Contents

Introduction

My thesis consists of two loosely connected parts. Part I, composed of the first four chapters, is devoted to discussing various ways how to introduce Laplace's operator on infinite dimensional spaces, especially on Hilbert spaces. In Part II, summability of sequences in Banach spaces is dealt with. This part is made up only of the last fifth chapter.

Chapter 1 is preliminary. There we give a review of some well–known notions and results needed later on from the theory of differentiation in Banach spaces and the theory of linear operators on Hilbert spaces.

Chapter 2 is concerned with the theory of the so–called Lévy laplacian studied by P. Lévy in his books [Lev0] and [Lev1]. In Section 2.1, for a real functional f over a domain in a real separable Hilbert space $\mathbf{H}$, the spherical mean of f in the sense of Gâteaux–Lévy is defined by the limit of the sequence of its spherical means on suitable n-dimensional subspaces in $\mathbf{H}$ as n tends to ∞. The peculiar feature of the theory consists in the fact that the spherical mean in the sense of Gâteaux–Lévy is not a mean with respect to a countably additive measure. By means of this notion of spherical mean the Lévy laplacian is defined. In comparison with finite dimensional case, the definition of spherical mean depends, in general, on the choice of an orthonormal basis in $\mathbf{H}$ and so does the Lévy laplacian. Futhermore, on the class of functionals having second Fréchet derivatives, the Lévy laplacian $\boldsymbol{L}^E$ can be thought of as the limit of arithmetic means of the sequence of non-mixed second partial derivatives with respect to a given orthonormal basis E in $\mathbf{H}$, i.e., for a real functional f having the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{H}$,

$$\boldsymbol{L}^E f(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} D_{jj} f(x) \tag{1}$$

if the limit exists (where $D_{jj}f(x) = f''(x)(e_j, e_j)$).

In Section 2.2, weakly continuous functionals are shown to have the invariant mean value property, i.e., the mean value property for any orthonormal basis in $\mathbf{H}$. In Section 2.3, it is shown that for uniformly continuous functionals volume and spherical means coincide. There we prove also a striking theorem due to P. Lévy which tells us that a uniformly continuous functional f on a sphere in $\mathbf{H}$ is equal to its spherical mean almost everywhere on the sphere in the sense of P. Lévy provided that the spherical mean exists. In the remainder of the section, the Dirichlet and the Poisson problem are discussed for the class of uniformly continuous functionals. In Section 2.4, we are interested in conditions under which a harmonic functional, i.e., a functional with its Lévy laplacian equal identically to zero, has the mean value property. We give an example of an infinitely Fréchet differentiable harmonic functional on $\mathbf{H}$ which has only the local mean value property on $\mathbf{H}$.

In Section 2.5, examples of irregular differential operators of the second order are given. What means that a differential operator of the second order in $\mathbf{H}$ is irregular ? Before giving an explanation, let us note that the Lévy laplacian of any functional depending only on a finite number of coordinates in $\mathbf{H}$ is identically zero, as easily seen. If this is true for a differential operator $\boldsymbol{D}$ of the second order in $\mathbf{H}$, then $\boldsymbol{D}$ is called irregular. The notion of irregular differential operator was first introduced in the author's article presented below in Chapter 4.

In Section 2.6, dependence of the spherical mean and the Lévy laplacian on the choice of an orthonormal basis in $\mathbf{H}$ is thoroughly studied. It is shown there that if the Lévy laplacian $\boldsymbol{L}^E f(x)$ of a functional f at a point $x \in \mathbf{H}$ exists for any orthonormal basis E in $\mathbf{H}$, then the value $\boldsymbol{L}^E f(x)$ is independent of the choice of an orthonormal basis E in $\mathbf{H}$. On the other hand, if this is not the case, then there is an orthonormal basis E in $\mathbf{H}$ such that the values of the Lévy laplacian of f at x fill a non-degenerate closed interval while the orthonormal basis E is being rearranged.

In Chapter 3, we give a sketch of the potential theory on abstract Wiener spaces developed by L. Gross in his famous paper [Gro2] where the probabilistic approach to the classical potential theory was generalized to infinite dimensional spaces, namely, to separable Banach spaces. This approach enables us to establish notions useful in finite dimensional case, including Brownian motion, spherical mean, laplacian, Green measure and so on, on separable Banach spaces and to solve the Dirichlet and the Poisson problem in a much more satisfactory way than we have done for the Lévy laplacian in Section 2.3. At the end of this chapter, it is shown that on Hilbert spaces the laplacian introduced by L. Gross is the same as that studied first by Yu. L. Daletzky. Daletzky's laplacian is an important example of regular differential operators of the second order. In fact, a differential operator $\boldsymbol{D}$ of the second order in $\mathbf{H}$ is said to be regular provided that there are an orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$ in $\mathbf{H}$ and a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers such that $\{a_n\}_{n=1}^{\infty} \in \ell^1$ (i.e., $\sum |a_n| < \infty$) and

$$\boldsymbol{D}f(x) = \sum_{n=1}^{\infty} a_n D_{nn} f(x) \tag{2}$$

whenever a functional f has the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{H}$. If any a_n is strictly positive, then such a differential operator $\boldsymbol{D}$ is called a Daletzky laplacian.

In Chapter 4, an author's article published in *Comment. Math. Univ. Carolinae* 39,1(1998), 115–135 is presented. The article deals with differential operators of the second order in a real separable Hilbert space $\mathbf{H}$. It is shown there that every differential operator of the second order in $\mathbf{H}$ can be decomposed into a regular and an irregular differential operator. In spite of being differential operators of the second order irregular differential operators behave like those of the first order. A characterization of irregularity of differential operators by means of independence on the choice of an orthonormal basis in

$\mathbf{H}$ is also given. Moreover, differential operators satisfying maximum principle are characterized and, for an irregular differential operator satisfying maximum principle, the corresponding Dirichlet and the Poisson problem are discussed; in particular, weakly continuous functionals are 'harmonic'.

In Chapter 5, an author's article prepared for publication is presented. In the article, the following statements are proved: Given a sequence $\{a_n\}_{n=1}^{\infty}$ in a Banach space $\mathbf{X}$ enjoying the weak Banach-Saks property, there is a subsequence (or a permutation) $\{b_n\}_{n=1}^{\infty}$ of the sequence $\{a_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} b_j = a$$

whenever a belongs to the closed convex hull of the set of weak limit points of $\{a_n\}_{n=1}^{\infty}$. In case $\mathbf{X}$ has the Banach-Saks property and $\{a_n\}_{n=1}^{\infty}$ is bounded the converse assertion holds too. Moreover, in case $\mathbf{X} = \mathbf{R}$ it tells us that, for a bounded sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, there is a permutation $\{b_n\}_{n=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} b_j = a$$

whenever a real number a lies between $\liminf a_n$ and $\limsup a_n$. This assertion plays an important role in our discussing dependence of the Lévy laplacian on the choice of an orthonormal basis in $\mathbf{H}$ in Section 2.6. Futhermore, a new characterization of reflexive spaces in terms of limit points and cores of bounded sequences is also given.

Throughout the text, all Banach spaces are supposed to be real and only real separable Hilbert spaces are considered.

Bibliography is given at the end of the text. Let us emphasize that the articles incorporated into Chapters 4 and 5 have their own references marked only by numbers, such as [4], whereas throughout the other text references are marked by letters sometimes in conjunction with numbers, such as [BaS] and [Gro2]. All sections of Chapters 2 and 3 and the articles from Chapters 4 and 5 are followed by Comments where additional notes are added and an author's contribution is explained if any.

Part I

Laplacians in Hilbert spaces

Chapter 1

Preliminaries

1.1 Fréchet derivatives

Let $\mathbf{X}$ and $\mathbf{Y}$ be real Banach spaces. The set of all bounded linear operators $T : \mathbf{X} \to \mathbf{Y}$ is denoted by $\mathcal{L}(\mathbf{X}, \mathbf{Y})$. Moreover, denote by $\mathcal{N}_s^2(\mathbf{X}, \mathbf{Y})$ the set of all bilinear operators $a : \mathbf{X} \times \mathbf{X} \to \mathbf{Y}$ that are symmetric, i.e., $a(h, k) = a(k, h),\ h, k \in \mathbf{X}$, and bounded, i.e.,

$$\|a\| = \sup_{\|h\|_{\mathbf{X}}=1, \|k\|_{\mathbf{X}}=1} \|a(h, k)\|_{\mathbf{Y}} < +\infty$$

(where $\|\cdot\|_{\mathbf{X}}$ is the norm in $\mathbf{X}$ and $\|\cdot\|_{\mathbf{Y}}$ is the norm in $\mathbf{Y}$). It is well-known that $(\mathcal{N}_s^2(\mathbf{X}, \mathbf{Y}), \|\cdot\|)$ is a Banach space. Let $f : \mathbf{X} \to \mathbf{Y}$ be defined on a ball with center at a point $x \in \mathbf{X}$. Then the *derivative* of f at x *in the direction* $h \in \mathbf{X}$ is defined by

$$D_h f(x) = \lim_{t \to 0+} \frac{f(x + th) - f(x)}{t}$$

if the limit exists. Furthermore, if there is $L \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ such that

$$f(x + h) - f(x) = Lh + o(\|h\|_{\mathbf{X}})$$

as $h \to 0$, then $f'(x) = L$ is called the *Fréchet derivative of f at x*. If $f' : \mathbf{X} \to \mathcal{L}(\mathbf{X}, \mathbf{Y})$ has the Fréchet derivative $T \in \mathcal{L}(\mathbf{X}, \mathcal{L}(\mathbf{X}, \mathbf{Y}))$ at x, then $f''(x)(h, k) = (Th)k,\ h, k \in \mathbf{X}$ is called the *second Fréchet derivative* of f at x. It is well-known that $f''(x) \in \mathcal{N}_s^2(\mathbf{X}, \mathbf{Y})$ and $f''(x)(h, k) = D_h(D_k f(x)),\ h, k \in \mathbf{X}$. The Fréchet derivatives of higher orders are introduced similarly. If $n \in \mathbf{N}$ and $G \subset \mathbf{X}$ is open, then denote by $\mathcal{C}^n(G, \mathbf{Y})$ the set of all $f : G \to \mathbf{Y}$ that have continuous Fréchet derivatives $f^{(k)}$ of k-th order on G for each $k \le n$ and, finally, put

$$\mathcal{C}^\infty(G, \mathbf{Y}) = \bigcap_{n=1}^{\infty} \mathcal{C}^n(G, \mathbf{Y}).$$

In case $\mathbf{Y} = \mathbf{R}$ we shall write $\mathcal{C}^n(G)$ instead of $\mathcal{C}^n(G, \mathbf{R})$.

