Steady compressible Navier–Stokes–Fourier equations with Dirichlet boundary condition for the temperature

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Steady compressible Navier–Stokes–Fourier system I

$\Omega \subset \mathbb{R}^3$, bounded, smooth ($C^2$)

- Balance of mass

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \tag{1}$$

ϱ: $\Omega \mapsto \mathbb{R}$ ... density of the fluid
$\mathbf{u}$: $\Omega \mapsto \mathbb{R}^3$ ... velocity field

- Balance of momentum

$$\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S} + \nabla p = \varrho \mathbf{f} \tag{2}$$

$\mathbb{S}$ ... viscous part of the stress tensor (symmetric tensor)
$\mathbf{f}$: $\Omega \mapsto \mathbb{R}^3$ ... specific volume force (given)
p... pressure (scalar quantity)

- Balance of total energy

$$\operatorname{div}\left(\varrho E \mathbf{u}\right) + \operatorname{div}(\mathbf{q} + p\mathbf{u}) = \varrho \mathbf{f} \cdot \mathbf{u} + \operatorname{div}\left(\mathbb{S}\mathbf{u}\right) \tag{3}$$

$E = \frac{1}{2}|\mathbf{u}|^2 + e$... specific total energy
e ... specific internal energy (scalar quantity)
$\mathbf{q}$ ... heat flux (vector field)
(no energy sources assumed)

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Steady compressible Navier–Stokes–Fourier system II

- Boundary conditions at $\partial\Omega$ for velocity: either homogeneous Dirichlet

$$\mathbf{u} = \mathbf{0} \tag{4}$$

or Navier (partial slip)

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad (\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}} + \lambda \mathbf{u}_{\boldsymbol{\tau}} = 0, \quad \lambda \geq 0. \tag{5}$$

- Boundary conditions at $\partial\Omega$ for temperature:

$$\vartheta = \vartheta_D, \tag{6}$$

$\vartheta_D \geq \vartheta_0$, bounded, sufficiently smooth, assumed to be extended with the same properties to the whole Ω.

- Total mass

$$\int_\Omega \varrho \, \mathrm{d}x = M > 0 \tag{7}$$

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Thermodynamics I

We will work with basic quantities: density ϱ and temperature ϑ

We assume: $e = e(\varrho, \vartheta)$, $p = p(\varrho, \vartheta)$

Gibbs' relation

$$\frac{1}{\vartheta}\left(De(\varrho,\vartheta) + p(\varrho,\vartheta)D\left(\frac{1}{\varrho}\right)\right) = Ds(\varrho,\vartheta) \tag{8}$$

with $s(\varrho, \vartheta)$ the specific entropy.

The specific entropy fulfills formally the entropy balance

$$\operatorname{div}(\varrho s\mathbf{u}) + \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma = \frac{\mathbb{S} : \nabla\mathbf{u}}{\vartheta} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2} \tag{9}$$

Second law of thermodynamics

$$\sigma = \frac{\mathbb{S} : \nabla\mathbf{u}}{\vartheta} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2} \geq 0 \tag{10}$$

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Thermodynamics II

Another possibility is to work with internal energy balance (heat equation)

Balance of internal energy

$$\operatorname{div}\left(\varrho e \mathbf{u}\right) + \operatorname{div}\mathbf{q} + p\operatorname{div}\mathbf{u} = \mathbb{S} : \nabla\mathbf{u}$$

The troublemaker is the nonlinear term on the rhs. Anyway, this equation plays an important role in the construction of weak solutions.

Weak solution I

- Weak formulation of the continuity equation
  Both ϱ, $\mathbf{u}$ extended by zero outside Ω

$$\int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0 \qquad \forall \psi \in C_0^1(\mathbb{R}^3) \tag{11}$$

- Renormalized continuity equation

$$\int_{\mathbb{R}^3} b(\varrho)\mathbf{u} \cdot \nabla \psi \, \mathrm{d}x + \int_{\mathbb{R}^3} \big(\varrho b'(\varrho) - b(\varrho)\big) \operatorname{div} \mathbf{u} \psi \, \mathrm{d}x = 0 \; \forall \psi \in C_0^1(\mathbb{R}^3) \tag{12}$$

  for all $b \in C^1([0,\infty))$ with $b'(z) = 0$ for $z \geq K > 0$.

