$. Then we have $\nu^*(H) \leq c\mu^*(M)$ and

$$\mu(M \setminus H) = \mu^*(M \setminus H) \leq \sum_{k=1}^{\infty} \mu^*(M \setminus H_k) = 0.$$

□

Theorem 1.8. *Let ν and μ be Radon measures on $\mathbf{R}^n$ and μ satisfies Vitali theorem. Then $D(\nu,\mu,x)$ is finite μ-a.e.*

Proof. Denote

$$\begin{aligned}D &= \{x \in \mathbf{R}^n;\ D(\nu,\mu,x) \in \langle 0,\infty)\},\\ N_1 &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu,\mu,x) \text{ is not defined}\},\\ N_2 &= \{x \in \mathbf{R}^n;\ \underline{D}(\nu,\mu,x) \text{ is not defined}\},\\ N_3 &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu,\mu,x) = \infty\},\\ N_4 &= \{x \in \mathbf{R}^n;\ \underline{D}(\nu,\mu,x) < \overline{D}(\nu,\mu,x)\}.\end{aligned}$$

Then we have

- $D = \mathbf{R}^n \setminus (N_1 \cup N_2 \cup N_3 \cup N_4)$,
- $\mu(N_1) = \mu(N_2) = 0$ (Theorem 1.6).

Further we define

$$\begin{aligned} A_k &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu, \mu, x) > k\}, \\ A(r,s) &= \{x \in \mathbf{R}^n;\ \underline{D}(\nu, \mu, x) < s < r < \overline{D}(\nu, \mu, x)\}, \quad s, r \in \mathbf{Q}^+, s < r. \end{aligned}$$

The we have

$$\begin{aligned} N_3 &= \bigcap_{k=1}^{\infty} A_k, \\ N_4 &= \bigcup \{A(r,s);\ r, s \in \mathbf{Q}^+, s < r\}. \end{aligned}$$

We show $\mu(N_3) = 0$. Choose $Q \subset N_3$ bounded. By Theorem 1.7(i) we have

$$k\mu^*(Q) \leq \nu^*(Q) < \infty$$

for every $k \in \mathbf{N}$. Therefore $\mu^*(Q) = 0$ and thus also $\mu(N_3) = 0$, since N_3 is a countable union of bounded sets.

We show $\mu(N_4) = 0$. It is sufficient to show $\mu(A(r,s)) = 0$ for every $s, r \in \mathbf{Q}^+, s < r$. Choose $Q \subset A(r,s)$ bounded. By Theorem 1.7(ii) there exists $H \subset Q$ such that $\mu(Q \setminus H) = 0$ and $\nu^*(H) \leq s\mu^*(Q)$. By Theorem 1.7(i) we have $r\mu^*(H) \leq \nu^*(H)$. We may conclude

$$r\mu^*(Q) = r\mu^*(H) \leq \nu^*(H) \leq s\mu^*(Q) < \infty.$$

Since $r > s > 0$, we have $\mu^*(Q) = 0$. This implies $\mu(A(r,s)) = 0$. □

Lemma 1.9. *Let ν and μ be Radon measures on $\mathbf{R}^n$ and μ satisfies Vitali theorem. Then the mappings $x \mapsto \overline{D}(\nu, \mu, x)$, $x \mapsto \underline{D}(\nu, \mu, x)$ are μ-measurable.*

Proof. We start with the following observation.

The set

$$M(r, \alpha) = \left\{x \in \mathbf{R}^n;\ \exists B \in \mathcal{B}\colon\ \operatorname{diam} B < r \wedge x \in B \wedge \tfrac{\nu(B)}{\mu(B)} < \alpha\right\}$$

is open for every $r > 0$ and $\alpha \in \mathbf{R}$.

If $x \in M(r, \alpha)$, then there exist $y \in \mathbf{R}^n$ and $s > 0$ with $x \in \overline{B}(y, s)$, $2s < r$,

$$\frac{\nu\big(\overline{B}(y,s)\big)}{\mu\big(\overline{B}(y,s)\big)} < \alpha.$$

We find $s' > s$ such that $2s' < r$, $\nu\big(\overline{B}(y,s')\big)/\mu\big(\overline{B}(y,s')\big) < \alpha$. Now we have $x \in B(y,s') \subset M(r,\alpha)$. This finishes the proof of the observation.

——— The end of the lecture no. 5, 29. 10. 2024 ———

Denote $D = \{x \in \mathbf{R}^n;\ \underline{D}(\nu,\mu,x) \text{ exists finite}\}$. The set D is μ-measurable by Theorem 1.8. For every $x \in D$ we have

$$\begin{aligned}
&\underline{D}(\nu,\mu,x) < \alpha \\
&\Leftrightarrow \exists \tau \in \mathbf{Q}, \tau > 0\ \forall r \in \mathbf{Q}, r > 0\ \exists B \in \mathcal{B}\colon\ \operatorname{diam} B < r,\ x \in B, \frac{\nu(B)}{\mu(B)} < \alpha - \tau \\
&\Leftrightarrow \exists \tau \in \mathbf{Q}, \tau > 0\ \forall r \in \mathbf{Q}, r > 0 :\ x \in M(r, \alpha - \tau).
\end{aligned}$$

The set $\{x \in \mathbf{R}^n;\ \underline{D}(\nu,\mu,x) < \alpha\}$ is intersection of D with a Borel set. This implies that the mapping $x \mapsto \underline{D}(\nu,\mu,x)$ is μ-measurable.

Measurability of the mapping $x \mapsto \overline{D}(\nu,\mu,x)$ can be proved analogously. □

Theorem 1.10. *Let ν and μ be Radon measures on $\mathbf{R}^n$, μ satisfy Vitali theorem, $\nu \ll \mu$, and $B \subset \mathbf{R}^n$ be μ-measurable. Then we have*

$$\int_B D(\nu,\mu,x)\,d\mu(x) = \nu(B).$$

Proof. Choose $\beta \in \mathbf{R}$, $\beta > 1$. Define

$$\begin{aligned}
B_k &= \{x \in B;\ \beta^k < D(\nu,\mu,x) \le \beta^{k+1}\}, \qquad k \in \mathbf{Z}, \\
N &= \{x \in B;\ D(\nu,\mu,x) = 0\}.
\end{aligned}$$

These sets are μ-measurable by Lemma 1.9. Using Theorem 1.8 we have

$$\mu\Big(B \setminus \big(\bigcup_{k=-\infty}^{\infty} B_k \cup N\big)\Big) = 0.$$

Then we have

$$\begin{aligned}
\int_B D(\nu,\mu,x)\,d\mu(x) &= \sum_{k=-\infty}^{\infty} \int_{B_k} D(\nu,\mu,x)\,d\mu(x) \le \sum_{k=-\infty}^{\infty} \beta^{k+1}\mu(B_k) \\
&\overset{\text{Theorem 1.7(i)}}{\le} \sum_{k=-\infty}^{\infty} \beta^{k+1}\beta^{-k}\nu(B_k) \le \beta\nu(B).
\end{aligned}$$

Going $\beta \to 1+$ we get

$$\int_B D(\nu,\mu,x)\,d\mu(x) \le \nu(B).$$

Now let $\beta > 1$ again. Define

$$C_k = \{x \in B;\ \beta^k \le D(\nu,\mu,x) < \beta^{k+1}\}, \qquad k \in \mathbf{Z}.$$

Besides the equality

$$\mu\Big(B \setminus (\bigcup_{k=-\infty}^{\infty} C_k \cup N)\Big) = 0,$$

we have also $\nu\big(B \setminus (\bigcup_{k=-\infty}^{\infty} C_k \cup N)\big) = 0$, since $\nu \ll \mu$. By Theorem 1.7(ii) and absolute continuity of ν with respect to μ we obtain $\nu^*(Q) \le c\mu^*(Q) < \infty$ for any $c > 0$ and $Q \subset N$ bounded. Similarly as in the proof of Theorem 1.8 we get $\nu(N) = 0$. Then we have

$$\int_B D(\nu,\mu,x)\,d\mu(x) \ge \sum_{k=-\infty}^{\infty} \int_{C_k} D(\nu,\mu,x)\,d\mu(x) \ge \sum_{k=-\infty}^{\infty} \beta^k \mu(C_k)$$
$$\overset{\text{Theorem 1.7(ii)}}{\ge} \sum_{k=-\infty}^{\infty} \beta^k \beta^{-(k+1)} \nu(C_k) = \frac{1}{\beta}\nu(B).$$

Now it follows $\int_B D(\nu,\mu,x)\,d\mu(x) \ge \nu(B)$. □

1.3 Lebesgue points

Definition. Let μ be a Radon measure on $\mathbf{R}^n$. The symbol $\mathcal{L}^1_{loc}(\mu)$ denotes the set of all functions $f\colon \mathbf{R}^n \to \mathbf{C}$, which are μ-measurable and for every $x \in \mathbf{R}^n$ there exists $r > 0$ such that $\int_{B(x,r)} |f(t)|\,d\mu(t) < \infty$.

Definition. Let $f \in \mathcal{L}^1_{loc}(\mu)$. We say that $x \in \mathbf{R}^n$ is **Lebesgue point of f (with respect to μ)**, if it holds

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall B \in \mathcal{B}, x \in B, \operatorname{diam} B < \delta\colon\ \frac{\int_B |f(t) - f(x)|\,d\mu(t)}{\mu(B)} < \varepsilon.$$

Theorem 1.11. *Let μ be a Radon measure on $\mathbf{R}^n$ satisfying Vitali theorem and $f \in \mathcal{L}^1_{loc}(\mu)$. Then μ-a.e. points of f are Lebesgue points.*

Proof. Without any loss of generality we may assume that $\mu(\mathbf{R}^n) < \infty$ and $f \in \mathcal{L}^1(\mu)$. Let (C_k) be a sequence of closed discs in $\mathbf{C}$, which forms a basis of $\mathbf{C}$. We denote

$$g_k(x) := \operatorname{dist}(f(x), C_k), \qquad x \in \mathbf{R}^n.$$

The function g_k is nonnegative μ-measurable function satisfying $g_k \in \mathcal{L}^1(\mu)$. Let $\nu_k = \int g_k\,d\mu$. By Theorem 1.10 we have $D(\nu_k,\mu,x) = g_k(x)$ μ-a.e. Denote

$$P_k = \{x \in f^{-1}(C_k);\ \neg(D(\nu_k,\mu,x) = 0)\}.$$

We have $g_k = 0$ on $f^{-1}(C_k)$, therefore $\mu(P_k) = 0$. We show that every point from $\mathbf{R}^n \setminus \bigcup_{k=1}^{\infty} P_k$ is a Lebesgue point of f.

Let $x \in \mathbf{R}^n \setminus \bigcup_{k=1}^{\infty} P_k$. Choose $\varepsilon > 0$. We find C_k such that $f(x) \in C_k$ and $C_k \subset B(f(x), \varepsilon/2)$. For any $t \in \mathbf{R}^n$ it holds

$$|f(t) - f(x)| \leq g_k(t) + \varepsilon.$$

There exists $\delta > 0$ such that

$$\forall B \in \mathcal{B},\ x \in B,\ \operatorname{diam} B < \delta: \ \frac{\int_B g_k(t)\, d\mu(t)}{\mu(B)} < \varepsilon,$$

since $D(\nu_k, \mu, x) = 0$. Take $B \in \mathcal{B}$ with $x \in B$, $\operatorname{diam} B < \delta$ we get

$$\frac{\int_B |f(t) - f(x)|\, d\mu(t)}{\mu(B)} \leq \frac{\int_B g_k(t)\, d\mu(t) + \varepsilon\mu(B)}{\mu(B)} < 2\varepsilon.$$

This finishes the proof. □

1.4 Density theorem

Definition. Let μ be a measure on $\mathbf{R}^n$, $A \subset \mathbf{R}^n$ be μ-measurable, and $x \in \mathbf{R}^n$. We say that $c \in [0,1]$ is **μ-density of the set A at x**, if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall B \in \mathcal{B},\ x \in B,\ \operatorname{diam} B < \delta \colon \left| \frac{\mu(A \cap B)}{\mu(B)} - c \right| < \varepsilon.$$

We denote $\mathrm{d}_\mu(A, x) = c$.

The end of the lecture no. 6, 12. 11. 2024

Theorem 1.12 (Lebesgue)**.** *Let μ be a Radon measure on $\mathbf{R}^n$ satisfying Vitali theorem and $M \subset \mathbf{R}^n$ be μ-measurable. Then*

- $\mathrm{d}_\mu(M, x) = 1$ *for μ-a.e. $x \in M$,*
- $\mathrm{d}_\mu(M, x) = 0$ *for μ-a.e. $x \in \mathbf{R}^n \setminus M$.*

Proof. Define ν on $\mathbf{R}^n$ by

$$\nu(A) = \mu(A \cap M) \quad \text{for every } A \subset \mathbf{R}^n \ \mu\text{-measurable}.$$

Then we have

- $\mathrm{d}_\mu(M, x) = D(\nu, \mu, x)$, if at least one term is well defined,
- $\nu \ll \mu$,
- $\nu = \int \chi_M \, d\mu$.

By Theorem 1.10 we have $\nu = \int D(\nu, \mu, x)\, d\mu(x)$ therefore $\mathrm{d}_\mu(M, x) = D(\nu, \mu, x) = \chi_M(x)$ μ-a.e. □

1.5 AC and BV functions

Remark. For $a, c, b \in \mathbf{R}, a < c < b$, it holds

- $\mathrm{V}_a^b f = \mathrm{V}_a^c f + \mathrm{V}_c^b f$,
- $|f(b) - f(a)| \leq \mathrm{V}_a^b f$.

Example. Let f be a function with continuous derivative on an interval $[a, b]$. Then $\mathrm{V}_a^b f = \int_a^b |f'(x)|\, dx$.