1.2 Linear operators on Hilbert spaces

In this section, we shall give a review of some results from theory of linear operators on Hilbert spaces needed later on. Such results can be found in several books, see e.g. [Kuo1, Chapter I, §1] and [GK], [SF]. In what follows, $(\mathbf{H}, (\,.\,,\,.\,))$ is a real separable Hilbert space with the induced norm $\|x\| = \sqrt{(x,x)},\ x \in \mathbf{H}$. Denote by $\mathcal{L} = \mathcal{L}(\mathbf{H})$ the Banach space of all bounded linear operators $S : \mathbf{H} \to \mathbf{H}$ endowed with the operator norm

$$\|S\| = \sup\{\|Sx\|;\ x \in \mathbf{H}, \|x\| = 1\}.$$

Denote by $\mathcal{L}_s = \mathcal{L}_s(\mathbf{H})$ the Banach space of all bounded linear operators S on $\mathbf{H}$ that are self-adjoint (i.e., $S = S^*$ where S^* is the adjoint operator to S) endowed with the operator norm. Recall that a linear operator $S : \mathbf{H} \to \mathbf{H}$ is compact provided that the closure of $S(M)$ in $\mathbf{H}$ is compact for any bounded set $M \subset \mathbf{H}$. Moreover, denote by $\mathcal{L}_\infty = \mathcal{L}_\infty(\mathbf{H})$ the Banach space of all compact linear operators S on $\mathbf{H}$ endowed with the operator norm. Of course, $\mathcal{L}_\infty \subset \mathcal{L}$. Given $T \in \mathcal{L}_\infty \cap \mathcal{L}_s$, it is well–known that there are an orthonormal basis $E = \{e_n\}_{n=1}^\infty$ of $\mathbf{H}$ consisting of eigenvectors of T and a sequence $\{\mu_n\}_{n=1}^\infty$ of real numbers such that

$$Tx = \sum_{n=1}^\infty \mu_n (x, e_n) e_n,\ x \in \mathbf{H}. \tag{1}$$

Futhermore, μ_n is an eigenvalue of T corresponding to the eigenvector e_n. If T is, in addition, positively definite, i.e., $(Th, h) \geq 0,\ h \in \mathbf{H}$, then $\mu_n \geq 0,\ n \in \mathbf{N}$ and it is easily seen that

$$Sx = \sum_{n=1}^\infty \sqrt{\mu_n} (x, e_n) e_n,\ x \in \mathbf{H}$$

is a unique positively definite operator of $\mathcal{L}_\infty \cap \mathcal{L}_s$ with $T = S^2$. Consequently, the operator S will be denoted by $\sqrt{T}$.

A linear operator $S : \mathbf{H} \to \mathbf{H}$ is said to be *of Hilbert-Schmidt type* provided that there is an orthonormal basis $E = \{e_n\}_{n=1}^\infty$ such that

$$\|S\|_2 = \left(\sum_{n=1}^\infty \|Se_n\|^2 \right)^{\frac{1}{2}} < +\infty. \tag{2}$$

It is easy to see that $\|S\|_2$ does not depend on the choice of E. Denote by $\mathcal{L}_2 = \mathcal{L}_2(\mathbf{H})$ the Banach space of all operators of Hilbert-Schmidt type on $\mathbf{H}$ endowed with the norm $\|\cdot\|_2$. These operators have a few relevant properties:

(H1) Given an orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$ in $\mathbf{H}$, $\mathcal{L}_2$ constitutes a Hilbert space with the inner product

$$((A, B)) = \sum_{n=1}^{\infty} (Ae_n, Be_n), \ \ A, B \in \mathcal{L}_2.$$

(H2) For each $A \in \mathcal{L}_2$, $A^* \in \mathcal{L}_2$ and $\|A\|_2 = \|A^*\|_2$.

(H3) Let $A \in \mathcal{L}_2$ and $B \in \mathcal{L}$. Then $AB \in \mathcal{L}_2$, $\|AB\|_2 \leq \|A\|_2\|B\|$ and $BA \in \mathcal{L}_2$, $\|BA\|_2 \leq \|A\|_2\|B\|$.

(H4) Each $A \in \mathcal{L}_2$ is compact and $\|A\|_2 \geq \|A\|$.

Given $S \in \mathcal{L}_{\infty}$, it is easily seen that $S^*S \in \mathcal{L}_{\infty} \cap \mathcal{L}_s$ is positively definite. Putting $T = \sqrt{S^*S}$, there are an orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$ and a sequence $\{\mu_n\}_{n=1}^{\infty}$ of positive numbers such that (1) holds. Obviously,

$$\|S\|_2 = \left(\sum_{n=1}^{\infty} \mu_n^2\right)^{\frac{1}{2}}. \tag{3}$$

Futhermore, define

$$\|S\|_1 = \sum_{n=1}^{\infty} \mu_n. \tag{4}$$

A compact operator S on $\mathbf{H}$ is said to be *nuclear* (or *of trace class type*) provided that $\|S\|_1 < +\infty$. Denote by $\mathcal{L}_1$ the Banach space of all nuclear operators on $\mathbf{H}$ endowed with the norm $\|\cdot\|_1$. Obviously, if $T \in \mathcal{L}_1 \cap \mathcal{L}_s$ is positively definite, then $\sqrt{T} \in \mathcal{L}_2$. Nuclear operators have the following properties:

(N1) Each $A \in \mathcal{L}_1$ belongs to $\mathcal{L}_2$ and $\|A\|_2 \leq \|A\|_1$.

(N2) If $A \in \mathcal{L}_1$ and $E = \{e_n\}_{n=1}^{\infty}$ is an orthonormal basis in $\mathbf{H}$, then the series

$$tr(A) = \sum_{n=1}^{\infty} (Ae_n, e_n) \tag{5}$$

absolutely converges. The sum $tr(A)$ is called the *trace* of A and does not depend on the choice of E. Of course, $tr(A) = tr(A^*)$ for each $A \in \mathcal{L}_1$.

(N3) For each $A \in \mathcal{L}_1$, $A^* \in \mathcal{L}_1$.

(N4) Let $A, B \in \mathcal{L}_2$. Then $AB \in \mathcal{L}_1$, $tr(AB) = tr(BA)$ and $\|AB\|_1 \leq \|A\|_2\|B\|_2$.

(N5) Let $A \in \mathcal{L}_1$ and $B \in \mathcal{L}$. Then $AB \in \mathcal{L}_1$, $BA \in \mathcal{L}_1$, $\|AB\|_1 \leq \|A\|_1\|B\|$, $\|BA\|_1 \leq \|A\|_1\|B\|$ and $tr(AB) = tr(BA)$.

(N6) The dual space $\mathcal{L}_{\infty}^*$ of $\mathcal{L}_{\infty}$ is isometric to $\mathcal{L}_1$, i.e., for each $\tau \in \mathcal{L}_{\infty}^*$ there exists a unique $A \in \mathcal{L}_1$ such that $\|\tau\| = \|A\|_1$ and

$$\tau(B) = tr(AB), \ \ B \in \mathcal{L}_{\infty}. \tag{6}$$

A linear operator S on $\mathbf{H}$ is said to be *finite-dimensional* provided that $S(\mathbf{H})$ is a finite dimensional subspace of $\mathbf{H}$. We denote by $\mathcal{L}_f$ the linear space of all finite–dimensional operators on $\mathbf{H}$. Obviously,

$$\mathcal{L}_f \subset \mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathcal{L}_\infty \subset \mathcal{L}.$$

Futhermore, $\mathcal{L}_f$ is dense in $\mathcal{L}_1$ with respect to $\|\cdot\|_1$, $\mathcal{L}_f$ is dense in $\mathcal{L}_2$ with respect to $\|\cdot\|_2$ and $\mathcal{L}_f$ is dense in $\mathcal{L}_\infty$ with respect to the operator norm $\|\cdot\|$.

In conclusion, a well-known theorem due to J. von Neumann will be mentioned.

Theorem 1. *(J. von Neumann) For each $S \in \mathcal{L}_s$ and $\varepsilon > 0$, there is $T \in \mathcal{L}_2 \cap \mathcal{L}_s$ such that $\|T\|_2 < \varepsilon$ and, putting $D = S - T$, there are an orthonormal basis $E = \{e_n\}_{n=1}^\infty$ and a (bounded) sequence $\{\mu_n\}_{n=1}^\infty$ of real numbers such that*

$$Dx = \sum_{n=1}^\infty \mu_n (x, e_n) e_n, \; x \in \mathbf{H}. \tag{7}$$

Proof: See [SF, p. 111]. □

By this theorem, each $S \in \mathcal{L}_s$ can be written as the sum of a compact operator T and an operator D of type (7). Thus it is easy to see that the following assertion holds.