- Weak formulation of the momentum equation

$$\begin{aligned} &\int_{\Omega} \big(- \varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \varphi - p(\varrho, \vartheta) \operatorname{div} \varphi + \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \varphi \big) \, \mathrm{d}x \\ &+ \lambda \int_{\partial \Omega} \mathbf{u} \cdot \varphi \, \mathrm{d}S = \int_{\Omega} \varrho \mathbf{f} \cdot \varphi \, \mathrm{d}x \quad \forall \varphi \in C_0^1(\Omega; \mathbb{R}^3) (\in C_{0,\mathbf{n}}^1(\Omega; \mathbb{R}^3)) \end{aligned} \tag{13}$$

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Weak formulation of the total energy balance

$$\int_\Omega -\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x$$
$$= \int_\Omega \left(\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\right)\mathrm{d}x$$
$$+ \int_\Omega \left(\left(-\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u}\right)\cdot\nabla\psi + \mathbf{q}\cdot\nabla\psi\right)\mathrm{d}x$$
$$- \int_{\partial\Omega}\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}S - \lambda\int_{\partial\Omega}|\mathbf{u}|^2\psi\,\mathrm{d}S \qquad \forall\psi\in C^1(\overline{\Omega}) \tag{14}$$

Weak formulation of the entropy inequality

$$\int_\Omega \Big(\frac{\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u}}{\vartheta} - \frac{\nabla\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}\Big)\psi\,\mathrm{d}x - \int_{\partial\Omega}\frac{\mathbf{q}\cdot\mathbf{n}}{\vartheta}\psi\,\mathrm{d}S$$
$$\leq \int_\Omega \Big(\frac{-\mathbf{q}\cdot\nabla\psi}{\vartheta} - \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\mathrm{d}x \qquad \forall \text{ nonnegative } \psi\in C^1(\overline{\Omega}) \tag{15}$$

This formulation works fine for given heat flux on the boundary, but not for given temperature on the boundary.

Problem: boundary integrals with the heat flux!

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Weak formulation of the total energy balance

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&\int_\Omega -\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi \,\mathrm{d}x \\
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Weak solution III

One possible remedy: consider test functions with compact support.

Problem: We are not able to deduce a priori estimates!

Solution is based on the same idea used by Chaudhuri and Feireisl for the evolutionary case, to consider so-called ballistic energy inequality.

Chaudhuri, N., Feireisl, E.: Navier–Stokes–Fourier system with Dirichlet boundary conditions. ArXiv:2106.05315 (2021).

Use as test function $\psi = 1$ in weak formulation of the energy equality and $\psi = \tilde{\vartheta}$ in weak formulation of the entropy inequality, where $\tilde{\vartheta}$ is an arbitrary extension of the boundary data which is sufficiently regular and strictly positive in $\overline{\Omega}$.

Weak formulation of the ballistic energy inequality

$$\int_\Omega \left(\frac{\tilde{\vartheta}}{\vartheta} \mathbb{S} : \nabla \mathbf{u} - \mathbf{q} \cdot \nabla \vartheta \frac{\tilde{\vartheta}}{\vartheta^2} \right) \mathrm{d}x + \lambda \int_{\partial\Omega} |\mathbf{u}|^2 \, \mathrm{d}S \leq \int_\Omega \left(\varrho \mathbf{f} \cdot \mathbf{u} - \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla \tilde{\vartheta} - \frac{\mathbf{q}}{\vartheta} \cdot \nabla \tilde{\vartheta} \right) \mathrm{d}x. \tag{16}$$

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Weak solution IV

Definition

The triple $(\varrho, \mathbf{u}, \vartheta)$ is called a renormalized weak solution to our system (1)–(7) if $\varrho \geq 0$, $\vartheta > 0$ a.e. in Ω, $\mathbf{u} = \mathbf{0}$ (or $\mathbf{u} \cdot \mathbf{n} = 0$), $\vartheta = \vartheta_D$ on $\partial\Omega$, $\int_\Omega \varrho \, \mathrm{d}x = M$, and the weak and renormalized formulation of the continuity equation, weak formulation of the total energy balance with compactly supported test functions, weak formulation of the entropy inequality balance with compactly supported test functions and the ballistic energy inequality for any $\tilde{\vartheta}$ specified above hold true.

Problem with integrability for some interesting cases:

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Both definitions are reasonable in the sense that any smooth weak or entropy variational solutions are actually classical solutions to (1)–(7) (weak-strong compatibility).

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Constitutive relations I

- Newtonian fluid

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{d}\operatorname{div}\mathbf{u}\mathbb{I}\right] + \xi(\vartheta)\operatorname{div}\mathbf{u}\mathbb{I} \quad (17)$$

$\mu(\cdot)\colon \mathbb{R}^+ \to \mathbb{R}^+$,
$\xi(\cdot)\colon \mathbb{R}^+ \to \mathbb{R}^+_0$: viscosity coefficients

- Fourier's law

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta \quad (18)$$

$\kappa(\cdot)\colon \mathbb{R}^+ \to \mathbb{R}^+ \ldots$ heat conductivity

- Pressure law

$$p = p(\varrho, \vartheta) = (\gamma - 1)\varrho e(\varrho, \vartheta), \qquad \gamma > 1 \quad (19)$$

If $p \in C^1((0,\infty)^2)$, then it is of the form

$$p(\varrho, \vartheta) = \vartheta^{\frac{\gamma}{\gamma-1}} P\left(\frac{\rho}{\vartheta^{\frac{1}{\gamma-1}}}\right) \quad (20)$$

with $P \in C^1((0,\infty))$.