Remark. Let I be a closed nonempty interval. Then we have

(a) $f, g \in AC(I) \Rightarrow f + g \in AC(I)$,

(b) $f \in AC(I), \alpha \in \mathbf{R} \Rightarrow \alpha f \in AC(I)$.

Theorem 1.13. *Let $f\colon [a, b] \to \mathbf{R}$, $a < b$. Then f is absolutely continuous on $[a, b]$ if and only if f is difference of of two nondecreasing absolutely continuous functions on $[a, b]$.*

Proof. $\Rightarrow$ We denote $v(x) = \mathrm{V}_a^x f$, $x \in [a, b]$. The function v is well defined since $f \in BV([a, x])$, $x \in [a, b]$. For every $x, y \in I := [a, b]$, $x < y$, we have $v(y) - v(x) = V_x^y f$.

The function v is nondecreasing. This is obvious.

The function $v - f$ is nondecreasing. For every $x, y \in I, x < y$ we have

$$\big(v(y) - f(y)\big) - \big(v(x) - f(x)\big) = \big(v(y) - v(x)\big) - \big(f(y) - f(x)\big) = V_x^y f - \big(f(y) - f(x)\big) \geq 0.$$

The function v is absolutely continuous. Choose $\varepsilon > 0$. We find $\delta > 0$ such that

$$\sum_{j=1}^{m} |f(b_j) - f(a_j)| < \varepsilon,$$

whenever $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$ are points from $I = [a, b]$ with $\sum_{j=1}^{m}(b_j - a_j) < \delta$. Now assume that we have points $A_1 < B_1 \leq A_2 < B_2 \leq \cdots \leq A_p < B_p$ from I satisfying $\sum_{j=1}^{p}(B_j - A_j) < \delta$. For each $j \in \{1, \dots, p\}$ we find points

$$A_j = a_1^j < b_1^j = a_2^j < b_2^j = \cdots < b_{m_j}^j = B_j$$

such that

$$v(B_j) - v(A_j) = V_{A_j}^{B_j} f < \sum_{i=1}^{m_j} |f(b_i^j) - f(a_i^j)| + \frac{\varepsilon}{p}.$$

The we have

$$\sum_{j=1}^{p} \sum_{i=1}^{m_j} (b_i^j - a_i^j) = \sum_{j=1}^{p} (B_j - A_j) < \delta$$

and

$$\sum_{j=1}^{p}|v(B_j)-v(A_j)|<\sum_{j=1}^{p}\Big(\sum_{i=1}^{m_j}|f(b_i^j)-f(a_i^j)|+\frac{\varepsilon}{p}\Big)<\varepsilon+\varepsilon=2\varepsilon$$

Now we can write $f=v-(v-f)$. □

Remark. Let $F\colon \mathbf{R}\to\mathbf{R}$ be nondecreasing function which is continuous at each point from the right. Then there exists a Radon measure ν_F such that F is the distribution function of ν_F, i.e.,

$$\nu_F\big((a,b]\big)=F(b)-F(a),\qquad a,b\in\mathbf{R},a<b.$$

Lemma 1.14. *Let $f\colon (a,b)\to\mathbf{R}$, $x_0\in(a,b)$, and $f'(x_0)\in\mathbf{R}$. Then we have*

$$\lim_{\substack{[x_1,x_2]\to[x_0,x_0]\\ x_1\le x_0\le x_2,x_1\neq x_2}}\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(x_0).$$

Lemma 1.15. *Let $f\colon (a,b)\to\mathbf{R}$ be nondecreasing on (a,b), $C(f)$ be the set of all points of continuity of f, and $A\in\mathbf{R}$. Then for every $x_0\in C(f)$ it holds*

$$f'(x_0)=A\Leftrightarrow\lim_{\substack{[x_1,x_2]\to[x_0,x_0]\\ x_1\le x_0\le x_2,x_1\neq x_2\\ x_1,x_2\in C(f)}}\frac{f(x_2)-f(x_1)}{x_2-x_1}=A.$$

The end of the lecture no. 7, 19. 11. 2024

Lemma 1.16. *Let f be a distribution function of a Radon measure μ on $\mathbf{R}$, $x_0\in C(f)$, $A\in\mathbf{R}$. Then*

$$f'(x_0)=A\Leftrightarrow D(\mu,\lambda_1,x_0)=A.$$

Theorem 1.17 (Lebesgue)**.** *Let f be a monotone function on an interval I. Then we have*

(a) *$f'(x)$ exists a.e. in I,*

(b) *f' is measurable and $\left|\int_a^b f'\right|\le|f(b)-f(a)|$, whenever $a,b\in I,a<b$,*

(c) *$f'\in\mathcal{L}^1_{loc}(I)$.*

Proof. Without any loss of generality we may assume that f is nondecreasing. Let $a,b\in I$, $a<b$. We define

$$g(x)=\begin{cases}\lim_{t\to a+}f(t), & x\in(-\infty,a]\\ \lim_{t\to x+}f(t), & x\in(a,b),\\ f(b), & x\in[b,\infty).\end{cases}$$

The function g is nondecreasing, continuous from the right at each point of $\mathbf{R}$, and $\{x\in(a,b)\ f(x)\neq g(x)\}$ is countable. By Remark there exists a Radon measure ν on $\mathbf{R}$ such that

$$\forall c,d\in\mathbf{R},c<d\colon \nu((c,d])=g(d)-g(c).$$

We find Radon measures μ,σ such that $\nu=\sigma+\mu$, $\sigma\ll\lambda$, and $\mu\perp\lambda$.

Claim. We have $D(\mu, \lambda, x) = 0$ λ-a.e.

Proof of Claim. There exists a Borel set N such that $\lambda(N) = 0$ and $\mu(\mathbf{R} \setminus N) = 0$. Denote $D = \{x \in \mathbf{R} \setminus N;\ D(\mu, \lambda, x) > c\}$. Then we have $0 = \mu(D) \geq c\lambda(D)$. This implies $\lambda(D) = 0$, and, consequently, $\lambda(\{x \in \mathbf{R} \setminus N;\ D(\mu, \lambda, x) > 0\}) = 0$. This gives the claim. □

Lemma 1.16 gives $g'(x) = D(\nu, \lambda, x)$ λ-a.e. in $[a, b]$, since g is continuous at each point $[a, b]$ except a countable set. For every $x_0 \in (a, b) \cap C(f)$ we have $f'(x_0) = A \in \mathbf{R}$ if and only if $g'(x_0) = A \in \mathbf{R}$ (Lemma 1.15), since $f(t) = g(t)$ whenever $t \in C(f) \cap (a, b)$. This implies (a).

(b) We have

$$\begin{aligned} f(b) - f(a) &\geq g(b) - g(a) = \nu((a, b]) \geq \sigma((a, b]) \\ &= \int_a^b D(\sigma, \lambda, x)\, d\lambda(x) \overset{\text{Claim}}{=} \int_a^b D(\nu, \lambda, x)\, d\lambda(x). \end{aligned}$$

(c) This follows from (b). □

Theorem 1.18. *Let I be a nonempty interval and $f \in BV(I)$. Then $f'(x)$ exists finite a.e. in I.*

Theorem 1.19. *Let $f\colon [a, b] \to \mathbf{R}$, $a < b$. Then the following assertions are equivalent.*

(i) $f \in AC([a, b])$.

(ii) *We have $\varphi \in \mathcal{L}^1([a, b])$ such that*

$$f(x) = f(a) + \int_a^x \varphi(t)\, dt, \qquad x \in [a, b].$$

(iii) *$f'(x)$ exists a.e. in $[a, b]$, $f' \in \mathcal{L}^1([a, b])$ and*

$$f(x) = f(a) + \int_a^x f'(t)\, dt, \qquad x \in [a, b].$$

——— The end of the lecture no. 8, 26. 11. 2024 ———

Theorem 1.20 (per partes for Lebesgue integral)**.** *Let* $f, g \in AC([a, b])$. *Then we have*

$$\int_a^b f'g = [fg]_a^b - \int_a^b fg'.$$

Theorem 1.21. *Let* g *be a nonnegative function on* $[a, b]$ *with* $g \in \mathcal{L}^1([a, b])$. *Let* f *be a continuous function on* $[a, b]$. *Then there exists* $\xi \in [a, b]$ *such that*

$$\int_a^b fg = f(\xi) \int_a^b g.$$

Theorem 1.22. *Let* $f \in \mathcal{L}^1([a, b])$ *and* g *be a monotone function on* $[a, b]$. *Then there exists* $\xi \in [a, b]$ *such that*

$$\int_a^b fg = g(a) \int_a^\xi f + g(b) \int_\xi^b f.$$

1.6 Rademacher theorem

Definition. Let $M \subset \mathbf{R}^n$. We say that $f\colon M \to \mathbf{R}$ is **Lipschitz (on M)**, if there exists $K > 0$ such that

$$\forall x, y \in M\colon |f(x) - f(y)| \leq K||x - y||.$$

Remark. If f is Lipschitz on M, then f is continuous on M.

Theorem 1.23. *Let $G \subset \mathbf{R}^n$ be open nonempty and $f\colon G \to \mathbf{R}$ be Lipschitz on G. Then f is differentiable a.e. on G.*

Lemma 1.24. *Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be continuous and $i \in \{1, \dots, n\}$. Then the set*

$$D_i = \left\{x \in \mathbf{R}^n;\ \tfrac{\partial f}{\partial x_i}(x) \text{ exists}\right\}$$

is Borel.

Proof. We have

$$\begin{aligned}
&\frac{\partial f}{\partial x_i}(x) \text{ exists}\\
&\Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0\ \forall t_1, t_2 \in (-\delta, \delta) \setminus \{0\}\colon \left|\tfrac{f(x+t_1e_i)-f(x)}{t_1} - \tfrac{f(x+t_2e_i)-f(x)}{t_2}\right| < \varepsilon\\
&\Leftrightarrow \forall \varepsilon \in \mathbf{Q}^+\ \exists \delta \in \mathbf{Q}^+\ \forall t_1, t_2 \in \big((-\delta, \delta) \cap \mathbf{Q}\big) \setminus \{0\}\colon \left|\tfrac{f(x+t_1e_i)-f(x)}{t_1} - \tfrac{f(x+t_2e_i)-f(x)}{t_2}\right| < \varepsilon.
\end{aligned}$$

For $\varepsilon > 0$ and nonzero t_1, t_2 denote

$$D(\varepsilon, t_1, t_2) = \left\{x \in \mathbf{R}^n;\ \left|\tfrac{f(x+t_1e_i)-f(x)}{t_1} - \tfrac{f(x+t_2e_i)-f(x)}{t_2}\right| < \varepsilon\right\}.$$

The set $D(\varepsilon, t_1, t_2)$ is open since f is continuous. We have

$$D_i = \bigcap_{\varepsilon \in \mathbf{Q}^+} \bigcup_{\delta \in \mathbf{Q}^+} \bigcap_{\substack{t_1 \in (-\delta,\delta)\cap\mathbf{Q}\\ t_1 \neq 0}} \bigcap_{\substack{t_2 \in (-\delta,\delta)\cap\mathbf{Q}\\ t_2 \neq 0}} D(\varepsilon, t_1, t_2),$$

therefore D_i is Borel. □

The end of the lecture no. 9, 3. 12. 2024

Lemma 1.25. *Let $\beta > 0$, $A \neq \emptyset$, $f_\alpha, \alpha \in A$, be β-Lipschitz function on $\mathbf{R}^n$ and $x \in \mathbf{R}^n$ be such that $\sup_{\alpha\in A} f_\alpha(x)$ is finite. Then the function $z \mapsto \sup_{\alpha\in A} f_\alpha(z)$ is β-Lipschitz on $\mathbf{R}^n$.*

Proof. Let $u, v \in \mathbf{R}^n$. Then $|f_\gamma(u) - f_\gamma(x)| \leq \beta||u-x||$ for any $\gamma \in A$, therefore

$$f_\gamma(u) \leq f_\gamma(x) + \beta||u-x|| \leq \sup_{\alpha\in A} f_\alpha(x) + \beta||u-x||.$$

This implies

$$\sup_{\gamma\in A} f_\gamma(u) \leq \sup_{\alpha\in A} f_\alpha(x) + \beta||u-x||,$$

thus $\sup_{\gamma\in A} f_\gamma(u) \in \mathbf{R}$. Further we have

$$f_\gamma(u) \leq f_\gamma(v) + \beta||u-v|| \leq \sup_{\alpha\in A} f_\alpha(v) + \beta||u-v|| \quad \text{for every } \gamma \in A.$$

We get

$$\sup_{\gamma\in A} f_\gamma(u) \leq \sup_{\alpha\in A} f_\alpha(v) + \beta||u-v||.$$

Thus we have

$$\sup_{\alpha\in A} f_\alpha(u) - \sup_{\alpha\in A} f_\alpha(v) \leq \beta||u-v||.$$

Interchanging the roles of u and v we obtain

$$\sup_{\alpha\in A} f_\alpha(v) - \sup_{\alpha\in A} f_\alpha(u) \leq \beta||u-v||,$$

which proves β-Lipschitzness. □

Lemma 1.26. *Let $\beta > 0$, $E \subset \mathbf{R}^n$ be nonempty and $f\colon E \to \mathbf{R}$ be β-Lipschitz. Then there exists β-Lipschitz function $\tilde{f}\colon \mathbf{R}^n \to \mathbf{R}$ with $\tilde{f}|_E = f$.*

Proof. The function $f_x\colon y \mapsto f(x) - \beta \cdot ||y-x||$ is β-Lipschitz for every $x \in E$ since

$$|f_x(u) - f_x(v)| = \big|\beta \cdot ||u-x|| - \beta \cdot ||v-x||\big| \leq \beta||u-v||$$

for every $u, v \in \mathbf{R}^n$. For every $y \in E$ we have $\sup_{x\in E} f_x(y) \leq f(y)$. Using Lemma 1.25 we get the mapping defined by

$$\tilde{f}(y) = \sup_{x\in E}(f(x) - \beta||y-x||)$$

is β-Lipschitz on $\mathbf{R}^n$. For $z \in E$ we have $\tilde{f}(z) \geq f_z(z) = f(z)$. Moreover $f_x(z) = f(x) - \beta||z-x|| \leq f(z)$, which gives $\tilde{f}(z) \leq f(z)$. Thus we prove $\tilde{f}(z) = f(z)$. □

Proof of Theorem 1.23. By Lemma 1.26 we may suppose that f is Lipschitz with the constant β on $\mathbf{R}^n$, i.e.,

$$\forall x, y \in \mathbf{R}^n\colon |f(x) - f(y)| \leq \beta||x-y||.$$

We show that f is differentiable a.e. This gives also the statement of the theorem. Let $E \subset \mathbf{R}^n$ be a set of those points where at least one partial derivative does not exist. The set $\mathbf{R}^n \setminus D_i$ is by Lemma 1.24 measurable. We use Fubini theorem and Rademacher theorem for $n = 1$ (see Remark) to get $\lambda_n(\mathbf{R}^n \setminus D_i) = 0$. Then we have $\lambda_n(E) = 0$, since $E = \bigcup_{i=1}^n (\mathbf{R}^n \setminus D_i)$.