Corollary. *An $S \in \mathcal{L}_s$ is compact if and only if $\lim_{n\to\infty}(Se_n, e_n) = 0$ for each orthonormal basis $E = \{e_n\}_{n=1}^\infty$ in $\mathbf{H}$.*

Chapter 2

The Lévy laplacian

2.1 Spherical means and the Lévy laplacian

First of all, we introduce spherical and volume means of functionals defined on a Hilbert space. In what follows, $(\mathbf{H}, (\,.\,,\,.\,))$ is a real separable Hilbert space with the induced norm $\|x\| = \sqrt{(x,x)}$, $x \in \mathbf{H}$. For $x \in \mathbf{H}$ and $r > 0$, define the sphere and the ball in $\mathbf{H}$ with center x and radius r by

$$S(x,r) = \{y \in \mathbf{H};\ \|y-x\| = r\} \text{ and } B(x,r) = \{y \in \mathbf{H};\ \|y-x\| < r\},$$

respectively. Let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathbf{H}$. For $n \in \mathbf{N}$ let σ_n and λ_n stand for the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$ and the Lebesgue measure in $\mathbf{R}^n$, respectively. Furthermore, for $x \in \mathbf{R}^n$ and $r > 0$ denote by $S_n(x,r)$ and $B_n(x,r)$ the sphere and the ball in $\mathbf{R}^n$ with center x and radius r, respectively. For a real functional f defined on a sphere $S(x,r)$ in $\mathbf{H}$ put, for each $n \in \mathbf{N}$,

$$\mathcal{M}_n(f,x,r) = \fint_{S_n(0,1)} f(x + r\sum_{j=1}^{n} y_j e_j)\,\sigma_n(dy)$$

$$\left(= \frac{1}{\sigma_n(S_n(0,1))} \int_{S_n(0,1)} f(x + r\sum_{j=1}^{n} y_j e_j)\,\sigma_n(dy) \right)$$

whenever the integral on the right-hand side exists, and define

$$\mathcal{M}^E(f,x,r) = \lim_{n\to\infty} \mathcal{M}_n(f,x,r) \tag{1}$$

if the right-hand side exists. Similarly, for a real functional f defined on a ball $B(x,r)$ in $\mathbf{H}$ put, for each $n \in \mathbf{N}$,

$$\mathcal{A}_n(f,x,r) = \fint_{B_n(0,1)} f(x + r\sum_{j=1}^{n} y_j e_j)\,\lambda_n(dy)$$

$$\left(= \frac{1}{\lambda_n(B_n(0,1))} \int_{B_n(0,1)} f(x + r\sum_{j=1}^{n} y_j e_j)\, \lambda_n(dy) \right)$$

whenever the integral on the right-hand side exists, and define

$$\mathcal{A}^E(f,x,r) = \lim_{n\to\infty} \mathcal{A}_n(f,x,r) \tag{2}$$

if the right-hand side exists. The numbers $\mathcal{M}^E(f,x,r)$ and $\mathcal{A}^E(f,x,r)$ are called the *spherical mean* of f on $S(x,r)$ and the *volume mean* of f on $B(x,r)$ with respect to the orthonormal basis E, respectively. If the number $\mathcal{M}^E(f,x,r)$ does not depend on the choice of the orthonormal basis E, i.e., there is $m \in \mathbf{R}$ such that $m = \mathcal{M}^F(f,x,r)$ for each orthonormal basis F of $\mathbf{H}$, then we shall write $\mathcal{M}(f,x,r)$ instead of $\mathcal{M}^E(f,x,r)$ and refer to $\mathcal{M}(f,x,r)$ as the *invariant spherical mean* of f on $S(x,r)$. The *invariant volume mean* is defined similarly.

Remark 1. It is easy to see that the following assertions hold. Let E be an orthonormal basis of $\mathbf{H}$, $x \in \mathbf{H}$, $r > 0$ and f,g be real functionals defined on $\mathbf{H}$. If $a,b \in \mathbf{R}$ then

$$\mathcal{M}^E(af + bg, x, r) = a\mathcal{M}^E(f,x,r) + b\mathcal{M}^E(g,x,r)$$

whenever the right-hand side makes sense. If $\psi_{x,r}(y) = x + ry,\ y \in \mathbf{H}$, then

$$\mathcal{M}^E(f,x,r) = \mathcal{M}^E(f \circ \psi_{x,r}, 0, 1)$$

whenever one side exists. Moreover, $|\mathcal{M}^E(f,x,r)| \le \mathcal{M}^E(|f|,x,r)$ if both sides of the inequality make sense.

Definition 1. *Let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathbf{H}$ and f be a real functional defined on a ball in $\mathbf{H}$ with center at a point $x \in \mathbf{H}$. Then the Lévy laplacian of f at x (with respect to E) is defined by*

$$\Delta_L^E f(x) = 2 \lim_{r\to 0+} \frac{\mathcal{M}^E(f,x,r) - f(x)}{r^2} \tag{3}$$

whenever the limit exists. If the number $\Delta_L^E f(x)$ does not depend on the choice of E, i.e., there is $m \in \mathbf{R}$ such that $m = \Delta_L^F f(x)$ for each orthonormal basis F, then we shall write $\Delta_L f(x)$ instead of $\Delta_L^E f(x)$.

Remark 2. A real functional f defined on an open set $G \subset \mathbf{H}$ is said to have the *mean value property* on G (relative to an orthonormal basis E of $\mathbf{H}$) provided that, for each ball $B(x,r)$ contained with its (strong) closure in G, $\mathcal{M}^E(f,x,r) = f(x)$. Of course, such a functional f satisfies $\Delta_L^E f = 0$ on G. If f has the mean value property on G relative to each orthonormal basis E of $\mathbf{H}$, then f is said to have the *invariant mean value property* on G. Of course, f has the invariant mean value property on G if and only if $\mathcal{M}(f,x,r) = f(x)$ for

each ball $B(x,r)$ contained with its closure in G. Obviously, such an f satisfies $\Delta_L f = 0$ on G.
Example 1. If $L : \mathbf{H} \to \mathbf{R}$ is linear (not necessarily bounded), then L has the invariant mean value property on $\mathbf{H}$. Indeed, if $E = \{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathbf{H}$, $x \in \mathbf{H}$ and $r > 0$, then for each $n \in \mathbf{N}$,

$$\mathcal{M}_n(L,x,r) = Lx + r\sum_{j=1}^{n} Le_j \fint_{S_n(0,1)} y_j\,\sigma_n(dy) = Lx$$

because $\fint_{S_n(0,1)} y_j\,\sigma_n(dy) = 0$ by the symmetry of σ_n.
Example 2. If $a : \mathbf{H}\times\mathbf{H} \to \mathbf{R}$ is a bilinear functional (not necessarily bounded), $f(y) = a(y,y),\ y \in \mathbf{H}$ and $E = \{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathbf{H}$, then for each $x \in \mathbf{H}$ and $r > 0$,

$$\mathcal{M}^E(f,x,r) = f(x) + r^2 \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} a(e_j,e_j)$$

if one side exists. Indeed, by the symmetry of σ_n,

$$\fint_{S_n(0,1)} y_j y_k\,\sigma_n(dy) = 0$$

if $j \neq k$ and

$$\fint_{S_n(0,1)} y_1^2\,\sigma_n(dy) = \ldots = \fint_{S_n(0,1)} y_n^2\,\sigma_n(dy).$$

Since

$$\fint_{S_n(0,1)} (y_1^2 + \ldots + y_n^2)\,\sigma_n(dy) = 1$$

we have

$$\fint_{S_n(0,1)} y_j^2\,\sigma_n(dy) = \frac{1}{n}$$

for $j = 1,\ldots,n$. For each $n \in \mathbf{N}$,

$$\mathcal{M}_n(f,x,r) = a(x,x) + r\sum_{k=1}^{n} a(x,e_k)\fint_{S_n(0,1)} y_k\,\sigma_n(dy) +$$

$$+ r\sum_{j=1}^{n} a(e_j,x)\fint_{S_n(0,1)} y_j\,\sigma_n(dy) + r^2\sum_{j,k=1}^{n} a(e_j,e_k)\fint_{S_n(0,1)} y_j y_k\,\sigma_n(dy)$$

and, as a result,

$$\mathcal{M}_n(f,x,r) = f(x) + r^2\cdot\frac{1}{n}\sum_{j=1}^{n} a(e_j,e_j),$$

which completes the proof. In particular, if f is a quadratic form of a compact linear operator $T : \mathbf{H} \to \mathbf{H}$, i.e., $f(x) = (Tx, x),\ x \in \mathbf{H}$, then f has the invariant mean value property on $\mathbf{H}$ because $\lim_{n\to\infty}(Te_n, e_n) = 0$ for each orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$ of $\mathbf{H}$.

It is easy to see that if f is a real functional having the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{H}$ and $E = \{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathbf{H}$, then

$$f''(x)(h, k) = \sum_{i,j=1}^{\infty} D_{ij} f(x)(h, e_i)(k, e_j),\ \ h, k \in \mathbf{H}$$

where $D_{ij} f(x) = D_{e_i}(D_{e_j} f(x))$.

Definition 2. *For an orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$ and a real functional f having the Fréchet derivative of 2-nd order at a point $x \in \mathbf{H}$ put*

$$\boldsymbol{L}^E f(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} D_{jj} f(x)$$

whenever the limit exists. If the number $\boldsymbol{L}^E f(x)$ does not depend on the choice of E, i.e., there is $m \in \mathbf{R}$ such that $m = \boldsymbol{L}^F f(x)$ for each orthonormal basis F, then we shall write $\boldsymbol{L} f(x)$ instead of $\boldsymbol{L}^E f(x)$.