Constitutive relations I

- Newtonian fluid

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{d} \operatorname{div} \mathbf{u} \mathbb{I}\right] + \xi(\vartheta) \operatorname{div} \mathbf{u} \mathbb{I} \qquad (17)$$

$\mu(\cdot)\colon \mathbb{R}^+ \to \mathbb{R}^+$,
$\xi(\cdot)\colon \mathbb{R}^+ \to \mathbb{R}^+_0$: viscosity coefficients

- Fourier's law

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta \qquad (18)$$

$\kappa(\cdot)\colon \mathbb{R}^+ \to \mathbb{R}^+ \dots$ heat conductivity

- Pressure law

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Constitutive relations II

We assume

$$\begin{gathered} P(\cdot) \in C^1([0,\infty)) \cap C^2((0,\infty)), \\ P(0) = 0, \quad P'(0) = p_0 > 0, \quad P'(Z) > 0, \quad Z > 0, \\ \lim_{Z\to\infty} \frac{P(Z)}{Z^\gamma} = p_\infty > 0, \\ 0 < \frac{\gamma P(Z) - ZP'(Z)}{Z} \le c_5 < \infty, \quad Z > 0. \end{gathered} \tag{21}$$

We have for $K > 0$ a fixed constant

$$\begin{aligned} c_6 \varrho\vartheta &\le p(\varrho,\vartheta) \le c_7 \varrho\vartheta, \text{ for } \varrho \le K\vartheta^{\frac{1}{\gamma-1}}, \\ c_8 \varrho^\gamma &\le p(\varrho,\vartheta) \le c_9 \begin{cases} \vartheta^{\frac{\gamma}{\gamma-1}}, & \text{for } \varrho \le K\vartheta^{\frac{1}{\gamma-1}}, \\ \varrho^\gamma, & \text{for } \varrho > K\vartheta^{\frac{1}{\gamma-1}}. \end{cases} \end{aligned} \tag{22}$$

$$\begin{gathered} \frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0 \qquad \text{in } (0,\infty)^2, \\ p = d\varrho^\gamma + p_m(\varrho,\vartheta), \quad d > 0, \qquad \text{with} \quad \frac{\partial p_m(\varrho,\vartheta)}{\partial \varrho} > 0 \qquad \text{in } (0,\infty)^2. \end{gathered} \tag{23}$$

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Constitutive relations III

The specific internal energy

$$\left.\begin{aligned} \frac{1}{\gamma-1} p_\infty \varrho^{\gamma-1} \leq e(\varrho,\vartheta) \leq c_{10}(\varrho^{\gamma-1}+\vartheta), \\ \frac{\partial e(\varrho,\vartheta)}{\partial \varrho}\varrho \leq c_{11}(\varrho^{\gamma-1}+\vartheta) \end{aligned}\right\} \text{ in } (0,\infty)^2. \tag{24}$$

The specific entropy $s(\varrho,\vartheta)$ defined by the Gibbs law

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\right) \quad \text{with} \quad S'(Z) = -\frac{1}{\gamma-1}\frac{\gamma P(Z) - ZP'(Z)}{Z^2} < 0. \tag{25}$$

Furthermore

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Therefore, for a fixed $\varrho > 0$ we know that $\lim_{\vartheta\to 0^+} s(\varrho,\vartheta)$ exists. We assume that limit is finite. Then we may always choose the additive constant in the definition of the specific entropy in such a way that for any $\varrho \geq 0$

$$\lim_{\vartheta\to 0^+} s(\varrho,\vartheta) = 0. \tag{27}$$

Then

$$|s(\varrho,\vartheta)| \leq c_{12}(1 + |\ln\varrho| + [\ln\vartheta]^+) \quad \text{in } (0,\infty)^2. \tag{28}$$

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Therefore, for a fixed $\varrho > 0$ we know that $\lim_{\vartheta \to 0^+} s(\varrho, \vartheta)$ exists. We assume that limit is finite. Then we may always choose the additive constant in the definition of the specific entropy in such a way that for any $\varrho \geq 0$

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Constitutive relations V

- Heat conductivity

$$\kappa(\vartheta) \sim (1 + \vartheta)^m \tag{29}$$

$m \in \mathbb{R}^+$

- Viscosity coefficients

$$\begin{aligned} C_1(1 + \vartheta) \leq \mu(\vartheta) \leq C_2(1 + \vartheta) \\ 0 \leq \xi(\vartheta) \leq C_2(1 + \vartheta) \end{aligned} \tag{30}$$