For $p, q \in \mathbf{Q}^n$, $m \in \mathbf{N}$, denote

$$S(p,q,m) = \Big\{x \in \mathbf{R}^n;\ \forall i \in \{1,\dots,n\}\ \forall t \in (-1/m, 1/m) \setminus \{0\} \colon p_i \le \tfrac{f(x+te_i)-f(x)}{t} \le q_i\Big\}.$$

It is easy to verify that the set $S(p,q,m)$ is Borel. Let $\tilde{S}(p,q,m)$ be the set of all points of $S(p,q,m)$, where $S(p,q,m)$ has density 1. Then Theorem 1.12 gives

$$\lambda_n\big(S(p,q,m) \setminus \tilde{S}(p,q,m)\big) = 0.$$

The set

$$N = \bigcup \{S(p,q,m) \setminus \tilde{S}(p,q,m);\ p, q \in \mathbf{Q}^n, m \in \mathbf{N}\}$$

is of measure zero.

We show that f is differentiable at each point $x \in \mathbf{R}^n \setminus (E \cup N)$. Take $x \in \mathbf{R}^n \setminus (E \cup N)$ and $\varepsilon \in (0,1)$. Choose $p, q \in \mathbf{Q}^n$ such that

$$q_i - \varepsilon < p_i < \frac{\partial f}{\partial x_i}(x) < q_i, \quad i = 1, \dots, n.$$

Then there is $m \in \mathbf{N}$ such that $x \in S(p,q,m)$. Since $x \notin N$, the point x is a point of density of the set $S(p,q,m)$. Denote $S = S(p,q,m)$.

We find $\delta \in (0, 1/m)$ such that

$$\lambda_n\big(B(x,r) \setminus S\big) \le \left(\tfrac{\varepsilon}{2}\right)^n \lambda_n(B(x,r))$$

for every $r \in (0, 2\delta)$. Notice that the set $B(x, (1+\varepsilon)\tau) \setminus S$ does not contain a ball with radius $\varepsilon\tau$, whenever $\tau \in (0,\delta)$. Otherwise it would hold

$$c_n(\varepsilon\tau)^n \le (\varepsilon/2)^n c_n (1+\varepsilon)^n \tau^n,$$

a contradiction. (The symbol c_n denotes n-dimensional measure of the unit ball.)

Choose $y \in B(x,\delta)$, $y \ne x$. Denote

$$y^i = [y_1, y_2, \dots, y_i, x_{i+1}, \dots, x_n].$$

For every $i \in \{0, \dots, n\}$ define a ball $B_i = B(y^i, \varepsilon||y-x||)$. Using the preceding observation we have $B_i \cap S \ne \emptyset$. Find points $z^i \in S \cap B_i$, $i = 0, \dots, n-1$, and denote $w^i = z^{i-1} + (y_i - x_i)e_i$, $i = 1, \dots, n$.

Then we have

$$p_i \le \frac{f(w^i) - f(z^{i-1})}{y_i - x_i} \le q_i \quad \text{if } x_i \ne y_i,$$
$$p_i < \frac{\partial f}{\partial x_i}(x) < q_i,$$

therefore

$$\left|f(w^i) - f(z^{i-1}) - \frac{\partial f}{\partial x_i}(x)(y_i - x_i)\right| \le (q_i - p_i)|y_i - x_i| \le \varepsilon\|y - x\|.$$

Then we have

$$\begin{aligned}
&\left|f(y) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)(y_i - x_i)\right| \\
&\le \sum_{i=1}^{n}\left|f(w^i) - f(z^{i-1}) - \frac{\partial f}{\partial x_i}(x)(y_i - x_i)\right| + \sum_{i=1}^{n}(|f(y^i) - f(w^i)| + |f(z^{i-1}) - f(y^{i-1})|) \\
&\le n\varepsilon||y - x|| + 2n\beta\varepsilon||y - x|| = \varepsilon(n + 2n\beta)||y - x||,
\end{aligned}$$

thus the proof is finished. □

Remark. Let us mention the following two deep results of D. Preiss ([3]).

1. Let H be a Hilbert space and $f\colon H \to \mathbf{R}$ be Lipschitz. Then there exists $x \in H$, where f is *Fréchet differentiable*, i.e., there exists a continuous linear mapping $L\colon H \to \mathbf{R}$ such that
$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - L(h)|}{||h||} = 0.$$

2. There exists a closed measure zero set $F \subset \mathbf{R}^2$ such that any Lipschitz function on $\mathbf{R}^2$ is differentiable at some point of F.

The end of the lecture no. 10, 10. 12. 2024

1.7 Lipschitz functions and $W^{1,\infty}$

Remark. We have

$$W^{1,\infty}(\Omega) = \big\{u \in L^\infty(\Omega);\ \partial_i u \in L^\infty(\Omega) \text{ (in the sense of distributions)}, i \in \{1, \dots, n\}\big\}.$$

Theorem 1.27. *Let $U \subset \mathbf{R}^n$ be open. Then $f\colon U \to \mathbf{R}$ is local Lipschitz on U if and only if $f \in W^{1,\infty}_{\mathrm{loc}}(U)$.*

Without proof.

1.8 Maximal operator

Definition. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be measurable. For $x \in \mathbf{R}^n$ we define

$$Mf(x) = \sup_{B \in \mathcal{B}, x \in B} \frac{1}{\lambda_n(B)} \int_B |f|.$$

Theorem 1.28 (Hardy-Littlewood-Wiener)**.**

(a) *If $f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, then Mf is finite a.e.*

(b) *There exists $c > 0$ such that for every $f \in L^1(\mathbf{R}^n)$ and $\alpha > 0$ we have*

$$\lambda_n(\{x \in \mathbf{R}^n;\ Mf(x) > \alpha\}) \leq \frac{c}{\alpha}\|f\|_1.$$

(c) *Let $p \in (1, \infty]$. Then there exists A such that for every $f \in L^p(\mathbf{R}^n)$ we have $\|Mf\|_p \leq A\|f\|_p$.*

Chapter 2

Hausdorff measures

2.1 Basic notions

Convention. We will assume that (P, ρ) is a metric space.

Definition. Let $p > 0$, $A \subset P$. Denote

$$\begin{aligned}\mathcal{H}_p(A, \delta) &= \inf\Big\{\sum_{j=1}^{\infty}(\operatorname{diam} A_j)^p;\ A \subset \bigcup_{j=1}^{\infty} A_j,\ \operatorname{diam} A_j \le \delta\Big\}, \qquad \delta > 0;\\ \mathcal{H}_p(A) &= \sup_{\delta>0} \mathcal{H}_p(A, \delta).\end{aligned}$$

The function $A \mapsto \mathcal{H}_p(A)$ is called **p-dimensional outer Hausdorff measure**.

Remark. Definice $\mathcal{H}_s$ se nezmění, pokud budeme uvažovat A_n uzavřené (resp. otevřené).

Definition. Outer measure γ on P is called **metric**, if for every $A, B \subset P$ with $\inf\{\rho(x, y);\ x \in A, y \in B\} > 0$ we have $\gamma(A \cup B) = \gamma(A) + \gamma(B)$.

Theorem 2.1. *Let γ be a metric outer measure on P. Then every Borel subset of P is γ-measurable.*

Proof. We have that γ-measurable sets form σ-algebra. Therefore it is sufficient to prove that closed sets are γ-measurable. Nechť tedy $F \subset P$ je uzavřená. Vezměme testovací množinu $T \subset P$. Bez újmy na obecnosti můžeme předpokládat, že $\gamma(T) < \infty$, neboť chceme dokázat nerovnost

$$\gamma(T) \ge \gamma(T \cap F) + \gamma(T \setminus F).$$

Označme

$$\begin{aligned}P_0 &= \{x \in T;\ \operatorname{dist}(x, F) \ge 1\},\\ P_j &= \{x \in T;\ \tfrac{1}{j+1} \le \operatorname{dist}(x, F) < \tfrac{1}{j}\} \text{ pro } j \in \mathbf{N}.\end{aligned}$$

Množiny $P_0, P_2, P_4, \dots$ mají od sebe navzájem kladné vzdálenosti, a tedy pro libovolné $m \in \mathbf{N}$ platí

$$\sum_{j=0}^{m} \gamma(P_{2j}) = \gamma(\bigcup_{j=0}^{m} P_{2j}) \leq \gamma(T).$$

Podobně

$$\sum_{j=0}^{m} \gamma(P_{2j+1}) = \gamma(\bigcup_{j=0}^{m} P_{2j+1}) \leq \gamma(T).$$

Dostáváme tak, že řada $\sum_{j=0}^{\infty} \gamma(P_j)$ je konvergentní. Dále platí, že vzdálenost $T \cap F$ a $\bigcup_{j=0}^{m} P_j$ je kladná. Máme tedy

$$\gamma((T \cap F) \cup \bigcup_{j=0}^{m} P_j) = \gamma(T \cap F) + \gamma(\bigcup_{j=0}^{m} P_j).$$

Dostáváme tak

$$\begin{aligned}
\gamma(T \cap F) + \gamma(T \setminus F) &= \gamma(T \cap F) + \gamma(\bigcup_{j=0}^{\infty} P_j) \\
&\leq \gamma(T \cap F) + \gamma(\bigcup_{j=0}^{m} P_j) + \gamma(\bigcup_{j=m+1}^{\infty} P_j) \\
&\leq \gamma((T \cap F) \cup \bigcup_{j=0}^{m} P_j) + \gamma(\bigcup_{j=m+1}^{\infty} P_j) \\
&\leq \gamma((T \cap F) \cup \bigcup_{j=0}^{m} P_j) + \sum_{j=m+1}^{\infty} \gamma(P_j) \\
&\leq \gamma(T) + \sum_{j=m+1}^{\infty} \gamma(P_j).
\end{aligned}$$

Pro $m \to \infty$ se poslední člen blíží k nule, a dostáváme tak dokazovanou nerovnost. □

Theorem 2.2. *$\mathcal{H}_p$ is a metric outer measure.*

Proof. Není těžké ukázat, že funkce $A \mapsto \mathcal{H}_p(A, \delta)$ je vnější míra. Limitní přechod $\delta \to 0+$, pak dává, že $\mathcal{H}_p$ je vnější míra.

Nechť nyní $A, B \subset P$ a $\inf\{\rho(a, b);\ a \in A, b \in B\} = \delta_0 > 0$. Pokud nyní $C \subset A \cup B$ a $\operatorname{diam} C < \delta_0$, pak $C \subset A$ nebo $C \subset B$. Máme tedy

$$\mathcal{H}_p(A \cup B, \delta) = \mathcal{H}_p(A, \delta) + \mathcal{H}_p(B, \delta)$$

pro libovolné $\delta \in (0, \delta_0)$. Odtud

$$\mathcal{H}_p(A \cup B) = \mathcal{H}_p(A) + \mathcal{H}_p(B).$$

□

Corollary 2.3. *Every Borel subset of P is $\mathcal{H}_p$-measurable.*

Theorem 2.4. *Let $k, n \in \mathbf{N}$, $k \leq n$, $K = [0,1)^k \times \{0\}^{n-k} \subset \mathbf{R}^n$. Then $0 < \mathcal{H}_k(K) < \infty$.*

Proof. Zvolme $\delta > 0$. K němu nalezneme $m \in \mathbf{N}$ takové, že $\frac{\sqrt{k}}{m} < \delta$. Krychli $[0,1]^k$ rozdělíme na m^k nepřekrývajících se krychlí $K_1, K_2, \dots, K_{m^k}$, jejichž hrany délky $1/m$ jsou rovnoběžné se souřadnými osami. Diametr těchto krychlí je $\sqrt{k}/m$. Potom

$$\mathcal{H}_k(K,\delta) \leq \sum_{j=1}^{m^k} (\operatorname{diam}(K_j \times \{0\}^{n-k})^k = m^k \cdot \frac{k^{k/2}}{m^k} = k^{k/2}.$$

Odtud plyne $\mathcal{H}_k(K) < \infty$.

Nechť $\pi : \mathbf{R}^n \to \mathbf{R}^k$ je projekce $\pi(x_1, \dots, x_n) = [x_1, \dots, x_k]$. Označme $\lambda(A) = \lambda_k(\pi(A \cap K))$. Pokud $A \subset \mathbf{R}^n$, pak

$$\lambda(A) \leq 2^k (\operatorname{diam} A)^k.$$

Nechť (A_j) je posloupnost podmnožin K taková, že $\bigcup A_j = K$. Potom

$$\sum_{j=1}^{\infty} (\operatorname{diam} A_j)^k \geq 2^{-k} \sum_{j=1}^{\infty} \lambda(A_j) \geq 2^{-k} \lambda(K) = 2^{-k}.$$

Takže platí $\mathcal{H}_k(K) \geq 2^{-k}$. □

Remark. It can be shown that $\kappa_k := \mathcal{H}_k\big([0,1]^k \times \{0\}^{n-k}\big) = (4/\pi)^{k/2} \Gamma(1 + \frac{k}{2})$.