Remark 3. If E is an orthonormal basis of $\mathbf{H}$, a real functional f has the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{H}$ and g is a quadratic form of $f''(x)$, i.e., $g(y) = f''(x)(y, y),\ y \in \mathbf{H}$, then

$$\boldsymbol{L}^E f(x) = \mathcal{M}^E(g, 0, 1)$$

whenever one side exists, see Example 2.

The following theorem shows that $\boldsymbol{L}^E$ is an extention of Δ_L^E on the class of functionals having the second Fréchet derivatives. Therefore we shall refer to $\boldsymbol{L}^E$ as the Lévy laplacian, too.

Theorem 1. *If E is an orthonormal basis of $\mathbf{H}$, f is a real functional having the Fréchet derivative of 2-nd order at a point $x \in \mathbf{H}$ and there is $R > 0$ such that f has $\mathcal{M}^E(f, x, r)$ for all $0 < r < R$, then $\Delta_L^E f(x)$ and $\boldsymbol{L}^E f(x)$ are both defined and $\Delta_L^E f(x) = \boldsymbol{L}^E f(x)$.*

Proof: Since f has the second Fréchet derivative at x, it holds that

$$f(x + h) - f(x) = f'(x)h + \frac{1}{2} f''(x)(h, h) + o(\|h\|^2)$$

as $h \to 0$. Therefore, for each $n \in \mathbf{N}$, we have

$$\mathcal{M}_n(f, x, r) - f(x) = \frac{r^2}{2} \cdot \frac{1}{n} \sum_{j=1}^{n} D_{jj} f(x) + g_n(r)$$

with $\lim_{r\to 0+} g_n(r)/r^2 = 0$ uniformly for $n \in \mathbf{N}$, i.e.,

$$2\lim_{r\to 0+} \frac{\mathcal{M}_n(f,x,r)-f(x)}{r^2} = \frac{1}{n}\sum_{j=1}^{n} D_{jj}f(x) \tag{4}$$

uniformly for $n \in \mathbf{N}$. Futhermore, for all $0 < r < R$,

$$2\lim_{n\to\infty} \frac{\mathcal{M}_n(f,x,r)-f(x)}{r^2} = 2\frac{\mathcal{M}^E(f,x,r)-f(x)}{r^2}. \tag{5}$$

By (4) and (5),

$$\Delta_L^E f(x) = 2\lim_{r\to 0+}\left(\lim_{n\to\infty} \frac{\mathcal{M}_n(f,x,r)-f(x)}{r^2}\right) =$$

$$= 2\lim_{n\to\infty}\left(\lim_{r\to 0+} \frac{\mathcal{M}_n(f,x,r)-f(x)}{r^2}\right) = \boldsymbol{L}^E f(x).$$

□

Comments

The notion of means of functionals over domains in the Hilbert space $L^2(0,1)$ of real square-summable functions on $(0,1)$ was studied for the first time by R. Gâteaux [Gat]. His ideas were developed by P. Lévy in his books [Lev0] and [Lev1] (unfortunately, [Lev0] not available to the author). Moreover, P. Lévy introduced spherical and volume means of functionals defined on an arbitrary Hilbert space $\mathbf{H}$ in the same way as we have done at the beginning of the section, see [Lev1, p. 247]. The peculiar feature of the theory of means in the sense of Gâteaux–Lévy consists in the fact that there is no σ-additive Borel measure μ on $S(0,1)$ such that $\mu(S(0,1)) = 1$ and

$$\mathcal{M}(f,0,1) = \int_{S(0,1)} f\, d\mu$$

for any bounded Borel functional f on $S(0,1)$ for which $\mathcal{M}(f,0,1)$ exists. Indeed, assume on the contrary that such a μ exists. By Example 2, given an orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$ in $\mathbf{H}$,

$$1 = \int_{S(0,1)} 1\, d\mu = \int_{S(0,1)} (\sum_{n=1}^{\infty}(y,e_n)^2)\, \mu(dy) = \sum_{n=1}^{\infty}\int_{S(0,1)} (y,e_n)^2\, \mu(dy) = 0,$$

which is a contradiction. Thus this theory is quite different from that developed by N. Wiener in [Wie]. Compare also with the notion of the average $L(f,x,r)$ of a functional f on a sphere $S(x,r)$, as given in Section 3.2.

P. Lévy studied first the laplacian named now after him, see [Lev1, p. 247, 249 and 317]. Of course, P. Lévy was aware of the fact that, for a given orthonormal basis E of $\mathbf{H}$, both definitions of his laplacian Δ_L^E and $\boldsymbol{L}^E$ coincide on a wide class of functionals having second Fréchet derivatives. In [Fel1], it was shown that, given a functional f having the second Fréchet derivative at a point $x \in \mathbf{H}$, $\Delta_L^E f(x) = \boldsymbol{L}^E f(x)$ in the sense that if one side exists, so does the other and the equality holds. See also [Oba1], [KOS] and cf. [Kuo2, p.214]. In Theorem 1, we have proved a bit stronger result.

Question: Is there a functional f having the second Fréchet derivative at a point $x \in \mathbf{H}$ such that, for a given orthonormal basis E of $\mathbf{H}$, $\boldsymbol{L}^E f(x)$ exists, but there is no $R > 0$ for which $\mathcal{M}^E(f, x, r)$ exists for each $r \in (0, R)$ (which means, in particular, that $\Delta_L^E f(x)$ is not defined)?

2.2 Weakly continuous functionals

Now our objective is to prove that weakly continuous functionals have the invariant mean value property. For $K \subset \mathbf{H}$ denote by $\mathcal{C}_w(K)$ the set of all continuous functionals $f : (K, w) \to \mathbf{R}$ where w is the weak topology on $\mathbf{H}$. Such functionals are called *weakly continuous* on K.

Proposition 1. *Let E be an orthonormal basis in $\mathbf{H}$, $x \in \mathbf{H}$, $r > 0$ and for each $m \in \mathbf{N}$ let f_m be a real functional having $\mathcal{M}^E(f_m, x, r)$. If $\lim_{m\to\infty} f_m = f$ uniformly on $S(x, r)$, then f has $\mathcal{M}^E(f, x, r)$ and*

$$\mathcal{M}^E(f, x, r) = \lim_{m\to\infty} \mathcal{M}^E(f_m, x, r).$$

PROOF: Since $\lim_{m\to\infty} f_m = f$ uniformly on $S(x, r)$ we have

$$\lim_{m\to\infty} \mathcal{M}_n(f_m, x, r) = \mathcal{M}_n(f, x, r) \tag{1}$$

uniformly for $n \in \mathbf{N}$. Moreover, for each $m \in \mathbf{N}$,

$$\lim_{n\to\infty} \mathcal{M}_n(f_m, x, r) = \mathcal{M}^E(f_m, x, r). \tag{2}$$

By (1) and (2),

$$\lim_{m\to\infty} \mathcal{M}^E(f_m, x, r) = \lim_{n\to\infty} \mathcal{M}_n(f, x, r) = \mathcal{M}^E(f, x, r),$$

which completes the proof. □

Example 1. Denote by I the set of all sequences $\alpha : \mathbf{N} \to \mathbf{N}_0 (= \mathbf{N} \cup \{0\})$ such that $\alpha(n) = 0$ except for a finite number of values of n. Let $E = \{e_n\}_{n=1}^\infty$ be an orthonormal basis of $\mathbf{H}$. For $\alpha \in I$ put

$$p_\alpha(x) = \prod_{n=1}^{\infty} (x, e_n)^{\alpha(n)}, \ x \in \mathbf{H}. \tag{3}$$

Furthermore, let $\mathcal{P}_E$ be the set of all (finite) linear combinations of p_α, $\alpha \in I$. Then each $f \in \mathcal{P}_E$ has the mean value property relative to E. By Remark 1 in Section 2.1, in order to prove this fact it is sufficient to show that each p_α, $\alpha \in I$ has the mean value property on $\mathbf{H}$ relative to E. First of all, we show that if f, g together with f^2, g^2 have the mean value property on $\mathbf{H}$ relative to E, then so does $f \cdot g$. Indeed, for $x \in \mathbf{H}$ and $r > 0$, we have

$$|(f - f(x))(g - g(x))| \leq (f - f(x))^2 + (g - g(x))^2 =$$

$$= f^2 - 2f(x)f + f(x)^2 + g^2 - 2g(x)g + g(x)^2, \ x \in \mathbf{H}$$

and hence

$$\mathcal{M}^E(f \cdot g, x, r) - f(x)g(x) = 0.$$

Moreover, for each $i, m \in \mathbf{N}$,

$$\lim_{n\to\infty} ⨍_{S_n(0,1)} y_i^m \, \sigma_n(dy) = 0. \tag{4}$$

For $m = 1, 2$ see Examples 1 and 2 in Section 2.1. For $m > 2$ and $n \geq i$ we have

$$\left| ⨍_{S_n(0,1)} y_i^m \, \sigma_n(dy) \right| \leq ⨍_{S_n(0,1)} y_i^2 \, \sigma_n(dy) = \frac{1}{n},$$

which tends to 0 as $n \to \infty$. Now we prove that, for each $i \in \mathbf{N}$ and $N \in \mathbf{N}_0$,

$$f(x) = (x, e_i)^N, \ x \in \mathbf{H} \tag{5}$$

has the mean value property on $\mathbf{H}$ relative to E. Indeed, for $x \in \mathbf{H}$ and $r > 0$,

$$\mathcal{M}_n(f, x, r) = ⨍_{S_n(0,1)} ((x, e_i) + r y_i)^N \, \sigma_n(dy) =$$

$$= \sum_{m=0}^{N} \binom{N}{m} (x, e_i)^{N-m} r^m ⨍_{S_n(0,1)} y_i^m \, \sigma_n(dy)$$

for all $n \geq i$ and hence, by (4), $\mathcal{M}^E(f, x, r) = f(x)$. Finally, for each $\alpha \in I$ the functional p_α is a finite product of functionals of the type (5) and, consequently, p_α has the mean value property on $\mathbf{H}$ relative to E.