$\mu(\cdot)$ globally Lipschitz continuous, $\xi(\cdot)$ continuous

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Main result

Theorem

Let $\Omega \subset \mathbb{R}^d$ *be a* C^2 *bounded domain and* $\vartheta_D \in W^{2,q}(\Omega)$ *for some* $q > d$.
a) Dirichlet condition for the velocity: Let $\gamma > \frac{5}{3}$ *and* $m > \max\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\}$ *or* $\gamma \in \left(\frac{4}{3}, \frac{5}{3}\right]$ *and* $m > \max\{\frac{2}{3}, \frac{2(3\gamma-2)}{9(3\gamma-4)}\}$ *Then there exists a variational entropy solution to our problem, where* $\mathbf{u} \in W_0^{1,2}(\Omega;\mathbb{R}^3)$, $\vartheta \in L^{3m}(\Omega) \cap W^{1,r}(\Omega)$ *for some* $1 < r \leq 2$ *and* $\varrho \in L^{\gamma+\Theta}(\Omega)$ *for some* $\Theta > 0$. *If, additionally,* $\gamma > \frac{5}{3}$ *and* $m > 1$, *then the solution is also the weak solution.*

b) Navier condition for the velocity: Let $\gamma > 1$ *and* $m > \max\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\}$. *Let* $\lambda > 0$ *or* $\lambda \geq 0$ *and* Ω *is not axially symmetric. Then there exists a variational entropy solution to our problem, where* $\mathbf{u} \in W_{0,\mathbf{n}}^{1,2}(\Omega;\mathbb{R}^3)$, $\vartheta \in L^{3m}(\Omega) \cap W^{1,r}(\Omega)$ *for some* $1 < r \leq 2$ *and* $\varrho \in L^{\gamma+\Theta}(\Omega)$ *for some* $\Theta > 0$. *If, additionally,* $\gamma > \frac{5}{4}$, $m > 1$, $m > \frac{6\gamma}{15\gamma-16}$ *if* $\gamma \in \left(\frac{5}{4}, \frac{4}{3}\right]$ *and* $m > \frac{18-6\gamma}{9\gamma-7}$ *if* $\gamma \in \left(\frac{4}{3}, \frac{5}{3}\right)$, *then the solution is also the weak solution.*

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A priori estimates I

We take the ballistic energy inequality and choose a particular function $\tilde{\vartheta} := \vartheta_L$ such that

$$\Delta \vartheta_L = 0 \qquad \text{in } \Omega$$
$$\vartheta_L = \vartheta_D \qquad \text{on } \partial\Omega.$$

We have

$$\vartheta_L \in W^{2,q}(\Omega) \hookrightarrow C^1(\overline{\Omega})$$
$$\underline{\theta} \leq \vartheta_L \leq \overline{\theta} \qquad \text{in } \Omega.$$

Then

$$\int_\Omega \Big(\frac{\tilde{\vartheta}}{\vartheta} \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - \mathbf{q}(\vartheta, \nabla\vartheta) \cdot \nabla \vartheta \frac{\tilde{\vartheta}}{\vartheta^2} \Big) \, \mathrm{d}x$$
$$\geq \underline{\theta} \int_\Omega \Big(\frac{\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u}}{\vartheta} + \frac{\kappa(\theta) |\nabla \vartheta|^2}{\vartheta^2} \Big) \, \mathrm{d}x$$
$$\geq C \big(\|\mathbf{u}\|_{1,2}^2 + \|\ln \vartheta\|_{1,2}^2 + \|\vartheta^{\frac{m}{2}}\|_{1,2}^2 \big) - C(\vartheta_D).$$

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A priori estimates II

Therefore

$$\|\mathbf{u}\|_{1,2}^2 + \|\ln\vartheta\|_{1,2}^2 + \|\vartheta^{\frac{m}{2}}\|_{1,2}^2 \leq C\int_\Omega \left(\varrho\mathbf{f}\cdot\mathbf{u} - \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\vartheta_L + \frac{\kappa(\vartheta)\nabla\vartheta\cdot\nabla\vartheta_L}{\vartheta}\right)\,\mathrm{d}x + C(\vartheta_D). \tag{31}$$

We have

$$\int_\Omega \frac{\kappa(\vartheta)\nabla\vartheta\cdot\nabla\vartheta_L}{\vartheta}\,\mathrm{d}x = \int_\Omega \nabla K(\vartheta)\cdot\nabla\vartheta_L\,\mathrm{d}x$$

$$= -\int_\Omega K(\vartheta)\Delta\vartheta_L\,\mathrm{d}x + \int_{\partial\Omega} K(\vartheta)\frac{\partial\vartheta_L}{\partial\mathbf{n}}\,\mathrm{d}\sigma = \int_{\partial\Omega} K(\vartheta_D)\frac{\partial\vartheta_L}{\partial\mathbf{n}}\,\mathrm{d}\sigma,$$

where $K'(z) = \frac{\kappa(z)}{z}$.