Definition. Let $k \in \mathbf{N}$. The **k-dimensional normalized Hausdorff measure** is defined by $H^k = \frac{1}{\kappa_k} \mathcal{H}_k$.

Theorem 2.5 (regularity of Hausdorff measure)**.** *Let $k, n \in \mathbf{N}, k \leq n$, and $A \subset \mathbf{R}^n$. Then there exists a Borel set $B \subset \mathbf{R}^n$ such that $A \subset B$ and $H^k(A) = H^k(B)$.*

Theorem 2.6. *Let $n \in \mathbf{N}$ and $A \subset \mathbf{R}^n$. Then $H^n(A) = \lambda^{n*}(A)$.*

2.2 Area formula

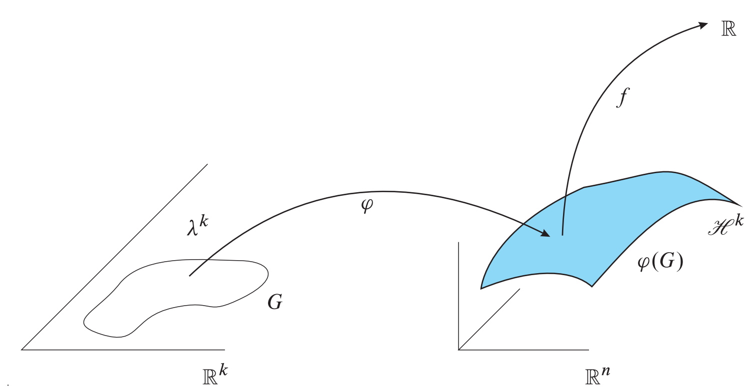
Notation. Let $k, n \in \mathbf{N}, k \leq n$, and $L \colon \mathbf{R}^k \to \mathbf{R}^n$ be a linear mapping. We denote $\operatorname{vol} L = \sqrt{\det L^T L}$.

Definition. Let $k, n \in \mathbf{N}$, $k \leq n$, and $G \subset \mathbf{R}^k$ be open. A mapping $f \colon G \to \mathbf{R}^n$ is said to be **regular**, if $f \in \mathcal{C}^1(G)$ and for every $x \in G$ the rank of $f'(a)$ is k.

Theorem 2.7 (area formula)**.** *Let $k, n \in \mathbf{N}, k \leq n$, $G \subset \mathbf{R}^k$ be an open set, $\varphi \colon G \to \mathbf{R}^n$ be an injective regular mapping and $f : \varphi(G) \to \mathbf{R}$ be H^k-measurable. Then we have*

$$\int_{\varphi(G)} f(x) \,\mathrm{d}\, H^k(x) = \int_G f(\varphi(t)) \operatorname{vol} \varphi'(t) \,\mathrm{d}\, \lambda^k(t),$$

if the integral at the right side converges.



2.3 Hausdorff dimension

Lemma 2.8. *Let* $0 < p < q$*,* $A \subset P$*, and* $\mathcal{H}_p(A) < \infty$*. Then* $\mathcal{H}_q(A) = 0$*.*

Proof. Let $\delta \in (0,1)$ and $\{A_j\}_{j=1}^{\infty}$ be a sequence of subsets of P such that $A \subset \bigcup_{j=1}^{\infty} A_j$, $\operatorname{diam} A_j \le \delta$ for every $j \in \mathbf{N}$, and $\sum_{j=1}^{\infty}(\operatorname{diam} A_j)^p < \mathcal{H}_p(A) + 1$. Then we have

$$\begin{aligned}\mathcal{H}_q(A,\delta) &\le \sum_{j=1}^{\infty}(\operatorname{diam} A_j)^q = \sum_{j=1}^{\infty}(\operatorname{diam} A_j)^p \cdot (\operatorname{diam} A_j)^{q-p} \\ &\le \sum_{j=1}^{\infty}(\operatorname{diam} A_j)^p \cdot \delta^{q-p} \le \delta^{q-p}(\mathcal{H}_p(A) + 1).\end{aligned}$$

Sending $\delta \to 0+$ we get $\mathcal{H}_q(A) = 0$. □

Definition. Let $A \subset P$. **Hausdorff dimension** of A is defined by

$$\dim A = \inf\{t \ge 0;\ \mathcal{H}_t(A) < \infty\}.$$

Remark. By Lemma 2.8 we have

$$\mathcal{H}_t(A) = \begin{cases} \infty & \text{for } t < \dim(A), \\ 0 & \text{for } t > \dim(A). \end{cases}$$

Corollary 2.9. (i) *For every* $A \subset B \subset P$ *we have* $\dim A \le \dim B$*.*

(ii) *For every* $A_i \subset P$*,* $i \in \mathbf{N}$*, we have* $\dim(\bigcup_{i=1}^{\infty} A_i) = \sup_i \dim A_i$*.*

(iii) *We have* $\dim([0,1]^k \times \{0\}^{n-k}) = k$*, in particular,* $\dim[0,1]^n = n$*.*

Example (Cantor set)**.** For $s \in \{\emptyset\} \cup \bigcup_{k=1}^{\infty}\{0,1\}^k$ we define inductively closed intervals I_s as follows

- $I_{\emptyset} = [0,1]$,

- if $I_s = [a, b]$, then $I_{s^\wedge i} = \begin{cases} [a, a + \frac{1}{3}(b-a)], & \text{if } i = 0, \\ [b - \frac{1}{3}(b-a), b], & \text{if } i = 1. \end{cases}$

Cantor set is defined by

$$C = \bigcap_{k=0}^{\infty} \bigcup_{s \in \{0,1\}^k} I_s.$$

The set C has the following properties:

- C is compact,
- C is nowhere dense,
- C is uncountable.

Theorem 2.10. *We have* $\dim C = \frac{\log 2}{\log 3}$.

Proof. Denote $d = \frac{\log 2}{\log 3}$.

We prove $\mathcal{H}_d(C) \leq 1$. We have $C \subset \bigcup_{s \in \{0,1\}^k} I_s$ and $\operatorname{diam} I_s \leq 3^{-k}$, $s \in \{0,1\}^k$. We infer

$$\sum_{s \in \{0,1\}^k} (\operatorname{diam} I_s)^d = 2^k \cdot (3^{-k})^d = 1.$$

Then we have $\mathcal{H}_d(C) \leq 1$.

We prove $\mathcal{H}_d(C) \geq 1/4$. It is sufficient to prove that

$$\sum_{j=1}^{\infty} (\operatorname{diam} I_j)^d \geq 1/4,$$

where I_j, $j \in \mathbf{N}$, are open intervals and $C \subset \bigcup_{j=1}^{\infty} I_j$. Convex envelope of an open set $G \subset \mathbf{R}$ is an open interval with the same diameter as G. The set C is compact, therefore there exist intervals $I_1, \dots, I_n$ covering C. Since C is nowhere dense, we may assume that, that the endpoints of $I_1, \dots, I_n$ are not in C. Then there exists $\delta > 0$ such that

$$\operatorname{dist}(C, \text{endpoints of } I_1, \dots, I_n) > \delta.$$

Let $k \in \mathbf{N}$ and $3^{-k} < \delta$. Then we have

$$\forall s \in \{0,1\}^k \; \exists j \in \{1, \dots, n\} \colon I_s \subset I_j. \tag{2.1}$$

Claim. Let $I \subset \mathbf{R}$ be an interval and $l \in \mathbf{N}$ we have

$$\sum_{\substack{I_s \subset I \\ s \in \{0,1\}^l}} (\operatorname{diam} I_s)^d \leq 4(\operatorname{diam} I)^d.$$

Proof of Claim. Suppose that the sum at the left side is nonzero. Let m be the smallest natural number such that I contains some $I_t, t \in \{0,1\}^m$. Then we have obviously $m \leq l$. Let $J_1, \dots, J_p$ are those intervals among $I_s,\ s \in \{0,1\}^m$, which intersect I. The we have $p \leq 4$ by the choice of m. Then we have

$$\begin{aligned} 4(\operatorname{diam} I)^d &\geq \sum_{i=1}^{p} (\operatorname{diam} J_i)^d = \sum_{i=1}^{p} \sum_{\substack{I_s \subset J_i \\ s \in \{0,1\}^l}} (\operatorname{diam} I_s)^d \\ &\geq \sum_{\substack{I_s \subset I \\ s \in \{0,1\}^l}} (\operatorname{diam} I_s)^d. \end{aligned}$$

Indeed, we have

$$\begin{aligned} (\operatorname{diam} J_i)^d &= (3^{-m})^d = 2^{-m}, \\ \sum_{\substack{I_s \subset J_i \\ s \in \{0,1\}^l}} (\operatorname{diam} I_s)^d &= 2^{l-m} \cdot (3^{-l})^d = 2^{-m}. \end{aligned}$$

□

Then we have

$$4 \sum_{j=1}^{\infty} (\operatorname{diam} I_j)^d \overset{\text{Claim}}{\geq} \sum_{j=1}^{n} \sum_{\substack{I_s \subset I_j \\ s \in \{0,1\}^k}} (\operatorname{diam} I_s)^d \overset{(2.1)}{\geq} \sum_{s \in \{0,1\}^k} (\operatorname{diam} I_s)^d = 1.$$

This finishes the proof. □

—— The end of Winter Semester ——

Part II

Summer semester

Chapter 3

Area and coarea formulae

Theorem 3.1. *Let (P_1, ρ_1) and (P_2, ρ_2) be metric spaces, $s > 0$, and $f\colon P_1 \to P_2$ be β-Lipschitz. Then $\mathcal{H}_s\big(f(P_1)\big) \le \beta^s \mathcal{H}_s(P_1)$.*

Proof. Choose $\delta > 0$. Let sets $A_j, j \in \mathbf{N}$, satisfy $P_1 = \bigcup_{j=1}^\infty A_j$ and $\operatorname{diam} A_j < \delta$ for every $j \in \mathbf{N}$. Then we have $f(P_1) = \bigcup_{j=1}^\infty f(A_j)$ and $\operatorname{diam} f(A_j) \le \beta \operatorname{diam} A_j \le \beta\delta$. Then we have

$$\mathcal{H}_s(f(P_1), \beta\delta) \le \sum_{j=1}^\infty (\operatorname{diam} f(A_j))^s \le \sum_{j=1}^\infty \beta^s (\operatorname{diam} A_j)^s.$$

This implies $\mathcal{H}_s(f(P_1), \beta\delta) \le \beta^s \mathcal{H}_s(P_1, \delta)$. Sending $\delta \to 0+$, we get $\mathcal{H}_s(f(P_1)) \le \beta^s \mathcal{H}_s(P_1)$. □

Lemma 3.2. *Let $k, n \in \mathbf{N}, k \le n$, a $L\colon \mathbf{R}^k \to \mathbf{R}^n$ be an injective linear mapping. Then for every λ^k-measurable set $A \subset \mathbf{R}^k$ it holds*

$$H^k\big(L(A)\big) = \sqrt{\det L^T L} \cdot \lambda^k(A). \tag{3.1}$$

Proof. The mapping L is linear and injective, therefore the dimension of the vector space $L(\mathbf{R}^k)$ is k. Thus there exists a linear isometry $Q\colon \mathbf{R}^k \to \mathbf{R}^n$ such that $Q(\mathbf{R}^k) = L(\mathbf{R}^k)$. Then we have

$$\begin{aligned} H^k\big(L(A)\big) &= H^k(Q^{-1} \circ L(A)) = \lambda^k\big(Q^{-1} \circ L(A)\big) \\ &= |\det(Q^{-1}L)| \cdot \lambda^k(A). \end{aligned} \tag{3.2}$$

$$\begin{aligned} \big(\det(Q^{-1}L)\big)^2 &= \det\big((Q^{-1}L)^T Q^{-1}L\big) \\ &= \det\big((\langle Q^{-1}Le_i, Q^{-1}Le_j\rangle)_{i,j=1}^n\big) \\ &= \det\big((\langle Le_i, Le_j\rangle)_{i,j=1}^n\big) = \det(L^T L). \end{aligned} \tag{3.3}$$

The desired equality (3.1) follows from (3.2) a (3.3). □

Notation. Let $k, n \in \mathbf{N}, k \le n$, and $L\colon \mathbf{R}^k \to \mathbf{R}^n$ be a linear mapping. We denote $\operatorname{vol} L = \sqrt{\det L^T L}$.

Remark. (a) The matrix $L^T L$ is called **Gram matrix**. By Lemma 3.2 we have $H^k\big(L([0,1]^k)\big) = \operatorname{vol} L$, thus $\operatorname{vol} L$ is k-dimensional volume of $L\big([0,1]^k\big)$. If $\varphi \in \mathcal{C}^1(G)$, then the mapping $t \mapsto \operatorname{vol} \varphi'(t)$ is continuous on the set G.

(b) If L is a matrix of the type $n \times k$, then the matrix $L^T L$ is symmetric and of the type $k \times k$.

(c) Gram determinant is nonnegative, since for every matrix A of the type $n \times k$ and for every $x \in \mathbf{R}^k$ we have $(L^T Lx, x) = (Lx, Lx) \geq 0$, thus $A^T A$ is positive semidefinite. Gram determinant is positive definite, whenever the rank of L is k.