Theorem 1. *If G is an open set in $\mathbf{H}$ and f is a weakly continuous functional on G, then f has the invariant mean value property on G.*

Proof: Given a ball $B = B(x, r)$ contained with its (strong) closure $\overline{B}$ in G and an orthonormal basis E of $\mathbf{H}$, we have to show that $\mathcal{M}^E(f, x, r) = f(x)$. Denote by $A = A(\overline{B})$ the set of functionals of $\mathcal{P}_E$ (defined above in Example 1) restricted to $\overline{B}$. Then A is an algebra contained in $\mathcal{C}_w(\overline{B})$, A contains the constant functionals and separates points on $\overline{B}$. Since $\overline{B}$ is weakly compact, by the Stone–Weierstrass approximation theorem, there are $\{f_n\}_{n=1}^{\infty} \subset A$ such that $\lim_{n\to\infty} f_n = f$ uniformly on $\overline{B}$. By Proposition 1,

$$\mathcal{M}^E(f,x,r) = \lim_{n\to\infty} \mathcal{M}^E(f_n,x,r) = \lim_{n\to\infty} f_n(x) = f(x).$$

$\square$

Of course, there is a functional f not continuous at any point of $\mathbf{H}$ but still having the invariant mean value property on $\mathbf{H}$. In fact, consider an unbounded linear functional $f : \mathbf{H} \to \mathbf{R}$, see Example 1 in Section 2.1.

Remark 1. Let us note that for a ball $B = B(x,r)$ in $\mathbf{H}$ the function algebra $A(\overline{B})$ defined above consists of norm uniformly continuous functionals on $\overline{B}$. Since uniform convergence preserves uniform continuity each weakly continuous functional f on an open set $G \subset \mathbf{H}$ is uniformly continuous on any closed ball in G.

Example 2. Denote by $\mathcal{R} = \mathcal{R}(\mathbf{H})$ the set of all $f : \mathbf{H} \to \mathbf{R}$ for which there are $m \in \mathbf{N}_0$, $\{a_1, ..., a_m\} \subset \mathbf{H}$ and a continuous function $F : \mathbf{R}^{m+1} \to \mathbf{R}$ such that

$$f(x) = F((a_1,x), ..., (a_m,x), \|x\|^2), \;\; x \in \mathbf{H}. \tag{6}$$

A functional $f \in \mathcal{R}$ is said to be *simple* provided, in addition, there is F independent of the last variable (i.e., $\partial_{m+1}F = 0$ on $\mathbf{R}^{m+1}$) such that (6) holds. In other words, simple functionals in $\mathbf{H}$ depend only on a finite number of coordinates in $\mathbf{H}$, see also Remark following Definition 10 in Chapter 4. Easily, simple functionals are weakly continuous and, therefore, have the invariant mean value property . Let $f \in \mathcal{R}$ be as in (6), $x \in \mathbf{H}$ and $r > 0$. Putting $\psi(y) = x + ry$, $y \in \mathbf{H}$, we get that

$$g(y) = F((a_1,x) + r(a_1,y), ..., (a_m,x) + r(a_m,y), \|x\|^2 + 2r(x,y) + r^2), \;\; y \in \mathbf{H}$$

is a simple functional equal to $f \circ \psi$ on $S(0,1)$ and, by Remark 1 in Section 2.1,

$$\mathcal{M}(f,x,r) = \mathcal{M}(g,0,1) = g(0) = F((a_1,x), ..., (a_m,x), \|x\|^2 + r^2). \tag{7}$$

Consequently, f has the invariant spherical mean on any sphere in $\mathbf{H}$ and

$$\Delta_L f(x) = 2\partial_{m+1} F((a_1,x), ..., (a_m,x), \|x\|^2) \tag{8}$$

if the right-hand side exists. It is easy to verify that

$$\mathcal{M}(\phi(h,g),x,r) = \phi(\mathcal{M}(h,x,r), \mathcal{M}(g,x,r)) \tag{9}$$

if $h, g \in \mathcal{R}$ and $\phi : \mathbf{R}^2 \to \mathbf{R}$ is continuous. Obviously,

$$\mathcal{M}(\mathcal{M}(f,\cdot,r_1),x,r_2) = \mathcal{M}(f,x,\sqrt{(r_1^2 + r_2^2)}) \tag{10}$$

whenever $r_1, r_2 > 0$.

Comments

The result from Proposition 1 is well-known, see [Fel1] and [Oba1]. In [Oba1, Lemma 3.1], the following fact is proved.

Proposition 2. *Let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of* $\mathbf{H}$, $p \in \mathbf{N}$, F_p *be a symmetric bounded p-linear functional on* $\mathbf{H}$ *and* $F(x) = F_p(x, x, ..., x)$, $x \in \mathbf{H}$.

(i) If p is odd, $\mathcal{M}^E(F, 0, r) = F(0) = 0$ for any $r \in \mathbf{R}$.

(ii) If $p = 2q$ with an integer $q \geq 1$,

$$\mathcal{M}^E(F, 0, r) = r^p (p-1)!! \lim_{n\to\infty} n^{-q} \sum_{i_1,\dots,i_q=1}^{n} F_p(e_{i_1}, e_{i_1}, e_{i_2}, e_{i_2}, ..., e_{i_q}, e_{i_q})$$

if one side exists. Here $(p-1)!! = 1 \cdot 3 \cdot 5 \cdots (p-1)$.

The proof of this proposition is a rather technically complicated. But it is easy to see that the result from Example 1 follows immediately from Proposition 2.

Theorem 1 seems to be due to G. E. Shilov. In [Shi1], G. E. Shilov noticed that the algebra $\mathcal{S}(K)$ of all simple functionals restricted to a weakly compact set $K \subset \mathbf{H}$ is contained in $\mathcal{C}_w(K)$, contains constant functionals and separates points on K and hence, by the Stone–Weierstrass theorem, for each $f \in \mathcal{C}_w(K)$, there are $f_n \in \mathcal{S}(K)$, $n \in \mathbf{N}$, such that $\lim_{n\to\infty} f_n = f$ uniformly on K. Consequently, taking Proposition 1 into account, in order to prove Theorem 1 it is sufficient to show that simple functionals have the invariant mean value property on $\mathbf{H}$, which was already known by P. Lévy [Lev1, p. 228] and follows also from a striking Lévy's theorem, see Remark 2 in Section 2.3. In proving Theorem 1, we used functionals of $\mathcal{P}_E$ instead of simple ones because we were able to establish that the former functionals have the mean value property in an elementary way.

In [Shi1], G. E. Shilov developed a theory of the Lévy laplacian on a rather narrow class of functionals defined on a special weakly compact set $K \subset \mathbf{H}$ which is the completion of the class of functionals f of type (6) having the Lévy laplacian endowed with the norm

$$|||f||| = \sup_{x\in K} |f(x)| + \sup_{x\in K} |\Delta_L f(x)|.$$

Results from Example 2 are due to G. E. Shilov, see [Shi1].

2.3 Uniformly continuous functionals

Now we shall be interested in uniformly continuous functionals.

Proposition 1. *If r_n, $n \in \mathbf{N}_0$ are strictly positive numbers with $\lim_{n\to\infty} r_n = r_0$, E is an orthonormal basis in $\mathbf{H}$ and f is a real uniformly continuous functional on a closed set $F \subset \mathbf{H}$ such that, for a point $x \in \mathbf{H}$, $\mathcal{M}^E(f,x,r_n)$ exists for each $n \in \mathbf{N}$, then $\mathcal{M}^E(f,x,r_0)$ exists and*

$$\mathcal{M}^E(f,x,r_0) = \lim_{n\to\infty} \mathcal{M}^E(f,x,r_n).$$

Proof: By Remark 1 in Section 2.1, for each $n \in \mathbf{N}_0$, putting $f_n = f \circ \psi_{x,r_n}$, we get

$$\mathcal{M}^E(f,x,r_n) = \mathcal{M}^E(f_n,0,1)$$

if one side exists. By the uniform continuity of f, $\lim_{n\to\infty} f_n = f$ uniformly on $S(0,1)$ and hence the proof is completed by Proposition 1 in Section 2.2. $\square$

Example 1. Let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathbf{H}$. By Proposition 1, if f is a real functional uniformly continuous on any ball in $\mathbf{H}$ and $x \in \mathbf{H}$, then

$$M^E(f;x) = \{r \in (0,+\infty)\ ; \mathcal{M}^E(f,x,r) \text{ exists}\}$$

is a closed set in $(0,+\infty)$. Conversely, if M is a closed set in $(0,+\infty)$, then there is a real functional f uniformly continuous on any ball in $\mathbf{H}$ such that $M^E(f;0) = M$. Indeed, there is a real continuous function F on $\mathbf{R}$ such that $M = \{r \in (0,+\infty);\ F(r) = 0\}$ and there is a bounded sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers such that $\frac{1}{n}\sum_{j=1}^{n} a_j$, $n \in \mathbf{N}$ does not converge. Put

$$f(x) = F(\|x\|)\sum_{n=1}^{\infty} a_n (x,e_n)^2,\ x \in \mathbf{H}.$$

Then it is easy to verify that f is uniformly continuous on any ball in $\mathbf{H}$ and, for each $n \in \mathbf{N}$ and $r > 0$,

$$\mathcal{M}_n(f,0,r) = F(r)\cdot r^2 \cdot \frac{1}{n}\sum_{j=1}^{n} a_j,$$

see Example 2 in Section 2.1. Therefore $\mathcal{M}^E(f,x,r)$ exists if and only if $F(r) = 0$, i.e., $r \in M$.