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A priori estimates III

We first consider $\gamma > \frac{5}{3}$. Then

$$\Big|\int_\Omega \varrho \mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x\Big| \leq C\|\mathbf{u}\|_6\|\varrho\|_{\frac{6}{5}} \leq \varepsilon\|\mathbf{u}\|_{1,2}^2 + C(\varepsilon)\|\varrho\|_{\frac{6}{5}}^2$$

$$\leq \varepsilon\|\mathbf{u}\|_{1,2}^2 + C(\varepsilon)\|\varrho\|_1^{\frac{5q-6}{3(q-1)}}\|\varrho\|_q^{\frac{q}{3(q-1)}} \leq \varepsilon\|\mathbf{u}\|_{1,2}^2 + C(\varepsilon)M^{\frac{5q-6}{3(q-1)}}\|\varrho\|_q^{\frac{q}{3(q-1)}}.$$

By properties of the entropy

$$\Big|\int_\Omega \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\vartheta_L\ \mathrm{d}x\Big| \leq C\int_\Omega \big(\varrho + \varrho[\ln\varrho]^+ + \varrho[\ln\vartheta]^+ + 1\big)|\mathbf{u}|\ \mathrm{d}x$$

$$\leq \varepsilon\|\mathbf{u}\|_{1,2}^2 + \varepsilon(\|\vartheta^{\frac{m}{2}}\|_{1,2}^2 + \|\vartheta_D\|_{1,2}^2) + C(\varepsilon,\delta)\|\varrho\|_q^{\frac{q}{3(q-1)}+\delta} + C(\varepsilon)$$

with $\delta > 0$, arbitrarily small.

Whence, for arbitrary $q > \frac{6}{5}$ and $\delta > 0$

$$\|\mathbf{u}\|_{1,2}^2 + \|\ln\vartheta\|_{1,2}^2 + \|\vartheta^{\frac{m}{2}}\|_{1,2}^2 \leq C(q,\delta,\vartheta_D)\big(1 + \|\varrho\|_q^{\frac{q}{3(q-1)}+\delta}\big). \tag{32}$$

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By properties of the entropy

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A priori estimates IV

Using as test function in the momentum balance the function $\boldsymbol{\varphi} := \mathcal{B}\big(\varrho^\Theta - \frac{1}{|\Omega|}\int_\Omega \varrho^\Theta \,\mathrm{d}x\big)$ for some $\Theta > 0$, where $\mathcal{B}$ is the standard Bogovskii operator, we end up with

$$\int_\Omega p(\varrho,\vartheta)\varrho^\Theta \,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega \varrho^\Theta \,\mathrm{d}x \int_\Omega p(\varrho,\vartheta)\,\mathrm{d}x + \int_\Omega \mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x - \int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x - \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x. \tag{33}$$

Then

$$\|\varrho\|_{\gamma+\Theta}^{\gamma+\Theta} \le C(\delta)\Big(1 + \|\varrho\|_{\gamma+\Theta}^{1+\Theta+\frac{\gamma+\Theta}{3(\gamma+\Theta-1)}+\delta} + \|\varrho\|_{\gamma+\Theta}^{\Theta+\frac{\gamma+\Theta}{6(\gamma+\Theta-1)}+\delta+\frac{\gamma+\Theta}{3m(\gamma+\Theta-1)}}\Big)$$

with $\Theta = \min\{2\gamma-3, \gamma\frac{3m-2}{3m+2}\}$.

We get the estimate ($r \le \min\{2, \frac{3m}{m+1}\}$)

$$\|\mathbf{u}\|_{1,2} + \|\ln\vartheta\|_{1,2} + \|\vartheta^{\frac{m}{2}}\|_{1,2} + \|\vartheta\|_{1,r} + \|\varrho\|_{\gamma+\Theta} \le C \tag{34}$$

provided

$$\gamma > \frac{5}{3}, \qquad m > \max\Big\{\frac{2}{3}, \frac{2(\gamma-1)}{6\gamma^2-11\gamma+3}\Big\}.$$

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$$\begin{aligned}\int_\Omega p(\varrho,\vartheta)\varrho^\Theta \,\mathrm{d}x = & \frac{1}{|\Omega|}\int_\Omega \varrho^\Theta \,\mathrm{d}x \int_\Omega p(\varrho,\vartheta)\,\mathrm{d}x + \int_\Omega \mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x \\ & - \int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x - \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x.\end{aligned} \tag{33}$$

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A priori estimates V

If $\gamma \leq \frac{5}{3}$, we cannot get the estimates of the density by the Bogovskii operator. We follow the ideas from the papers

A. Novotný, M. Pokorný: Weak and variational solutions to steady equations for compressible heat conducting fluids, SIAM Journal on Mathematical Analysis 43 (2011), 1158–1188.