Lemma 3.3. *Let $k, n \in \mathbf{N}, k \leq n$, $G \subset \mathbf{R}^k$ be open set, $\varphi\colon G \to \mathbf{R}^n$ be an injective regular mapping, $x \in G$, and $\beta > 1$. Then there exists a neighbourhood V of the point x such that*

(a) *the mapping $y \mapsto \varphi\big(\varphi'(x)^{-1}(y)\big)$ is β-Lipschitz on $\varphi'(x)(V)$,*

(b) *the mapping $z \mapsto \varphi'(x)\big(\varphi^{-1}(z)\big)$ is β-Lipschitz on $\varphi(V)$.*

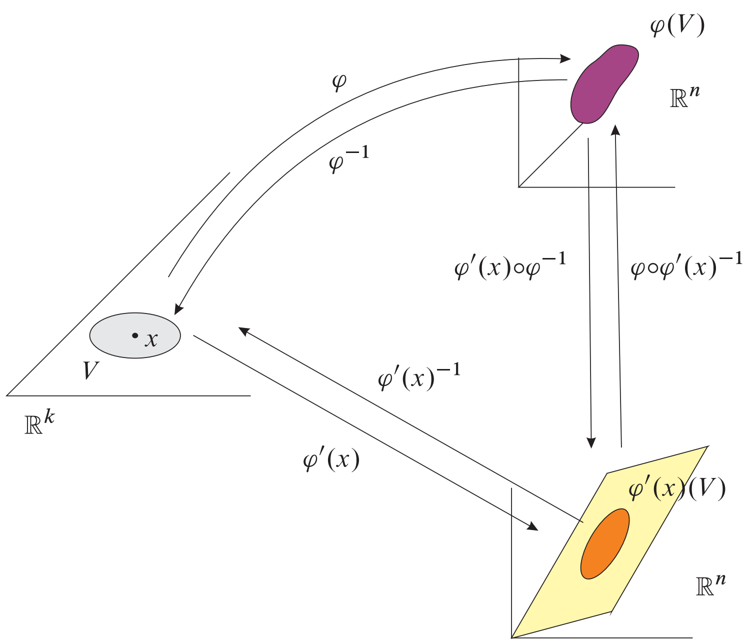


Figure 3.1:

Proof. First we infer several auxiliary inequalities. The linear mapping $v \mapsto \varphi'(x)(v)$ is injective, therefore there exists $\eta > 0$ such that

$$\forall v \in \mathbf{R}^k\colon \|\varphi'(x)(v)\| \geq \eta\|v\|. \tag{3.4}$$

We set $\eta = \inf\bigl\{\|\varphi'(x)(v)\|;\ v \in \mathbf{R}^k, \|v\| = 1\bigr\}$. The mapping $v \mapsto \varphi'(x)(v)$ is continuous and the unit sphere $\{v \in \mathbf{R}^k;\ \|v\| = 1\}$ is compact, therefore the infimum is attained at a point v_0. Since $\varphi'(x)(v_0) \neq 0$, η is positive.

We find $\varepsilon \in (0, \frac{1}{2}\eta)$ such that

$$\frac{2\varepsilon}{\eta} + 1 < \beta. \tag{3.5}$$

Further we find a ball V centered at the point x such that

$$\forall y \in V\colon \|\varphi'(y) - \varphi'(x)\| \leq \varepsilon.$$

We show that for every $u, v \in V$ it holds

$$\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| \leq \varepsilon\|u - v\|. \tag{3.6}$$

Fix $v \in V$ and consider the mapping

$$g\colon w \mapsto \varphi(w) - \varphi(v) - \varphi'(x)(w - v), \qquad w \in V.$$

For $w \in V$ we have $g'(w) = \varphi'(w) - \varphi'(x)$. Then we have

$$\begin{aligned}\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| &= \|g(u) - g(v)\| \\ &\leq \sup\{\|g'(w)\|;\ w \in V\} \cdot \|u - v\| \\ &\leq \varepsilon\|u - v\|,\end{aligned}$$

this implies (3.6).

Further we show that for every $u, v \in V$ we have

$$\|\varphi(u) - \varphi(v)\| \geq \frac{1}{2}\eta\|u - v\|. \tag{3.7}$$

For $u, v \in V$ we compute

$$\begin{aligned}\|\varphi(u) - \varphi(v)\| &\geq -\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \\ &\geq -\varepsilon\|u - v\| + \eta\|u - v\| \geq \frac{1}{2}\eta\|u - v\|,\end{aligned}$$

this gives (3.7).

(a) Choose $a, b \in \varphi'(x)(V)$. We find $u, v \in V$ such that $\varphi'(x)(u) = a$, $\varphi'(x)(v) = b$. We compute

$$\begin{aligned}\|\varphi\bigl(\varphi'(x)^{-1}(a)\bigr) - \varphi\bigl(\varphi'(x)^{-1}(b)\bigr)\| &= \|\varphi(u) - \varphi(v)\| \\ &\leq \|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \\ &\overset{(3.6)}{\leq} \varepsilon\|u - v\| + \|\varphi'(x)(u - v)\| \\ &\overset{(3.4)}{\leq} \tfrac{\varepsilon}{\eta}\|a - b\| + \|a - b\| = \bigl(\tfrac{\varepsilon}{\eta} + 1\bigr)\|a - b\| \\ &\overset{(3.5)}{\leq} \beta\|a - b\|.\end{aligned}$$

(b) Choose $p, q \in \varphi(V)$. We find $u, v \in V$ with $\varphi(u) = p$, $\varphi(v) = q$. Compute

$$\begin{aligned}
\|\varphi'(x)\big(\varphi^{-1}(p)\big) - \varphi'(x)\big(\varphi^{-1}(q)\big)\| &= \|\varphi'(x)(u) - \varphi'(x)(v)\| \\
&= \|\varphi'(x)(u-v)\| \\
&\leq \|\varphi(u) - \varphi(v) - \varphi'(x)(u-v)\| + \|\varphi(u) - \varphi(v)\| \\
&\overset{(3.6)}{\leq} \varepsilon\|u-v\| + \|p-q\| \\
&\overset{(3.7)}{\leq} \tfrac{2\varepsilon}{\eta}\|\varphi(u) - \varphi(v)\| + \|p-q\| = (\tfrac{2\varepsilon}{\eta} + 1)\|p-q\| \\
&\overset{(3.5)}{\leq} \beta\|p-q\|.
\end{aligned}$$

This finishes the proof. □

——— The end of the lecture no. 1, 20. 2. 2025 ———

Lemma 3.4. *Let $k, n \in \mathbf{N}, k \leq n$, $G \subset \mathbf{R}^k$ be an open set, $\varphi\colon G \to \mathbf{R}^n$ be an injective regular mapping, $x \in G$, and $\alpha > 1$. Then there exists a neighbourhood V of x such that for every λ^k-measurable $E \subset V$ we have*

$$\alpha^{-1}\int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t) \leq H^k\big(\varphi(E)\big) \leq \alpha\int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t).$$

Proof. Find $\beta > 1$ a $\tau > 1$ such that

$$\beta^k\tau < \alpha. \tag{3.8}$$

By Lemma 3.3 we find V_1 of x such that for φ and β (a) and (b) of the lemma holds. Using continuity of the mapping $t \mapsto \operatorname{vol}\varphi'(t)$ on G we find a neighbourhood V_2 of x such that

$$\forall t \in V_2\colon \tau^{-1}\operatorname{vol}\varphi'(x) \leq \operatorname{vol}\varphi'(t) \leq \tau\operatorname{vol}\varphi'(x). \tag{3.9}$$

Set $V = V_1 \cap V_2$. We show that V is the desired neighbourhood.

Let $E \subset V$ be λ^k-measurable. By (3.9) we get

$$\tau^{-1}\operatorname{vol}\varphi'(x)\cdot\lambda^k(E) \leq \int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t) \leq \tau\operatorname{vol}\varphi'(x)\cdot\lambda^k(E). \tag{3.10}$$

By Lemma 3.2 we have $\operatorname{vol}\varphi'(x)\cdot\lambda^k(E) = H^k\big(\varphi'(x)(E)\big)$, and we can write

$$\tau^{-1}H^k\big(\varphi'(x)(E)\big) \leq \int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t) \leq \tau H^k\big(\varphi'(x)(E)\big). \tag{3.11}$$

By Lemma 3.3(a) and by the choice of V_1 we get

$$\begin{aligned}
H^k\big(\varphi(E)\big) &= H^k\big(\varphi\circ\varphi'(x)^{-1}\circ\varphi'(x)(E)\big) \leq \beta^k H^k\big(\varphi'(x)(E)\big) \\
&\overset{(3.11)}{\leq} \beta^k\tau\int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t) \overset{(3.8)}{\leq} \alpha\int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t).
\end{aligned}$$

By Lemma 3.3(b) and by the choice of V_1 we get

$$H^k\big(\varphi(E)\big) \geq \beta^{-k} H^k\big(\varphi'(x) \circ \varphi^{-1} \circ \varphi(E)\big) = \beta^{-k} H^k\big(\varphi'(x)(E)\big) \overset{(3.11)}{\geq} \beta^{-k}\tau^{-1} \int_E \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t) \overset{(3.8)}{\geq} \alpha^{-1} \int_E \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t).$$

□

Theorem 3.5 (area formula)**.** *Let $k, n \in \mathbf{N}, k \leq n$, $G \subset \mathbf{R}^k$ be an open set, $\varphi\colon G \to \mathbf{R}^n$ be an injective regular mapping and $f\colon \varphi(G) \to \mathbf{R}$ be H^k-measurable. Then we have*

$$\int_{\varphi(G)} f(x) \, \mathrm{d}\, H^k(x) = \int_G f(\varphi(t)) \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t),$$

if the integral at the right side converges.

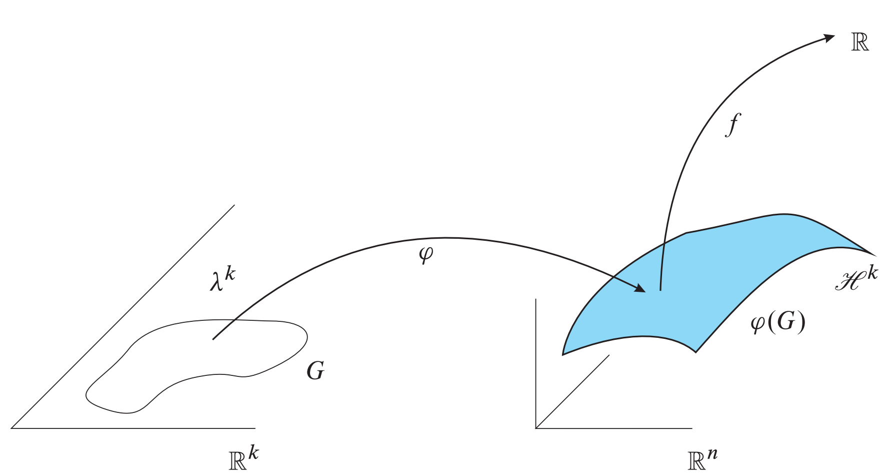


Figure 3.2:

Proof. The mapping φ is injective, therefore there exists an inverse mapping φ^{-1}. Each open set $H \subset G$ is a countable union of compact sets, therefore $\varphi(H)$ is a countable union of compact sets. Thus we get that φ^{-1} is Borel and the set $\varphi(G)$ is Borel.

The mappings φ is locally Lipschitz. Therefore $\varphi(G)$ is H^k-σ-finite by Theorem 3.1. The mappings φ^{-1} is also locally Lipschitz (by Lemma 3.3).

1. Suppose that $f = \chi_L$, where $L \subset \varphi(G)$ is H^k-measurable. We show

$$H^k(L) = \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t). \tag{3.12}$$

Choose $\alpha > 1$. By Lemma 3.4 we find for every $y \in G$ a neighbourhood $V_y \subset G$ of the point y such that for every λ^k-measurable set $E \subset V_y$ we have

$$\alpha^{-1} \int_E \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t) \le H^k(\varphi(E)) \le \alpha \int_E \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t). \tag{3.13}$$

It holds $\bigcup\{V_y;\ y \in G\} = G$. The space $\mathbf{R}^k$ is separable, therefore we can find a sequence $\{y_j\}$ of elements of G such that, we have $\bigcup_{j=1}^{\infty} V_{y_j} = G$. The measure H^k restricted to $\varphi(G)$ is σ-finite. Using this and Theorem 2.5 we find Borel sets $B, N \subset \varphi(G)$ such that $B \subset L \subset B \cup N$ and $H^k(N) = 0$. Using local lipschitzness of φ^{-1} we get $\lambda^k(\varphi^{-1}(N)) = H^k(\varphi^{-1}(N)) = 0$. Thus we obtain that the set $\varphi^{-1}(L)$ is λ^k-measurable. Set

$$A_j = \varphi^{-1}(L) \cap \big(V_{y_j} \setminus \bigcup_{i=1}^{j-1} V_{y_i}\big).$$

Then we have

(a) the set A_j is λ^k-measurable for every $j \in \mathbf{N}$,

(b) $A_j \subset V_{y_j}$ for every $j \in \mathbf{N}$,

(c) $\forall j, j' \in \mathbf{N}, j \neq j'\colon A_j \cap A_{j'} = \emptyset$,

(d) $\bigcup_{j=1}^{\infty} A_j = \varphi^{-1}(L)$,

(e) for every $j \in \mathbf{N}$ we have

$$\alpha^{-1} \int_{A_j} \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t) \le H^k\big(\varphi(A_j)\big) \le \alpha \int_{A_j} \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t),$$

(f) for every $j \in N$ the set $\varphi(A_j)$ is H^k-measurable.

From (a) and (c)–(f) we get

$$\alpha^{-1} \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t) \le H^k\big(\varphi(\varphi^{-1}(L))\big) \le \alpha \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t).$$

Since α has been chosen arbitrarily, we get (3.12).