Before proving that volume and spherical means coincide for uniformly continuous functionals a lemma will be shown.

Lemma 1. *Let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathbf{H}$, $x \in \mathbf{H}$, $r > 0$ and f be a real bounded functional on a ball $B(x,r)$ in $\mathbf{H}$ such that $\mathcal{A}_n(f,x,r)$ exists for each $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, denote $\alpha_n = \lambda_n(B_n(0,1))$ and let A_n be a Borel subset of $B_n(0,1)$ such that*

$$\lim_{n\to\infty} \lambda_n(A_n)/\alpha_n = 1.$$

Putting

$$f_n(y) = f(x + r\sum_{j=1}^{n} y_j e_j),\ y = (y_1, ..., y_n) \in B_n(0,1)$$

for each $n \in \mathbf{N}$, it holds that

$$\lim_{n\to\infty} (\mathcal{A}_n(f,x,r) - \fint_{A_n} f_n \, d\lambda_n) = 0.$$

Proof: It follows directly from the following equality

$$\mathcal{A}_n(f,x,r) = \left(1 - \frac{\lambda_n(A_n)}{\alpha_n}\right) \fint_{B_n(0,1)\setminus A_n} f_n \, d\lambda_n + \frac{\lambda_n(A_n)}{\alpha_n} \fint_{A_n} f_n \, d\lambda_n.$$

□

Theorem 1. *If E is an orthonormal basis of $\mathbf{H}$, $x \in \mathbf{H}$, $r > 0$ and f is a uniformly continuous functional on $\overline{B(x,r)}$, then*

$$\lim_{n\to\infty} (\mathcal{M}_n(f,x,r) - \mathcal{A}_n(f,x,r)) = 0;$$

in particular, $\mathcal{M}^E(f,x,r) = \mathcal{A}^E(f,x,r)$ if one side exists.

Proof: Given $\varepsilon > 0$, there is $0 < \delta < r$ such that $|f(y) - f(z)| \le \varepsilon$ for each $y, z \in \overline{B(x,r)}$, $\|y-z\| \le \delta$. Hence $|\mathcal{M}_n(f,x,s) - \mathcal{M}_n(f,x,r)| \le \varepsilon$ for each $r-\delta \le s \le r$ and $n \in \mathbf{N}$. Putting $t = (r-\delta)/r$ and $A_n = B_n(0,1) \setminus B_n(0,t)$, $n \in \mathbf{N}$, we have

$$\lim_{n\to\infty} \lambda_n(A_n)/\alpha_n = \lim_{n\to\infty} (1 - t^n) = 1.$$

Moreover, since each uniformly continuous functional on a bounded convex set in $\mathbf{H}$ is bounded, by Lemma 1, we get

$$\lim_{n\to\infty} (\mathcal{A}_n(f,x,r) - \fint_{A_n} f_n \, d\lambda_n) = 0$$

where f_n is defined as in Lemma 1. For each $n \in \mathbf{N}$,

$$\fint_{A_n} f_n \, d\lambda_n - \mathcal{M}_n(f,x,r) = \frac{n}{1-t^n} \int_t^1 s^{n-1} (\mathcal{M}_n(f,x,rs) - \mathcal{M}_n(f,x,r)) \, ds,$$

the norm of which is not greater than ε. Thus,

$$\lim_{n\to\infty} (\mathcal{A}_n(f,x,r) - \mathcal{M}_n(f,x,r)) = 0,$$

which completes the proof. □

Now our objective is to prove a striking theorem due to P. Lévy.

Lemma 2. *Let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in* $\mathbf{H}$*,* $x \in \mathbf{H}$*,* $r > 0$ *and* f *be a real bounded functional on a sphere* $S(x,r)$ *in* $\mathbf{H}$ *such that* $\mathcal{M}_n(f,x,r)$ *exists for each* $n \in \mathbf{N}$*. For each* $n \in \mathbf{N}$*, denote* $\beta_n = \sigma_n(S_n(0,1))$ *and let* A_n *be a Borel subset of* $S_n(0,1)$ *such that*

$$\lim_{n\to\infty} \sigma_n(A_n)/\beta_n = 1. \tag{1}$$

Putting

$$f_n(y) = f(x + r\sum_{j=1}^{n} y_j e_j),\ y = (y_1, ..., y_n) \in S_n(0,1) \tag{2}$$

for each $n \in \mathbf{N}$*, it holds that*

$$\lim_{n\to\infty} (\mathcal{M}_n(f,x,r) - ⨍_{A_n} f_n \, d\sigma_n) = 0.$$

Proof: The proof is quite analogous to that of Lemma 1. □

Let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathbf{H}$, f be a uniformly continuous functional on a sphere $S(x,r)$ in $\mathbf{H}$ and f_n be defined as in (2) for each $n \in \mathbf{N}$. Given an $n \in \mathbf{N}$, there is at least one $m_n \in \mathbf{R}$ such that

$$\sigma_n(\{f_n \le m_n\})/\beta_n \ge \frac{1}{2} \text{ and } \sigma_n(\{f_n \ge m_n\})/\beta_n \ge \frac{1}{2} \tag{3}$$

where $\{f_n \le m_n\} = \{y \in S_n(0,1);\ f_n(y) \le m_n\}$ and $\{f_n \ge m_n\}$ is defined similarly. Indeed, it is sufficient to take m_n as the supremum of $m \in \mathbf{R}$ for which

$$\sigma_n(\{f_n \le m\})/\beta_n \le 1/2.$$

Such a real number m_n is called an *n-th approximative median* of f on $S(x,r)$ relative to E. If $m = \lim_{n\to\infty} m_n$, then m is called an *asymptotic median* of f on $S(x,r)$ relative to E.

Let $n \in \mathbf{N}$. For $A \subset S_n(0,1)$ and $\delta > 0$ denote by $U_\delta(A)$ the set of points of $S_n(0,1)$ whose (Euclidean) distance from A is less than δ, i.e., $U_\delta(A)$ is δ-*neighbourhood* of A on $S_n(0,1)$.

Fact 1. *If A is a Borel subset of $S_n(0,1)$ such that $\sigma_n(A)/\beta_n \ge 1/2$, then*

$$\sigma_n(U_\delta(A))/\beta_n \ge 1 - e^{-n\delta^2/2} \tag{4}$$

for each $\delta > 0$.

Proof: See [Bal, p. 41–47]. □

Remark 1. The previous statement is rather surprising if one notice that, for a given $\delta > 0$, the term on the right-hand side of (4) tends to 1 as $n \to \infty$!

Since f is uniformly continuous on $S(x,r)$, for a given $\varepsilon > 0$ there is $\delta > 0$ such that $|f(z) - f(y)| < \varepsilon$ for each $z, y \in S(x,r)$, $\|z - y\| < \delta$. Thus $U_\delta(\{f_n \leq m_n\}) \subset \{f_n < m_n + \varepsilon\}$ and, by (3) and Fact 1,

$$\lim_{n\to\infty} \sigma_n(\{f_n < m_n + \varepsilon\})/\beta_n = 1. \tag{5}$$

Similarly,

$$\lim_{n\to\infty} \sigma_n(\{f_n > m_n - \varepsilon\})/\beta_n = 1. \tag{6}$$

By (5) and (6),

$$\lim_{n\to\infty} \sigma_n(\{|f_n - m_n| \geq \varepsilon\})/\beta_n = 0,$$

which is equivalent to

$$\lim_{n\to\infty} \sigma_n(\{|f_n - m_n| < \varepsilon\})/\beta_n = 1. \tag{7}$$

For each $n \in \mathbf{N}$, putting $A_n = \{|f_n - m_n| < \varepsilon\}$,

$$\mathcal{M}_n(f,x,r) - m_n = \mathcal{M}_n(f,x,r) - ⨍_{A_n} f_n \, d\sigma_n + ⨍_{A_n} (f_n - m_n) \, d\sigma_n. \tag{8}$$

The absolute value of the last term of (8) is not greater than ε. Consequently, according to Lemma 2 and (7), $\limsup |\mathcal{M}_n(f,x,r) - m_n| \leq \varepsilon$ for each $\varepsilon > 0$, i.e.,

$$\lim_{n\to\infty} (\mathcal{M}_n(f,x,r) - m_n) = 0. \tag{9}$$

Thus $\mathcal{M}^E(f,x,r)$ exists if and only if f has an asymptotic median on $S(x,r)$ relative to E. Futhermore, if $m = \lim_{n\to\infty} m_n$, then, by (7),

$$\lim_{n\to\infty} \sigma_n(\{|f_n - m| < \varepsilon\})/\beta_n = 1 \tag{10}$$

for each $\varepsilon > 0$. On the other hand, one can prove that (10) implies

$$\lim_{n\to\infty} (\mathcal{M}_n(f,x,r) - m) = 0,$$

i.e., $\mathcal{M}^E(f,x,r) = m$ in the same way as we have shown that (9) follows from (7). If (10) is satisfied, then we shall say that $f = m$ *almost everywhere* on $S(x,r)$ *in the sense of P. Lévy* relative to E and we shall write $f = m$ L-a.e. on $S(x,r)$ relative to E. Thus we have proved the following theorem.