(for the Dirichlet boundary conditions for the velocity)

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A priori estimates VI

After some complicated estimates, in particular near the boundary, we end up with

$$\sup_{y\in\overline{\Omega}} \int_{B_{R_0}(y)\cap\Omega} \frac{p(\varrho,\vartheta)}{|x-y|^\alpha}\,\mathrm{d}x \le C\big(1+\|p(\varrho,\vartheta)\|_1 + \|\mathbf{u}\|_{1,2}(1+\|\vartheta\|_{3m}) + \|\varrho|\mathbf{u}|^2\|_1\big), \tag{35}$$

for the Dirichlet boundary conditions for the velocity, and

$$\sup_{y\in\overline{\Omega}} \int_{B_{R_0}(y)\cap\Omega} \Big(\frac{p(\varrho,\vartheta)}{|x-y|^\alpha} + \frac{\varrho|\mathbf{u}|^2}{|x-y|^\alpha}\Big)\,\mathrm{d}x \le C\big(1+\|p(\varrho,\vartheta)\|_1 + \|\mathbf{u}\|_{1,2}(1+\|\vartheta\|_{3m}) + \|\varrho|\mathbf{u}|^2\|_1\big), \tag{36}$$

for the Navier boundary conditions for the velocity. We may combine these estimates with

$$\|\mathbf{u}\|_{1,2}^2 + \|\ln\vartheta\|_{1,2}^2 + \|\vartheta^{\frac{m}{2}}\|_{1,2}^2 \le C(\delta,\vartheta_D)\big(1+\|\varrho\mathbf{u}\|_{1+\delta}^{1+\delta}\big) \tag{37}$$

with $\delta > 0$, arbitrarily small. This yields after many several technical computations

$$\|\mathbf{u}\|_{1,2}^2 + \|\ln\vartheta\|_{1,2}^2 + \|\vartheta^{\frac{m}{2}}\|_{1,2}^2 + \|\varrho\|_{s\gamma} + \|\varrho|\mathbf{u}|^2\|_s \le C \tag{38}$$

for some $s > 1$; we also need to check when $s > \frac{6}{5}$.

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Weak compactness I

Let $\{\mathbf{f}_n\}_{n=1}^{\infty} \subset L^\infty(\Omega;\mathbb{R}^3)$ be a sequence of functions such that $\|\mathbf{f}_n\|_\infty \leq C$ and $\mathbf{f}_n \to \mathbf{f}$ (strongly) in $L^1(\Omega;\mathbb{R}^3)$. Assume that $(\varrho_n, \mathbf{u}_n, \vartheta_n)$ is a sequence of solutions to our problems generated by the sequence of the right-hand sides $\mathbf{f}_n$ which fulfils the bound

$$\|\mathbf{u}_n\|_{1,2} + \|\ln \vartheta_n\|_{1,2} + \|\vartheta_n^{\frac{m}{2}}\|_{1,2} + \|\vartheta_n\|_{1,r} + \|\varrho_n\|_{\gamma+\Theta} < C$$

with γ, m, r and Θ satisfying the restrictions deduced above.

Then

$$\begin{aligned}
\mathbf{u}_n &\rightharpoonup \mathbf{u} && \text{weakly in } W_0^{1,2}(\Omega;\mathbb{R}^3),\\
\mathbf{u}_n &\to \mathbf{u} && \text{strongly in } L^q(\Omega;\mathbb{R}^3), && \forall 1 \leq q < 6,\\
\vartheta_n &\rightharpoonup \vartheta && \text{weakly in } W^{1,r}(\Omega) \cap L^{3m}(\Omega),\\
\vartheta_n &\to \vartheta && \text{strongly in } L^q(\Omega), && \forall 1 \leq q < 3m,\\
\ln \vartheta_n &\to \ln \vartheta && \text{strongly in } L^q(\Omega), && \forall 1 \leq q < 6,\\
\varrho_n &\rightharpoonup \varrho && \text{weakly in } L^{\gamma+\Theta}(\Omega),\\
\mathbf{u}_n &\to \mathbf{u} && \text{strongly in } L^q(\partial\Omega;\mathbb{R}^3), && \forall 1 \leq q < 4.
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Weak compactness II

$\varrho_n \mathbf{u}_n \rightharpoonup \varrho \mathbf{u}$ in some $L^q(\Omega; \mathbb{R}^3)$; therefore

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0 \qquad \forall \psi \in C^1_0(\mathbb{R}^3). \tag{39}$$

Moreover, if $\gamma + \Theta \geq 2$ (not true for small m and γ), the pair is also a renormalized solution to the continuity equation.

$$\int_\Omega \left(-\varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \varphi - \overline{p(\varrho, \vartheta)} \operatorname{div} \varphi + \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \varphi \right) \mathrm{d}x$$
$$= \int_\Omega \varrho \mathbf{f} \cdot \varphi \, \mathrm{d}x \quad \forall \varphi \in C^1_0(\Omega; \mathbb{R}^3), \tag{40}$$

where $\overline{p(\varrho, \vartheta)}$ denotes the weak limit of $p(\varrho_n, \vartheta_n)$ in $L^1(\Omega)$ (or, equivalently, of $p(\varrho_n, \vartheta)$ due to the strong convergence of the temperature).