2. Suppose that f is a nonnegative simple λ^k-measurable function, i.e., $f = \sum_{j=1}^{p} c_j \chi_{L_j}$, where $L_j \subset \varphi(G)$ is H^k-measurable and $c_j \ge 0$, $j = 1, \dots, p$. Then by (3.12) we have

$$\begin{aligned}
\int_{\varphi(G)} f(x) \, \mathrm{d}\, H^k(x) &= \sum_{j=1}^{p} c_j H^k(L_j) = \sum_{j=1}^{p} c_j \int_{\varphi^{-1}(L_j)} \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t) \\
&= \sum_{j=1}^{p} c_j \int_G \chi_{L_j} \circ \varphi(t) \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t) \\
&= \int_G f \circ \varphi(t) \operatorname{vol} \varphi'(t) \, \mathrm{d}\, \lambda^k(t).
\end{aligned} \tag{3.14}$$

3. Let f be a nonnegative H^k-measurable function. We find a nonnegative simple H^k-measurable functions $f_j\colon \varphi(G) \to \mathbf{R}$, $j \in \mathbf{N}$, such that $f_j \to f$ a $f_j \leq f_{j+1}$. Then by Levi theorem we get

$$\lim_{j\to\infty} \int_{\varphi(G)} f_j(x)\,\mathrm{d}\,H^k(x) = \int_{\varphi(G)} f(x)\,\mathrm{d}\,H^k(x),$$
$$\lim_{j\to\infty} \int_G f_j(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t) = \int_G f(\varphi'(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t).$$

Using the point 2 we have for every $j \in \mathbf{N}$ the equality

$$\int_{\varphi(G)} f_j(x)\,\mathrm{d}\,H^k(x) = \int_G f_j(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t),$$

we get

$$\int_{\varphi(G)} f(x)\,\mathrm{d}\,H^k(x) = \int_G f(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t).$$

4. Let f be a H^k-measurable function and the integral $\int_G f(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t)$ converges. Set $f^+ = \max\{f, 0\}$ a $f^- = \max\{-f, 0\}$. By the point 3 it holds

$$\int_{\varphi(G)} f^+(x)\,\mathrm{d}\,H^k(x) = \int_G f^+(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t). \tag{3.15}$$

The last integral equals $\int_G (f(\varphi(t))\operatorname{vol}\varphi'(t))^+\,\mathrm{d}\,\lambda^k(t)$, thus it is finite by assumption. Similarly we get

$$\int_{\varphi(G)} f^-(x)\,\mathrm{d}\,H^k(x) = \int_G (f(\varphi(t))\operatorname{vol}\varphi'(t))^-\,\mathrm{d}\,\lambda^k(t), \tag{3.16}$$

the last integral is finite again. This implies

$$\int_{\varphi(G)} f(x)\,\mathrm{d}\,H^k(x) = \int_G f(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t).$$

□

Remark. Area formula holds even for locally Lipschitz φ (cf. [2, F.34]).

——— The end of the lecture no. 2, 27. 2. 2025 ———

Example. Compute $H^2(\mathbb{S}_2)$, where $\mathbb{S}_2 = \{x \in \mathbf{R}^3;\ \|x\| = 1\}$.

The set $\mathbb{S}_2$ can be written as a disjoint union $\mathbb{S}_2 = A_1 \cup A_2$, where

$$A_1 = \{x \in \mathbb{S}_2;\ x_2 = 0, x_1 \leq 0\},$$
$$A_2 = \mathbb{S}_2 \setminus A_1.$$

The set A_1 is a Lipschitz image of a closed interval. Thus $H^1(A_2) < \infty$. By Theorem 3.1 we get $H^2(A_2) = 0$.

Using area formula we compute $H^2(A_2)$. We use spherical coordinate system $\varphi\colon G \to \mathbf{R}^3$, where $G = (-\pi, \pi) \times (-\pi/2, \pi/2)$ a

$$\varphi(\alpha, \gamma) = [\cos(\gamma)\cos(\alpha), \cos(\gamma)\sin(\alpha), \sin(\gamma)].$$

The mapping φ is injective regular and it holds $\varphi(G) = A_3$. We infer $\operatorname{vol}\varphi'(\alpha, \gamma) = \cos\gamma$ for $(\alpha, \gamma) \in G$. Then we have

$$\begin{aligned} H^2\bigl(\varphi(G)\bigr) &= \int_{\varphi(G)} 1 \,\mathrm{d}\, H^2 = \int_G \operatorname{vol}\varphi' \,\mathrm{d}\,\lambda^2 \\ &= \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\gamma \,\mathrm{d}\,\gamma \,\mathrm{d}\,\alpha = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\gamma \,\mathrm{d}\,\gamma = 4\pi. \end{aligned}$$

We may conclude $H^2(\mathbb{S}_2) = 4\pi$.

Theorem 3.6 (coarea formula)**.** *Let $k, n \in \mathbf{N}, k \geq n$, $\varphi\colon \mathbf{R}^k \to \mathbf{R}^n$ be Lipschitz mapping, $f\colon \mathbf{R}^k \to \mathbf{R}$ be λ^k-integrable function. Then we have*

$$\begin{aligned} &\int_{\mathbf{R}^k} f(x)\sqrt{\det(\varphi'(x)\varphi'(x)^T)} \,\mathrm{d}\,\lambda^k(x) \\ &\qquad = \int_{\mathbf{R}^n}\Bigl(\int_{\varphi^{-1}(\{y\})} f(x) \,\mathrm{d}\, H^{k-n}(x)\Bigr) \,\mathrm{d}\,\lambda^n(y). \end{aligned}$$

Without proof.

Theorem 3.7. *Let $f\colon \mathbf{R}^k \to \mathbf{R}$ be λ^k-integrable function. Then we have*

$$\int_{\mathbf{R}^k} f(x) \,\mathrm{d}\,\lambda^k(x) = \int_0^\infty \Bigl(\int_{\{z \in \mathbf{R}^k;\ \|z\| = r\}} f(x) \,\mathrm{d}\, H^{k-1}(x)\Bigr) \,\mathrm{d}\,\lambda^1(r). \tag{3.17}$$

Proof. Define $\varphi\colon \mathbf{R}^k \to \mathbf{R}$ by $\varphi(x) = \|x\|$. Then we have

$$\begin{aligned} \varphi'(x) &= (\|x\|^{-1}x_1, \dots, \|x\|^{-1}x_k), \qquad x \in \mathbf{R}^k \setminus \{0\}, \\ \varphi'(x)\varphi'(x)^T &= 1. \end{aligned}$$

By Theorem 3.6 we have (3.17). □

Chapter 4

Semicontinuous functions

Definition. Let X be a topological space and $f\colon X \to \mathbf{R}^*$. We say that f is **lower semicontinuous**, if the set $\{x \in X;\ f(x) > a\}$ is open for every $a \in \mathbf{R}$. We say that f is **upper semicontinuous**, if the set $\{x \in X;\ f(x) < a\}$ is open for every $a \in \mathbf{R}$.

Notation. The abbreviations **lsc** and **usc** are used.

Remark. (a) The function $f\colon X \to \mathbf{R}$ is lsc if and only if $\liminf_{t\to x} f(t) \geq f(x)$ whenever $x \in X'$.

(b) If $f\colon K \to \mathbf{R}$ is lsc on a nonempty compact space K, then f attains its minimum on K.

Theorem 4.1. *Let X be a metrizable topological space and $f\colon X \to \mathbf{R}^*$ be bounded from below. Then the function f is lsc, if and only if there exists a nondecreasing sequence $\{f_n\}$ of continuous functions from X to $\mathbf{R}$ such that $f_n \to f$.*

Chapter 5

Functions of Baire class 1

Definition. Let X and Y be metrizable topological spaces. A function $f\colon X \to Y$ is of **Baire class 1** (B_1**-function**) if for every open set U the set $f^{-1}(U)$ is F_σ.

Theorem 5.1 (Lebesgue–Hausdorff–Banach)**.** *Let X be a metrizable topological space and $f\colon X \to \mathbf{R}$ be a B_1-function. Then there exists a sequence $\{f_n\}$ of continuous functions from X to $\mathbf{R}$ with $f_n \to f$.*

Lemma 5.2. *Let X be a metrizable topological space and $A \subset X$ be G_δ and F_σ set. Then χ_A is a pointwise limit of a sequence of continuous functions.*

——— The end of the lecture no. 3, 6. 3. 2025 ———

Lemma 5.3. *Let X be a metrizable topological space, $p_n\colon X \to \mathbf{R}$, $n \in \omega$, be a pointwise limit of continuous functions. If the sequence $\{p_n\}$ converges uniformly to p, then p is a pointwise limit of continuous functions.*

Lemma 5.4 (reduction for F_σ sets)**.** *Let X be a metrizable topological space, A_n be F_σ set for every $n \in \omega$. Then there are F_σ sets $A_n^* \subset A_n$, $n \in \omega$, such that $A_n^* \cap A_m^* = \emptyset$, whenever $n, m \in \omega$, $n \neq m$, and $\bigcup_{n\in\omega} A_n^* = \bigcup_{n\in\omega} A_n$.*

Remark. Theorem 5.1 holds also for X zero-dimensional and Y separable metrizable.

Theorem 5.5 (Baire)**.** *Let X, Y be metrizable topological spaces, Y be separable, and $f\colon X \to Y$ be B_1-function. Then the set of points of continuity of f is residual and G_δ.*

Lemma 5.6. *Let X be a Polish topological space, i.e., separable topological space metrizable by a complete metric, $A, B \subset X$, $A \cap B = \emptyset$. If there is no set C which is G_δ and F_σ with $A \subset C$ and $C \cap B = \emptyset$, then there exists a closed nonempty set F such that $A \cap F$, $B \cap F$ are dense in F.*

——— The end of the lecture no. 4, 13. 3. 2025 ———

Proof. We define $F_0 = X$, $F_{\alpha+1} = \overline{A \cap F_\alpha} \cap \overline{B \cap F_\alpha}$, whenever $\alpha < \omega_1$, and $F_\eta = \bigcap_{\alpha<\eta} F_\alpha$, whenever $\eta < \omega_1$ is a limit ordinal. Then $(F_\alpha)_{\alpha<\omega_1}$ is a nonincreasing sequence of closed sets in X. One can infer that there exists $\zeta < \omega_1$ such that $F_\zeta = F_{\zeta+1}$.

Claim. $F_\zeta \neq \emptyset$

Proof of Claim. We assume towards contradiction that $F_\zeta = \emptyset$. Then we can write

$$X = \bigcup_{\alpha<\zeta} (F_\alpha \setminus F_{\alpha+1}). \tag{5.1}$$

We set $C = \bigcup_{\alpha<\zeta} \left(\overline{A \cap F_\alpha} \setminus F_{\alpha+1}\right)$. Then one can get $A \subset C$ and $C \cap B = \emptyset$. We have that C is F_σ as well as G_δ. To check the latter fact we define G_δ sets

$$G_\alpha = \overline{A \cap F_\alpha} \cup (X \setminus F_\alpha) \cup F_{\alpha+1}, \qquad \alpha < \zeta,$$

and we verify that

$$C = \bigcap_{\alpha<\zeta} G_\alpha.$$

The inclusion $\subset$. For $x \in C$ there exists $\alpha_0 < \omega_1$ such that $x \in \overline{A \cap F_{\alpha_0}} \setminus F_{\alpha_0+1}$. Take $\alpha < \omega_1$. We distinguish the following three possibilities. If $\alpha < \alpha_0$, then

$$x \in \overline{A \cap F_{\alpha_0}} \subset F_{\alpha_0} \subset F_{\alpha+1} \subset G_\alpha.$$

If $\alpha = \alpha_0$, then

$$x \in \overline{A \cap F_{\alpha_0}} \subset G_{\alpha_0} = G_\alpha.$$

If $\alpha > \alpha_0$ then

$$x \in X \setminus F_{\alpha_0+1} \subset X \setminus F_\alpha \subset G_\alpha.$$

The inclusion $\supset$. Now suppose that $x \in \bigcap_{\alpha<\zeta} G_\alpha$. By (5.1) there exists $\beta < \zeta$ with $x \in F_\beta \setminus F_{\beta+1}$. We also have $x \in G_\beta$. This implies that $x \in \overline{A \cap F_\beta} \setminus F_{\beta+1} \subset C$.

Thus C is a G_δ and F_σ set separating A form B, a contradiction. This finishes the proof of Claim. □

Now it is sufficient to set $F = F_\zeta$. □

Remark. Theorem 5.1 holds also for X zero-dimensional and Y separable metrizable.

Theorem 5.7 (Baire)**.** *Let X, Y be metrizable topological spaces, Y be separable, and $f \colon X \to Y$ be B_1-function. Then the set of points of continuity of f is residual and G_δ.*

Proof. Let $\{V_n;\ n \in \omega\}$ be an open basis of Y. Then $x \in X$ is a point of discontinuity of f if and only if there is $n \in \omega$ such that $x \in f^{-1}(V_n) \setminus \text{interior} f^{-1}(V_n)$. Thus we have

$$\text{the set of points of discontinuity of} f = \bigcup_{n \in \omega} \big(f^{-1}(V_n) \setminus \text{interior} f^{-1}(V_n)\big).$$

Fix $n \in \omega$. Then there are closed sets $F_j, j \in \omega$ such that $f^{-1}(V_n) \setminus \text{interior} f^{-1}(V_n) = \bigcup_{n \in \omega} F_j$. Each F_j has empty interior, thus the set $f^{-1}(V_n) \setminus \text{interior} f^{-1}(V_n)$ is meager. This means that the set of points of discontinuity of f is meager as well and we are done. □

Theorem 5.8 (Baire)**.** *Let X be Polish, Y separable metrizable, and $f\colon X \to Y$. Then the following are equivalent*

(i) *f is a B_1-function.*

(ii) *$f|_F$ has a point of continuity for every $F \subset X$ closed.*

Proof. (i) ⇒ (ii) It follows from Theorem 5.7.