Theorem 2. *(P. Lévy) Let E be an orthonormal basis in* $\mathbf{H}$, $x \in \mathbf{H}$, $r > 0$ *and f be a uniformly continuous functional on $S(x,r)$. Then the following statements are equivalent to each other:*

(i) $\mathcal{M}^E(f,x,r)$ *exists;*

(ii) f *has an asymptotic median on* $S(x,r)$ *relative to* E*;*

(iii) *there is* $m \in \mathbf{R}$ *such that* $f = m$ L*-a.e. on* $S(x,r)$ *relative to* E*.*

In addition, if (iii) holds, then $\mathcal{M}^E(f,x,r) = m$ *and* m *is an asymptotic median of* f *on* $S(x,r)$ *relative to* E*. If this is not the case and* m_n *is* n*-th approximative median of* f *on* $S(x,r)$ *relative to* E *for each* $n \in \mathbf{N}$*, then it still holds that* $\lim_{n\to\infty}(\mathcal{M}_n(f,x,r) - m_n) = 0$.

Corollary 1. *Let* E *be an orthonormal basis in* $\mathbf{H}$, $x \in \mathbf{H}$, $r > 0$, $f : S(x,r) \to \mathbf{R}$ *be uniformly continuous and* $I = \overline{f(S(x,r))}$*. Let* $F : S(x,r) \times I \to \mathbf{R}$ *be uniformly continuous and* $g(y) = F(y, f(y))$, $y \in S(x,r)$*. If* $\mathcal{M}^E(f,x,r)$ *exists and* $m = \mathcal{M}^E(f,x,r)$*, then*

$$\mathcal{M}^E(g,x,r) = \mathcal{M}^E(F(\cdot,m),x,r)$$

if the right-hand side exists.

Proof: If $m = \mathcal{M}^E(f,x,r)$ and $M = \mathcal{M}^E(F(\cdot,m),x,r)$, then, by Theorem 2, $f = m$ and $F(\cdot,m) = M$ L-a.e. on $S(x,r)$ relative to E. Owing to the uniform continuity of F, for a given $\varepsilon > 0$ there is $\delta > 0$ such that $|F(y,t) - F(y,s)| < \varepsilon/2$ whenever $y \in S(x,r)$ and $|t - s| < \delta$; $t, s \in I$. Of course, I is a closed bounded interval in $\mathbf{R}$ and $m \in I$. For each $y \in S(x,r)$,

$$|F(y,f(y)) - M| \le |F(y,f(y)) - F(y,m)| + |F(y,m) - M|.$$

Consequently,

$$\{|g - M| \ge \varepsilon\} \subset \{|F(\cdot,f(\cdot)) - F(\cdot,m)| \ge \varepsilon/2\} \cup \{|F(\cdot,m) - M| \ge \varepsilon/2\}$$

where we have written $\{|g - M| \ge \varepsilon\}$ instead of $\{y \in S(x,r);\ |g(y) - M| \ge \varepsilon\}$ and so on. Finally, since $\{|F(\cdot,f(\cdot)) - F(\cdot,m)| \ge \varepsilon/2\} \subset \{|f - m| \ge \delta\}$ we have

$$\{|g - M| \ge \varepsilon\} \subset \{|f - m| \ge \delta\} \cup \{|F(\cdot,m) - M| \ge \varepsilon/2\},$$

which implies $g = M$ L-a.e. on $S(x,r)$ relative to E and the proof is completed by Theorem 2. □

Corollary 2. *Let* $m \in \mathbf{N}$, $F : \mathbf{R}^m \to \mathbf{R}$ *be continuous and* f_j *be a uniformly continuous functional on a sphere* $S(x,r)$ *in* $\mathbf{H}$ *for* $j = 1,\dots,m$*. If* E *is an orthonormal basis in* $\mathbf{H}$, $\mathcal{M}^E(f_j,x,r)$ *exists and* $M_j = \mathcal{M}^E(f_j,x,r)$ *for each* $j = 1,\dots,m$*, then, putting* $f(y) = F(f_1(y),\dots,f_m(y))$, $y \in S(x,r)$*, the mean* $\mathcal{M}^E(f,x,r)$ *exists and*

$$\mathcal{M}^E(f,x,r) = F(M_1,\dots,M_m).$$

PROOF: It follows directly from Corollary 1. Indeed, putting

$$\tilde{F}(y,t) = F(f_1(y), ..., f_{m-1}(y), t),\ y \in S(x,r) \text{ and } t \in \mathbf{R},$$

according to Corollary 1,

$$\mathcal{M}^E(f,x,r) = \mathcal{M}^E(\tilde{F}(\cdot, M_m), x, r)$$

if the right-hand side exists. In order to conclude the proof it is sufficient to repeat this step $(m-1)$ times. □

Remark 2. By Corollary 2, if $f_1, ..., f_m$ are uniformly continuous functionals having the mean value property on an open set $G \subset \mathbf{H}$ with respect to an orthonormal basis E and $F : \mathbf{R}^m \to \mathbf{R}$ is continuous, then $f = F(f_1, ..., f_m)$ has the mean value property on G with respect to E, too. Furthermore, by Corollary 2, it is easy to calculate spherical means of functionals of $\mathcal{R}(\mathbf{H})$ introduced in Example 2 in Section 2.2.

Now we shall discuss the Dirichlet and the Poisson problem in the class of uniformly continuous functionals. In what follows $E = \{e_n\}_{n=1}^{\infty}$ will be an orthonormal basis in $\mathbf{H}$. The Dirichlet and the Poisson problem will be solved on a bounded open convex set $G \subset \mathbf{H}$ satisfying the following conditions:

(A) for each $x \in G$, there is a uniformly continuous functional ρ_x on $S(0,1)$ such that, putting

$$\varphi_x(y) = x + \rho_x(y)y,\ y \in S(0,1),$$

$\partial G = \varphi_x(S(0,1))$;

(B) for each $x \in G$, $r(x) = \mathcal{M}^E(\rho_x, 0, 1)$ exists;

(C) there is a bounded uniformly continuous functional $s : \overline{G} \to \mathbf{R}$ having the mean value property on G with respect to E such that $r(x) = \sqrt{s(x) - \|x\|^2}$, $x \in G$ and $s(x) = \|x\|^2$, $x \in \partial G$. In particular, given a ball $B(x,R)$ contained with its closure in G, it is easily seen that, by Corollary 2,

$$\mathcal{M}^E(r,x,R) = \sqrt{s(x) - \|x\|^2 - R^2} = \sqrt{r(x)^2 - R^2}. \tag{11}$$

Moreover, putting $r = 0$ on ∂G, r is a bounded uniformly continuous functional on $\overline{G}$.

Example 2. It is easily seen that each ball satisfies (A), (B) and (C). Indeed, if $G = B(x_0, R)$ with $x_0 \in \mathbf{H}$ and $R > 0$, then, for each $x \in G$,

$$\rho_x(y) = \sqrt{R^2 - \|x - x_0\|^2 + (x - x_0, y)^2} - (x - x_0, y),\ y \in S(0,1)$$

and $r(x) = \sqrt{R^2 - \|x - x_0\|^2}$.

The *Dirichlet problem* means here: Given a bounded open set G in $\mathbf{H}$ and a real uniformly continuous functional f defined on ∂G, find a real uniformly continuous functional H on $\overline{G}$ such that

(D1) H has the mean value property on G with respect to E (and thus $\Delta_L^E H = 0$ on G) and

(D2) $H = f$ on ∂G.

Theorem 3. *(The Dirichlet problem) Let G be a bounded open convex set in $\mathbf{H}$ satisfying (A), (B) and (C) and f be a real functional uniformly continuous on any ball in $\mathbf{H}$ such that f has the spherical mean with respect to E on any sphere and*

$$\mathcal{M}^E(\mathcal{M}^E(f,\cdot,r_1),x,r_2) = \mathcal{M}^E(f,x,\sqrt{(r_1^2+r_2^2)}) \tag{12}$$

for each $x \in \mathbf{H}$ and $r_1, r_2 > 0$. Then the functional

$$H_f(x) = \mathcal{M}^E(f \circ \varphi_x, 0, 1),\ x \in G \ \text{ and } \ H_f = f \text{ on } \partial G \tag{13}$$

is defined correctly and satisfies (D1) and (D2). In addition,

$$H_f(x) = \mathcal{M}^E(f,x,r(x)),\ x \in \overline{G}$$

(where $\mathcal{M}^E(f,x,0) = f(x)$) and

$$\sup_G |H_f| \le \sup_{\partial G} |f|. \tag{14}$$

Proof: Obviously, we can assume that f is a bounded uniformly continuous functional on $\mathbf{H}$ because relevant values of f are ones f attains within a ball in $\mathbf{H}$ being big enough. By Corollary 1, it is easy to verify that

$$H_f(x) = \mathcal{M}^E(f(x + r(x)\cdot,0,1) = \mathcal{M}^E(f,x,r(x)),\ x \in \overline{G}. \tag{15}$$

Futhermore, put $F(x,t) = \mathcal{M}^E(f,x,t)$, $x \in \mathbf{H}$ and $t \ge 0$. Then F is uniformly continuous and so is $H_f(x) = F(x,r(x))$, $x \in \overline{G}$. Moreover, given a ball $B(z,R)$ contained with its closure in G, by Corollary 1 and (11), we have

$$\mathcal{M}^E(H_f,z,R) = \mathcal{M}^E(F(\cdot,r(\cdot)),z,R) = \mathcal{M}^E(\mathcal{M}^E(f,\cdot,\sqrt{(r(z)^2-R^2)}),z,R).$$

By (12), the last term above is equal to $H_f(z)$, which concludes the proof that H_f has the mean value property on G. □

The *Poisson problem* means here: Given a bounded open set G in $\mathbf{H}$ and a real uniformly continuous functional f defined on G, find a real uniformly continuous functional P on $\overline{G}$ such that

(P1) $-\Delta_L^E P = f$ on G and

(P2) $P = 0$ on ∂G.