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Weak compactness III

Passing to the limit in the entropy inequality we get, using the weak lower semicontinuity for the first two terms

$$\int_\Omega \Big(\frac{\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u}}{\vartheta}+\frac{\kappa(\vartheta)|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi\,\mathrm{d}x \le \int_\Omega \Big(\frac{\kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi}{\vartheta}-\overline{\varrho s(\varrho,\vartheta)}\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x \tag{41}$$

for all $\psi \in C^1_0(\Omega)$, non-negative in Ω.

Similarly, for the ballistic energy inequality

$$\int_\Omega \Big(\frac{\tilde\vartheta}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u}+\kappa(\vartheta)|\nabla\vartheta|^2\frac{\tilde\vartheta}{\vartheta^2}\Big)\,\mathrm{d}x+\lambda\int_{\partial\Omega}|\mathbf{u}|^2\,\mathrm{d}S \le \int_\Omega \Big(\varrho\mathbf{f}\cdot\mathbf{u}-\overline{\varrho s(\varrho,\vartheta)}\mathbf{u}\cdot\nabla\tilde\vartheta+\frac{\kappa(\vartheta)\nabla\vartheta}{\vartheta}\cdot\nabla\tilde\vartheta\Big)\,\mathrm{d}x \tag{42}$$

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Weak compactness IV

If $\gamma + \Theta \geq 2$ (or $s > \frac{6}{5}$) and $m > 1$, we can pass to the limit in the total energy balance

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\begin{aligned}
&- \int_\Omega \Big(\frac{1}{2} \varrho |\mathbf{u}|^2 + \overline{\varrho e(\varrho, \vartheta) + p(\varrho, \vartheta)} \Big) \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x \\
&= \int_\Omega \Big(\varrho \mathbf{f} \cdot \mathbf{u} \psi - \big(\mathbb{S}(\vartheta, \mathbf{u}) \mathbf{u} + \kappa(\vartheta) \nabla \vartheta \big) \cdot \nabla \psi \Big) \, \mathrm{d}x \qquad \forall \psi \in C^1_0(\Omega).
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To conclude, we need to show the strong convergence of the density. For this, we need the renormalized continuity equation for the limit density and velocity. If $\gamma + \Theta \geq 2$, we may use the approach due to P.L. Lions. We may follow

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Strong convergence of density I

Assume first that the density is bounded in $L^2(\Omega)$. We use test functions in the momentum equation for $n \in \mathbb{N}$ the function $\varphi := \nabla\Delta^{-1}(1_\Omega \varrho_n^\sigma)$ and for the limit problem $\varphi := \nabla\Delta^{-1}(1_\Omega \overline{\varrho^\sigma})$, where $0 < \sigma < \min\{1, \Theta\}$.

$$\overline{p(\varrho,\vartheta)\varrho^\sigma} - \Big(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\Big)\overline{\varrho^\sigma \operatorname{div} \mathbf{u}} = \overline{p(\varrho,\vartheta)}\,\overline{\varrho^\sigma} - \Big(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\Big)\overline{\varrho^\sigma} \operatorname{div} \mathbf{u}$$

a.e. in Ω.

Pass to the limit in the renormalized continuity equation

$$\int_{\mathbb{R}^3} \overline{\varrho^\sigma}\mathbf{u} \cdot \nabla\psi \,\mathrm{d}x + (1-\sigma)\int_{\mathbb{R}^3} \overline{\varrho^\sigma \operatorname{div} \mathbf{u}}\psi \,\mathrm{d}x = 0$$

for any $\psi \in C_0^1(\mathbb{R}^3)$.

We renormalize the equation above and combine it with the effective viscous flux identity.

$$\int_{\mathbb{R}^3} \overline{\varrho^\sigma}^{\frac{1}{\sigma}} \mathbf{u}\cdot\nabla\psi \,\mathrm{d}x + \frac{1-\sigma}{\sigma}\int_{\mathbb{R}^3} \frac{1}{\frac{4}{3}\mu(\vartheta)+\xi(\vartheta)}\big(\overline{p(\varrho,\vartheta)\varrho^\sigma} - \overline{p(\varrho,\vartheta)}\,\overline{\varrho^\sigma}\big)\overline{\varrho^\sigma}^{\frac{1}{\sigma}-1}\psi \,\mathrm{d}x = 0$$