(ii) ⇒ (i) Let $U \subset Y$ be open. We write $U = \bigcup_{n \in \omega} F_n$, where F_n's are closed. It is sufficient to show that for every $n \in \omega$ there exists $D_n \in \mathbf{\Delta}^0_2(X)$ such that $f^{-1}(F_n) \subset D_n$ and $D_n \bigcap f^{-1}(Y \setminus U) = \emptyset$. Towards contradiction, we assume that this is not the case. Thus there exists $n_0 \in \omega$ such that there is no $\mathbf{\Delta}^0_2$ set separating $f^{-1}(F_{n_0})$ from $f^{-1}(Y \setminus U)$. Using Lemma 5.6 we find a nonempty closed set F such that $f^{-1}(F_{n_0}) \cap F$ is dense in F and $f^{-1}(Y \setminus U) \cap F$ is dense in F. Let $x^* \in F$ be a point of continuity of $f|_F$. We find a sequence $\{x_n\}$ of points of $f^{-1}(F_{n_0}) \cap F$ converging to x^*. Then $\lim f(x_n) = f(x^*) \in F_{n_0}$. Similarly we find a sequence $\{x'_n\}$ of points of $f^{-1}(Y \setminus U) \cap F$ converging to x^*. Then $\lim f(x_n) = f(x^*) \in Y \setminus U$, a contradiction. □

The end of the lecture no. 5, 20. 3. 2025

Chapter 6

Density topology, approximate continuity and differentiability

Definition. Let f be a function from $\mathbf{R}$ to $\mathbf{R}$, $a \in \mathbf{R}$, and $L \in \mathbf{R}$. We say that f has **approximate limit** L at the point a if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall B \in \mathcal{B}, a \in B, \operatorname{diam} B < \delta\colon\ \lambda^*\big(\{x \in B;\ |f(x) - L| \geq \varepsilon\}\big) < \varepsilon \lambda(B).$$

Theorem 6.1. *Let f be a function from $\mathbf{R}$ to $\mathbf{R}$, $a \in \mathbf{R}$. Then f has at most one approximate limit at a.*

Proof. Towards contradiction assume that $L, L' \in \mathbf{R}$, $L \neq L'$, are approximate limit of f at $a \in \mathbf{R}$. Find $\varepsilon > 0$ such that $|L - L'| > 3\varepsilon$. We find $\delta > 0$ such that

$$\forall B \in \mathcal{B}, a \in B, \operatorname{diam} B < \delta\colon\ \frac{\lambda^*\big(\{x \in B;\ |f(x) - L| \geq \varepsilon\}\big)}{\lambda(B)} < \frac{1}{2} \wedge \frac{\lambda^*\big(\{x \in B;\ |f(x) - L'| \geq \varepsilon\}\big)}{\lambda(B)} < \frac{1}{2}.$$

Fix $B \in \mathcal{B}, a \in B, \operatorname{diam} B < \delta$. Then we have

$$B \subset \{x \in B;\ |f(x) - L| \geq \varepsilon\} \cup \{x \in B;\ |f(x) - L'| \geq \varepsilon\}.$$

Thus we get

$$1 = \frac{\lambda(B)}{\lambda(B)} \leq \frac{\{x \in B;\ |f(x) - L| \geq \varepsilon\}}{\lambda(B)} + \frac{\{x \in B;\ |f(x) - L'| \geq \varepsilon\}}{\lambda(B)} < \frac{1}{2} + \frac{1}{2} = 1,$$

a contradiction. □

Notation. Let f be a function from $\mathbf{R}$ to $\mathbf{R}$. The approximate limit of f at $a \in \mathbf{R}$ is denoted by ap-$\lim_{x \to a} f(x)$.

Definition. A function from $\mathbf{R}$ to $\mathbf{R}$ is **approximately continuous** at $a \in \mathbf{R}$ if ap-$\lim_{x \to a} f(x) = f(a)$.

Definition. We say that a measurable set $A \subset \mathbf{R}$ is d**-open**, if each point of A is a point of density of A.

Theorem 6.2. *The system of d-open sets forms a topology.*

Notation. The symbol τ_d stands for the **density topology** from the previous theorem.

Remark (properties of density topology)**.**

- The topology τ_d is finer than the standard topology.
- The topology τ_d is not metrizable.
- A set $K \subset \mathbf{R}$ is τ_d-compact if and only if K is finite.
- The topology τ_d is not normal.
- Baire theorem holds in $(\mathbf{R}, \tau_d)$.
- $(\mathbf{R}, \tau_d)$ is a connected space.

Remark (approximate continuity and τ_d-continuity)**.** Let $f\colon \mathbf{R} \to \mathbf{R}$ and $a \in \mathbf{R}$. Then f is τ_d-continuous at a if and only if f is approximately continuous at a.

$\Rightarrow$ Let $\varepsilon > 0$. Then there is a d-open set G such that $a \in G$ and for every $x \in G$ we have $|f(x) - f(a)| < \varepsilon$. Since G is d-open we have $\mathrm{d}(G, a) = 1$. Thus there exists $\delta > 0$ such that

$$\forall B \in \mathcal{B}, a \in B, \operatorname{diam} B < \delta\colon\ \lambda(G \cap B) > (1 - \varepsilon)\lambda(B).$$

Choose $B \in \mathcal{B}$ with $a \in B$ and $\operatorname{diam} B < \varepsilon$. Then we have

$$\lambda^*\big(\{x \in B;\ |f(x) - f(a)| \geq \varepsilon\}\big) \subset B \setminus G \qquad \text{and} \qquad \lambda(B \setminus G) < \varepsilon\lambda(B)$$

and we are done.

$\Leftarrow$ Let $\varepsilon > 0$. Denote $A = \{x \in \mathbf{R};\ |f(x) - f(a)| \geq \varepsilon\}$. For every $n \in \mathbf{N}$ find $\delta_n > 0$ such that for every $B \in \mathcal{B}$ with $a \in B$ and $\operatorname{diam} B < \delta_n$ we have

$$\lambda^*\big(B \cap A\big) < \tfrac{1}{n}\lambda(B). \tag{6.1}$$

We find a measurable set C_n such that

$$A \cap [a - \delta_n, a + \delta_n] \subset C_n \subset [a - \delta_n, a + \delta_n] \setminus \{a\}$$

and $\lambda^*(A \cap [a - \delta_n, a + \delta_n]) = \lambda(C_n)$. Then we set

$$D = \bigcup_{n=1}^{\infty}([a - \delta_n, a + \delta_n] \setminus C_n)$$

and $\tilde{D} = \{x \in D;\ \mathrm{d}(D,x) = 1\}$. By Lebesgue density theorem (Theorem 1.12) we have $\lambda(\tilde{D}) = \lambda(D)$. Thus the set $\tilde{D}$ is d-open. By (6.1) we have for every n and $B \in \mathcal{B}$ with $a \in B$ and $\operatorname{diam} B < \delta_n$

$$1 \geq \frac{\lambda(D \cap B)}{\lambda(B)} \geq \frac{\lambda(B \setminus C_n)}{\lambda(B)} = 1 - \frac{\lambda(B \cap C_n)}{\lambda(B)} = 1 - \frac{\lambda^*(B \cap A)}{\lambda(B)} \geq 1 - \frac{1}{n}.$$

Consequently, $\mathrm{d}(\tilde{D}, a) = 1$. Using this and $a \in D$ we get $a \in \tilde{D}$. For every $x \in \tilde{D}$ we also have $|f(x) - f(a)| < \varepsilon$ and we are done.

Theorem 6.3. *The topology τ_d is completely regular, i.e., if $F \subset \mathbf{R}$ is τ_d-closed and $x_0 \in \mathbf{R} \setminus F$, then there exists τ_d-continuous function $f \colon \mathbf{R} \to [0,1]$ such that $f(y) = 0$ for every $y \in F$ and $f(x_0) = 1$.*

Lemma 6.4. *Let $E \subset \mathbf{R}$ be measurable, $X \subset E$ is closed and $\mathrm{d}(E,x) = 1$ for every $x \in X$. Then there exists a closed set $P \subset \mathbf{R}$ such that*

- *$X \subset P \subset E$,*
- *$\forall x \in X\colon\ \mathrm{d}(P,x) = 1$,*
- *$\forall x \in P\colon\ \mathrm{d}(E,x) = 1$.*

The end of the lecture no. 6, 27. 3. 2025

Remark. Let f be a function from $\mathbf{R}$ to $\mathbf{R}$.

(a) The function f is approximately continuous at $a \in \mathbf{R}$ if and only if f is τ_d-continuous at a.

(b) The function f is approximately continuous at $a \in \mathbf{R}$ if and only there exists a measurable set $M \subset \mathbf{R}$ such that $\mathrm{d}(M,a) = 1$ and $\lim_{x \to a, x \in M} f(x) = f(a)$.

Theorem 6.5 (Denjoy)**.** *Let $f \colon \mathbf{R} \to \mathbf{R}$. Then the function f is approximately continuous a.e. if and only if f is measurable.*

Proof. $\Rightarrow$ We set

$$N = \{x \in \mathbf{R};\ f \text{ is not approximately continuous at } x\}.$$

Then we have $\lambda_1(N) = 0$. Choose $c \in \mathbf{R}$ and set $M = \{x \in \mathbf{R};\ f(x) > c\}$. The set $M \setminus N$ is d-open, therefore it is a measurable set. This implies that M is measurable. Consequently, we have that f is measurable.

$\Leftarrow$ Choose $\varepsilon > 0$. By Luzin theorem there exist a closed set $F \subset \mathbf{R}$ with $\lambda_1(\mathbf{R} \setminus F) < \varepsilon$ and a function $g \colon F \to \mathbf{R}$ which is continuous on F satisfying $f|_F = g$. We have that a.e. point in F is a density point of F, therefore f is approximately continuous at a.e. point in F. This implies that f is approximately continuous a.e. in $\mathbf{R}$. $\square$

The end of the lecture no. 7, 3. 4. 2025

Theorem 6.6. *Let $f\colon \mathbf{R}\to\mathbf{R}$ be a bounded approximately continuous function. Then f has an antiderivative on $\mathbf{R}$.*

Proof. Find $K\in\mathbf{R}$ such that $|f(x)|\le K$ for every $x\in\mathbf{R}$. We set $F(x)=\int_0^x f$. The function f is measurable by Theorem 6.5 and is bounded, therefore F is well defined. Choose $x\in\mathbf{R}$. Let $\varepsilon>0$. We find $\delta>0$ such that for every $h\in(0,\delta)$ it holds

$$\frac{1}{h}\lambda_1\big(\{y\in[x,x+h];\ |f(y)-f(x)|\ge\varepsilon\}\big)<\varepsilon.$$

Fix $h\in(0,\delta)$ and denote $M=\{y\in[x,x+h];\ |f(y)-f(x)|\ge\varepsilon\}$. It holds

$$\begin{aligned}\Big|\frac{1}{h}\big(F(x+h)-F(x)\big)-f(x)\Big| &= \frac{1}{h}\Big|\int_x^{x+h}(f(t)-f(x))\,dt\Big|\\ &\le \frac{1}{h}\int_M |f(t)-f(x)|\,dt+\frac{1}{h}\int_{[x,x+h]\setminus M}|f(t)-f(x)|\,dt\\ &\le \frac{1}{h}2K\cdot\varepsilon h+\frac{1}{h}\cdot h\varepsilon=(2K+1)\varepsilon.\end{aligned}$$

This implies $F'_+(x)=f(x)$. One can infer $F'_-(x)=f(x)$ analogously. □

Corollary 6.7. *Let $f\colon \mathbf{R}\to\mathbf{R}$ be a bounded approximately continuous function. Then f has Darboux property and is in B_1.*

Theorem 6.8. *There exists a differentiable function $f\colon\mathbf{R}\to\mathbf{R}$ such that the sets $\{x\in\mathbf{R};\ f'(x)>0\}$ and $\{x\in\mathbf{R};\ f'(x)<0\}$ are dense.*

Proof. Let $A,B\subset\mathbf{R}$ be countable, dense, and disjoint. Suppose that $A=\{a_n;\ n\in\mathbf{N}\}$ and $B=\{b_n;\ n\in\mathbf{N}\}$. Observe that A and B are τ_d-closed. Using Theorem 6.3 we find for every $n\in\mathbf{N}$ approximately continuous functions g_n and h_n such that

$$\begin{aligned}&g_n(a_n)=1, && h_n(b_n)=1,\\ &0\le g_n\le 1, && 0\le h_n\le 1,\\ &g_n|_B=0, && h_n|A=0.\end{aligned}$$

We define

$$\psi=\sum_{n=1}^{\infty}2^{-n}g_n-\sum_{n=1}^{\infty}2^{-n}h_n.$$

Then the function ψ is bounded, approximately continuous, positive on A, and negative on B. By Theorem 6.6 there is a function f such that $f'=\psi$ and we are done. □

Remark. We say that a differentiable function g is of **Köpcke type** if g' is bounded and the sets $\{g'>0\}$ and $\{g'<0\}$ are dense.

Chapter 7

More on derivatives

Theorem 7.1 (Caratheodory–Vitali, [4])**.** *Let* $f\colon \mathbf{R} \to \mathbf{R}$ *satisfy* $f \in L^1(\lambda)$ *and* $\varepsilon > 0$*. Then there exists* $u, v\colon \mathbf{R} \to \mathbf{R}^*$ *such that*

- $u \leq f \leq v$*,*
- u *is usc and bounded from above,*
- v *is lsc and bounded from below,*
- $\int (u - v) \,\mathrm{d}\,\lambda < \varepsilon$*.*

Proof. To be added. □

——— The end of the lecture no. 8, 10. 4. 2025 ———

Theorem 7.2. *Let* f *be differentiable at each point of* $[a, b] \subset \mathbf{R}$ *and* $f' \in L^1([a, b])$*. Then we have*

$$f(x) - f(a) = (L) \int_a^x f'(t) \,\mathrm{d}\, t, \qquad x \in [a, b].$$

Proof. We may assume that $x = b$. Choose $\varepsilon > 0$. Using Theorem 7.1 we find a lsc function g on $[a, b]$ such that $g > f'$ and $\int_a^b g < \int_a^b f' + \varepsilon$. We set

$$G_\eta(x) = (L) \int_a^x g - f(x) + f(a) + \eta(x - a)$$

□

Lemma 7.3 ([4, 7.21])**.** *Let* F *be a differentiable at each point of the interval* $[a, b] \subset \mathbf{R}$ *and* F' *is bounded from below. Then* F *is absolutely continuous on* $[a, b]$*.*

Proof. Let $K \in \mathbf{R}$ be such that $F'(x) \geq K$ for every $x \in [a,b]$. Then the function $x \mapsto F(x) - Kx$ is nondecreasing on $[a,b]$. By Theorem 1.17 we have $F' \in F' \in L^1([a,b])$. For every $x \in [a,b]$ we have

$$F(x) - F(a) = (N)\int_a^x F' = (L)\int_a^x F'.$$

By Theorem 1.18 we get that F is absolutely continuous on $[a,b]$. $\square$

Notation. Let I be a nonempty open interval. The set of all real functions defined on I which have an antiderivative on I is denoted by $\Delta'(I)$.

Remark. We have ap-$\mathcal{C}_b(I) \subset \Delta'(I) \subset \mathcal{D}B_1(I)$.

Theorem 7.4 (Denjoy-Clarkson). *Let I be a nonempty open interval and $f \in \Delta'(I)$ Then f has Denjoy-Clarkson property, i.e., for every open $G \subset \mathbf{R}$ we have that either $f^{-1}(G) = \emptyset$ or $\lambda(f^{-1}(G)) > 0$.*

Proof. Without any loss of generality we may assume that $G = (\alpha, \beta)$, $\alpha, \beta \in \mathbf{R}, \alpha < \beta$. Set

$$E = \{x \in I;\ f(x) \in (\alpha, \beta)\}.$$

Assume towards contradiction that $E \neq \emptyset$ and $\lambda(E) = 0$. Choose $x_0 \in E$ and find $\alpha_1, \beta_1 \in \mathbf{R}$ such that $\alpha < \alpha_1 < \beta_1 < \beta$. Define

$$E_1 = \{x \in I;\ f(x) \in (\alpha_1, \beta_1)\}, \qquad P_1 = \overline{E_1}.$$

There exists $x_1 \in P_1$ such that $f|_{P_1}$ is continuous at x_1. We find an open interval $I_1 \subset I$ such that $x_1 \in I_1$ and

$$\forall x \in I_1 \cap P_1\colon |f(x) - f(x_1)| < \min\{\alpha_1 - \alpha, \beta - \beta_1\} =: \varepsilon.$$

Since E_1 is dense in P_1 there exists $x_2 \in I_1 \cap E_1$. The we have $|f(x_1) - f(x)| < \varepsilon$ and $f(x_2) \in (\alpha_1, \beta_1)$. Thus $f(x_1) \in (\alpha, \beta)$ and we can find an open interval $I_2 \subset I$ such that $x_1 \in I_2$ and

$$\forall x \in I_2 \cap P_1\colon f(x) \in (\alpha, \beta).$$

Then we have $I_2 \cap P_1 \subset E$ and therefore $\lambda(I_2 \cap P_1) = \emptyset$. This implies that $I_2 \cap P_1$ is nowhere dense. We find a countable disjoint family of open nonempty intervals $\mathcal{J}$ such that $I_2 \setminus P_1 = \bigcup \mathcal{J}$. For every $J \in \mathcal{J}$ we have

$$\forall x \in J\colon f(x) \leq \alpha_1 \ \vee\ f(x) \geq \beta_1.$$

Since f has Darboux property we have for every $J \in \mathcal{J}$ either

$$\forall x \in \overline{J} \cap I_2\colon f(x) \leq \alpha_1$$

or

$$\forall x \in \overline{J} \cap I_2 \colon f(x) \geq \beta_1.$$

We set

$$\mathcal{J}_1 = \{J \in \mathcal{J};\ \forall x \in J \colon f(x) \leq \alpha_1\},$$
$$\mathcal{J}_2 = \{J \in \mathcal{J};\ \forall x \in J \colon f(x) \geq \beta_1\}.$$

The set $\bigcup\{\partial J;\ J \in \mathcal{J}\}$ is dense in P_1, since P_1 is nowhere dense. Using thi and continuity of $f|_{P_1}$ at x_1 we can find a closed interval I_3 such that $x_1 \in \operatorname{interior} I_3 \subset I_2$ and $\bigcup \mathcal{J}_1 \cap I_3 = \emptyset$ or $\bigcup \mathcal{J}_2 \cap I_3 = \emptyset$. Suppose that the forme possibility occurs. Then for every $x \in I_3$ we have $f(x) \geq \alpha$. By Lemma 7.3 we have that the antiderivative F of f satisfies $F \in AC(I_3)$. Further we have $F'(x) \geq \beta_1$ for a.e. $x \in I_3$. We also have $\operatorname{interior}(I_3) \cap P_1 \neq \emptyset$. Therefore $\operatorname{interior}(I_3) \cap E_1 \neq \emptyset$. Pick $x_3 \in I_3 \cap E_1$. Then we have

$$f(x_3) = \lim_{x \to x_3+} \frac{F(x) - F(x_3)}{x - x_3} = \lim_{x \to x_3+} \frac{(L) \int_{x_3}^{x} f(t)dt}{x - x_3} \geq \beta_1,$$

a contradiction. □

The end of the lecture no. 9, 17. 4. 2025

Remark. It was proved ([1]) that there exists a differentiable function $f \colon \mathbf{R}^2 \to \mathbf{R}$ such that the set $(\nabla f)^{-1}(B(0,1))$ is nonempty and has 2-dimensional Lebesgue measure zero.

Zahorski classes

Definition. Let $E \subset \mathbf{R}$ be an F_σ set. We say that E belongs to class

M_0 if every point of E is a point of bilateral accumulation of E,

M_1 if every point fo E is a point of bilateral condensation of E,

M_2 if each one sided neighbourhood of each $x \in E$ intersects E in a set of positive measure,

M_3 if for each $x \in E$ and each sequence $\{I_k\}$ of closed intervals converging to x such that $x \notin I_k$ and $\lambda(I_n \cap E) = 0$ for each n, we have

$$\lim_{n\to\infty} \frac{\lambda(I_n)}{\operatorname{dist}(x, I_n)} = 0,$$

M_4 if there exist sequences $\{K_n\}$ and $\{\eta_n\}$ of closed sets and positive numbers respectively such that $E = \bigcup_{n=1}^{\infty} K_n$ and we have

$$\forall n \in \mathbf{N}\ \forall x \in K_n \forall c > 0\ \exists \varepsilon > 0$$
$$\forall h, h_1 \in \mathbf{R}, hh_1 > 0, \frac{h}{h_1} < c, |h + h_1| < \varepsilon \colon \frac{\lambda\big(E \cap (x+h, x+h+h_1)\big)}{|h_1|} > \eta_n.$$

M_5 if every point of E is a point of density of E.

Chapter 8

Paradoxical objects

Definition. Sets $X, Y \subset \mathbf{R}^3$ are **equidecomposable** if there are decompositions $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ of X and Y respectively such that X_i is an isometric copy of Y_i, $i = 1, \dots, n$. We will use the notation $X \equiv Y$.

Theorem 8.1 (Hausdorff-Banach-Tarski)**.** *Let X and Y be bounded subsets of $\mathbf{R}^3$ with nonempty interior. Then X and Y are equidecomposable.*

Lemma 8.2. *There exist rotations $a, b \in SO(3)$ with angles π and $\frac{2}{3}\pi$ respectively and sets A, B, C, D decomposing $\mathbb{S}^2$ such that*

(a) *D is countable,*

(b) *$C = b(B) = b^2(A)$,*

(c) *$A = a(B \cup C)$.*

Proof. We set

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Then $a, b \in SO(3)$, $a^2 = e$, and $b^3 = e$.

Claim. The product $b^{n_1} a b^{n_2} a \dots b^{n_k} a$, where $k \in \mathbf{N}$, $n_1, \dots, n_k \in \{1, 2\}$, is of the form

$$\frac{1}{2^k} \begin{pmatrix} p_1 & p_2 & p_3\sqrt{3} \\ i_1 & p_4 & i_2\sqrt{3} \\ i_3\sqrt{3} & p_5\sqrt{3} & i_4 \end{pmatrix},$$

where $p_1, \dots, p_5$ are even integers and $i_1, \dots, i_4$ are odd integers.

Prof of Claim. Mathematical induction over k. □

——— The end of the lecture no. 10, 24. 4. 2025 ———

Let G be the group generated by the elements a, b.

Claim. For every element $r \in G \setminus \{e, a\}$ there exist uniquely determined $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$, $k \in \mathbf{N}$, $n_1, \dots, n_k \in \{1, 2\}$ such that

$$r = a^{\varepsilon_1} b^{n_1} a b^{n_2} a \dots b^{n_k} a^{\varepsilon_2}.$$

We define subset $G_1, G_2, G_3 \subset G$ as follows. G_1 contains all words of the form $(b^2a)^n$, G_2 contains all words of the form $a(b^2a)^n$, and G_3 contains all words of the form $ba(b^2a)^n$. The *other* reduced words are in G_1, G_2, G_3 if they start with a, b, b^2 respectively. We have $G_3 = bG_2 = b^2G_1$ and $G_1 = a(G_2 \cup G_3)$. We denote

$$D = \{x \in \mathbb{S}^2;\ \exists r \in G, r \neq e \colon rx = x\}.$$

Then D is countable and G-invariant, i.e., $GD = D$. On $\mathbb{S}^2 \setminus D$ we define an equivalence relation $\sim$ by $x \sim y$ if and only if there is $r \in G \setminus \{e\}$ with $rx = y$. Let E be the choice set for $\sim$. We set $A = G_1E$, $B = G_2E$, and $C = G_3E$. This gives the desired decomposition. □

Lemma 8.3. *Let* $X, X', Y, Y', Z \subset \mathbf{R}^3$.

(a) *If* $X \cap X' = \emptyset$, $Y \cap Y' = \emptyset$, $X \equiv X'$, *and* $Y \equiv Y' = \emptyset$, *then* $X \cup X' \equiv Y \cup Y'$.

(b) *If* $X \equiv Y$ *an* $Y \equiv Z$, *then* $X \equiv Z$.

(c) *If* $X' \subset X$, $Y' \subset Y$, $X \equiv Y'$, *and* $X' \equiv Y$, *then* $X \equiv Y$.

Proof. (a) This is obvious.

(b) The proof is straightforward.

(c) Let $f \colon X \to Y'$ and $g \colon X' \to Y$ be bijections corresponding to $X \equiv Y'$ and $X' \equiv Y$ respectively. Then there exists $X'' \subset X$ such that the mapping h defined by

$$h(x) = \begin{cases} f(x) & \text{for } x \in X' \setminus X'' \\ g(x) & \text{for } x \in X'', \end{cases}$$

is a bijection of X onto Y. It is sufficient to set

$$X'' = X \setminus \bigcup_{n \geq 0} (g^{-1} \circ f)^n (X \setminus X').$$

Then we have $X \setminus X'' \equiv h(X \setminus X'')$ and $X'' \equiv h(X'')$. Thus we get $X \equiv h(X) = Y$. □

Proof of Theorem 8.1. Let A, B, C, D be as in Lemma 8.2. Let S' and S'' be disjoint spheres in $\mathbf{R}^3$ with radius 1. We find sets A', B', C', D' and A'', B'', C'', D'' as in Lemma 8.2 for S'

and S'' respectively. We infer $A \equiv A' \cup A''$, $B \equiv B' \cup B''$, and $C \equiv C' \cup C''$. We obtain $\mathbb{S}^2 \setminus D \equiv (S' \setminus D') \cup (S'' \setminus D'')$. We choose $z \in \mathbb{S}^2 \setminus D$ with $-z \in \mathbb{S}^2 \setminus D$. The set

$$R = \{r \in SO(3);\ r \text{ is a rotation with axes determined by } z \text{ such that}$$
$$\forall n \in \mathbf{N}\ \forall x \in D\ \forall y \in D \colon r^n x \neq y\}$$

is nonempty. Fix $r \in R$. The sets $r^j D$, $j \in \mathbf{N} \cup \{0\}$ are pairwise disjoint. We denote $U = \bigcup_{n \geq 0} r^n D$. Then we have $rU \equiv U \setminus D$. Thus we have $U \equiv U \setminus D$. This implies

$$S = U \cup (S \setminus U) \equiv (U \setminus D) \cup (S \setminus U) = S \setminus D.$$

This gives $S \equiv S' \cup S''$. Let B, B', B'' be closed balls such that $\partial B = \mathbb{S}^2$, $\partial B' = S'$, and $\partial B'' = S''$. Now it is not difficult to infer that

$$B \setminus \{c(B)\} \equiv \big(B' \setminus \{c(B')\}\big) \cup \big(B'' \setminus \{c(B'')\}\big)$$

Since $B \setminus \{c(B)\} \equiv B$, we see that $B \equiv B' \cup B''$.

The end of the lecture no. 11, 5. 5. 2025

Now assume that X, Y are subset of $\mathbf{R}^3$ which are bounded and they have nonempty interiors. Find $X' \subset X$ and $Y' \subset Y$ closed balls with the same radius s. Further find Z' and Z such that $X \equiv Z'$, $Z' \subset Z$, and Z is a finite disjoint union of closed ball with radius s. Then we have $X' \equiv Z$, $X \equiv Z'$, and therefore $X \equiv Z \equiv X'$. Similarly we have $Y \equiv Y'$. Since $Y' \equiv X'$, we conclude $X \equiv Y$. $\square$

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