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$$\int_{\mathbb{R}^3} \overline{\varrho^\sigma}^{\frac{1}{\sigma}}\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x + \frac{1-\sigma}{\sigma}\int_{\mathbb{R}^3} \frac{1}{\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)}\Big(\overline{p(\varrho,\vartheta)\varrho^\sigma} - \overline{p(\varrho,\vartheta)}\,\overline{\varrho^\sigma}\Big)\overline{\varrho^\sigma}^{\frac{1}{\sigma}-1}\psi\,\mathrm{d}x = 0$$

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Strong convergence of density I

Assume first that the density is bounded in $L^2(\Omega)$. We use test functions in the momentum equation for $n \in \mathbb{N}$ the function $\boldsymbol{\varphi} := \nabla\Delta^{-1}(1_\Omega \varrho_n^\sigma)$ and for the limit problem $\boldsymbol{\varphi} := \nabla\Delta^{-1}(1_\Omega \overline{\varrho^\sigma})$, where $0 < \sigma < \min\{1, \Theta\}$.

$$\overline{p(\varrho,\vartheta)\varrho^\sigma} - \Big(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\Big)\overline{\varrho^\sigma \operatorname{div} \mathbf{u}} = \overline{p(\varrho,\vartheta)}\,\overline{\varrho^\sigma} - \Big(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\Big)\overline{\varrho^\sigma} \operatorname{div} \mathbf{u}$$

a.e. in Ω.

Pass to the limit in the renormalized continuity equation

$$\int_{\mathbb{R}^3} \overline{\varrho^\sigma}\mathbf{u} \cdot \nabla\psi \, \mathrm{d}x + (1-\sigma)\int_{\mathbb{R}^3} \overline{\varrho^\sigma \operatorname{div} \mathbf{u}}\psi \, \mathrm{d}x = 0$$

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Strong convergence of density II

We take $\psi \equiv 1$ in Ω

$$\int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta)+\xi(\vartheta)}\left(\overline{p(\varrho,\vartheta)\varrho^\sigma} - \overline{p(\varrho,\vartheta)}\,\overline{\varrho^\sigma}\right)\overline{\varrho^\sigma}^{\frac{1}{\sigma}-1}\psi \,\mathrm{d}x = 0.$$

We use properties of the pressure function to get

$$\int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta)+\xi(\vartheta)} d\left(\overline{\varrho^{\gamma+\sigma}} - \overline{\varrho^\gamma}\,\overline{\varrho^\sigma}\right)\overline{\varrho^\sigma}^{\frac{1}{\sigma}-1}\psi \,\mathrm{d}x$$
$$+\int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta)+\xi(\vartheta)}\left(\overline{p_m(\varrho,\vartheta)\varrho^\sigma} - \overline{p_m(\varrho,\vartheta)}\,\overline{\varrho^\sigma}\right)\overline{\varrho^\sigma}^{\frac{1}{\sigma}-1}\psi \,\mathrm{d}x = 0.$$

Both functions $\varrho \mapsto \varrho^\gamma$ and $\varrho \mapsto p_m(\varrho,\vartheta)$ are increasing and the temperature converges strongly, thus both integrals above are non-negative and thus zero.

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Strong convergence of density III

$\varrho_n \to 0$ strongly in $L^q(\{x \in \Omega : \overline{\varrho^\sigma} = 0\})$ for any $1 \leq q < \gamma + \Theta$.

Due to the monotonicity of the power function $\overline{\varrho^{\gamma+\sigma}} = \overline{\varrho^\gamma}\,\overline{\varrho^\sigma}$ which implies $\overline{\varrho^\gamma} = \overline{\varrho^\sigma}^{\frac{\gamma}{\sigma}}$.

This yields that $\varrho_n^\sigma \to \varrho^\sigma$ in $L^q(\Omega)$, $1 \leq q < \frac{\gamma+\Theta}{\sigma}$; whence also $\varrho_n \to \varrho$ in $L^q(\Omega)$, $1 < q < \gamma + \Theta$.

If the density is not bounded in $L^2(\Omega)$, we do not know whether the renormalized continuity equation is fulfilled. To this aim, following the idea of E. Feireisl from the evolutionary case, we additionally estimate the oscillation defect measure

$$\mathbf{osc_q}[\varrho_n \to \varrho](Q) = \sup_{k>1}\Big(\limsup_{n\to\infty}\int_Q |T_k(\varrho_n) - T_k(\varrho)|^q \,\mathrm{d}x\Big), \tag{44}$$

where $T_k(\cdot)$ is a suitable cut-off function. if we get the estimate for $q > 2$, it is possible to verify that if the sequence of densities and velocities satisfies the renormalized continuity equation, then the same holds also for the limit functions. The rest is similar as above, with certain small modifications.

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THANK YOU VERY MUCH

FOR YOUR ATTENTION!