Theorem 4. *(The Poisson problem) Let G and f be as in Theorem 3 and, for each $x \in \mathbf{H}$, put*

$$F_x(y,t) = \int_0^t v f(x+vy)\,dv,\ y \in \mathbf{H} \text{ and } t \in \mathbf{R}.$$

Then the functional

$$P_f(x) = \mathcal{M}^E(F_x(\cdot,\rho_x(\cdot)),0,1),\ x \in G \ \text{ and } \ P_f = 0 \text{ on } \partial G \tag{16}$$

is defined correctly and satisfies (P1) and (P2). In addition,

$$P_f(x) = \int_0^{r(x)} v\mathcal{M}^E(f,x,v)\,dv,\ x \in \overline{G}. \tag{17}$$

Futhermore, putting $C = \sup\{|\rho_x(y)|^2/2;\ x \in G, y \in S(0,1)\}(< +\infty)$,

$$\sup_G |P_f| \le C \sup_G |f|. \tag{18}$$

PROOF: We can assume again that f is a bounded uniformly continuous functional on $\mathbf{H}$. First we shall show that, given a point $x \in G$,

$$\mathcal{M}^E(F_x(\cdot, r(x)), 0, 1) = \int_0^{r(x)} v\mathcal{M}^E(f,x,v)\,dv. \tag{19}$$

Indeed, applying the Fubini theorem, we have

$$\mathcal{M}_n(F_x(\cdot, r(x)), 0, 1) = \int_0^{r(x)} v\mathcal{M}_n(f,x,v)\,dv$$

for each $n \in \mathbf{N}$. Letting $n \to \infty$, by the Lebesgue dominated convergence theorem, we get (19). Futhermore, letting $K = \sup\{|\rho_x(y)|;\ y \in S(0,1)\}(< +\infty)$, F_x is uniformly continuous on $\mathbf{H} \times [-K, K]$ and hence, by Corollary 1, $P_f(x) = \mathcal{M}^E(F_x(\cdot, r(x)), 0, 1)$ and, finally, by (19), we get (17).

Second we shall show that, given a ball $B(x,R)$ contained with its closure in G,

$$\mathcal{M}^E(P_f, x, R) = \int_R^{r(x)} v\mathcal{M}^E(f,x,v)\,dv. \tag{20}$$

Indeed, let $L > 0$ be such that $0 \le r \le L$ on $\overline{G}$. Putting

$$F(y,t) = \int_0^t v\mathcal{M}^E(f,y,v)\,dv,\ y \in \mathbf{H} \text{ and } t \in [0,L],$$

F is uniformly continuous and so is $P_f(y) = F(y, r(y))$, $y \in \overline{G}$. Futhermore, by Corollary 1, $\mathcal{M}^E(P_f, x, R) = \mathcal{M}^E(F(\cdot, r(\cdot)), x, R)$ exists and is equal to $\mathcal{M}^E(F(\cdot, \surd(r(x)^2 - R^2)), x, R)$ provided that the latter mean exists. Now we are going to show that

$$\mathcal{M}^E(F(\cdot, \surd(r(x)^2 - R^2), x, R) = \int_0^{\sqrt{r(x)^2 - R^2}} v\mathcal{M}^E(\mathcal{M}^E(f, \cdot, v), x, R)\,dv, \tag{21}$$

which, by (12), is the same as

$$\int_0^{\sqrt{r(x)^2 - R^2}} v\mathcal{M}^E(f, x, \surd(v^2 + R^2))\,dv = \int_R^{r(x)} v\mathcal{M}^E(f,x,v)\,dv.$$

Thus, in order to prove (20) it is sufficient to show (21). Applying the Fubini theorem, we have

$$\mathcal{M}_n(F(\cdot,\sqrt{r(x)^2-R^2}),x,R)=\int_0^{\sqrt{r(x)^2-R^2}} v\mathcal{M}_n(\mathcal{M}^E(f,\cdot,v),x,R)\,dv$$

for each $n\in\mathbf{N}$. Letting $n\to\infty$, by the Lebesgue dominated convergence theorem, we get (21).

Finally, by (20),

$$-\mathcal{M}^E(P_f,x,R)+P_f(x)-\frac{R^2}{2}f(x)=\int_0^R v(\mathcal{M}^E(f,x,v)-f(x))\,dv.$$

Since $\lim_{v\to 0+}\mathcal{M}^E(f,x,v)=f(x)$ the term on the right-hand side divided by $R^2/2$ approches 0 as $R\to 0+$, i.e., $-\Delta_L^E P_f(x)=f(x)$. □

Remark 3. Let $f\in\mathcal{R}$, i.e., $f:\mathbf{H}\to\mathbf{R}$ is a functional for which there are $m\in\mathbf{N}_0$, $\{a_1,...,a_m\}\subset\mathbf{H}$ and a continuous function $F:\mathbf{R}^{m+1}\to\mathbf{R}$ such that

$$f(x)=F((a_1,x),...,(a_m,x),\|x\|^2),\ x\in\mathbf{H},$$

see Example 2 in Section 2.2. Obviously, f satisfies the hypotheses of Theorems 3 and 4 and

$$H_f(x)=F((a_1,x),...,(a_m,x),s(x)),\ x\in\overline{G}$$

and

$$P_f(x)=\frac{1}{2}\int_{\|x\|^2}^{s(x)} F((a_1,x),...,(a_m,x),t)\,dt,\ x\in\overline{G}.$$

Futhermore, let $s_1,...,s_m$ be uniformly continuous functionals on $\overline{G}$ having the mean value property on G with respect to E and put now

$$f(x)=F(s_1(x),...,s_m(x),\|x\|^2),\ x\in\overline{G}.$$

By Corollary 2, it is easily seen that the functional

$$H(x)=F(s_1(x),...,s_m(x),s(x)),\ x\in\overline{G}$$

has the mean value property on G with respect to E and satisfies (D2). Now we are going to show that the functional

$$P(x)=\frac{1}{2}\int_{\|x\|^2}^{s(x)} F(s_1(x),...,s_m(x),t)\,dt,\ x\in\overline{G}$$

satisfies (P1) and (P2). Indeed, defining $K:\mathbf{R}^{m+2}\to\mathbf{R}$ by

$$K(t_1,...,t_{m+2})=\frac{1}{2}\int_{t_{m+2}}^{t_{m+1}} F(t_1,...,t_m,t)\,dt,$$

we have

$$P(x) = K(s_1(x), ..., s_m(x), s(x), \|x\|^2), \ x \in G.$$

Given a ball $B(x, R)$ contained with its closure in G, by Corollary 2,

$$\mathcal{M}^E(P, x, R) = K(s_1(x), ..., s_m(x), s(x), \|x\|^2 + R^2).$$

Hence $\Delta_L^E P(x) = 2\partial_{m+2} K(s_1(x), ..., s(x), \|x\|^2) = -f(x)$, which conclude the proof.

Recall that linear operators $T_t, \ t \geq 0$ on a Banach space $\mathcal{T}$ form a strongly continuous contraction semigroup on $\mathcal{T}$ provided that

(i) T_0 is the identity operator on $\mathcal{T}$ and, for each $t > 0$, $T_t : \mathcal{T} \to \mathcal{T}$ is a contraction, i.e., T_t is a linear operator with the operator norm $\|T_t\| \leq 1$;

(ii) for each $t_1, t_2 \geq 0$, $T_{t_1} \circ T_{t_2} = T_{t_1+t_2}$;

(iii) for each $f \in \mathcal{T}$, $\lim_{t\to 0+} T_t f = f$ in the sense of the norm topology.

Futhermore, the infinitesimal generator A of the semigroup $T_t, \ t \geq 0$ is defined by

$$Af = \lim_{t\to 0+} \frac{T_t f - f}{t}$$

for each $f \in \mathcal{T}$ such that the limit exists.

For a real bounded functional f on $\mathbf{H}$ define

$$\|f\|_0 = \sup\{|f(x)|; \ x \in \mathbf{H}\}.$$

Denote by $\mathcal{T} = \mathcal{T}_E(\mathbf{H})$ the set of bounded uniformly continuous functionals f on $\mathbf{H}$ satisfying the condition (12). Obviously, $(\mathcal{T}, \|\cdot\|_0)$ is a Banach space.

Proposition 2. *For $f \in \mathcal{T}$ and $t > 0$ put*

$$T_t f(x) = \mathcal{M}^E(f, x, \sqrt{t}), \ x \in \mathbf{H} \tag{22}$$

and $T_0 f = f$. Then the operators $T_t, \ t \geq 0$ form a strongly continuous contraction semigroup on the Banach space $\mathcal{T}$ with the infinitesimal generator $\Delta_L^E/2$. Moreover, for each $f \in \mathcal{T}$ and $x \in \mathbf{H}$,

$$\Delta_L^E f(x) = 2 \lim_{t\to 0+} \frac{T_t f(x) - f(x)}{t} \tag{23}$$

whenever the limit exists.

Proof: It is easily seen that (12) implies (ii). By Definition 1 in Section 2.1, (23) is true. Obviously, (i) and (iii) hold. □

Comments

Theorem 1, which tells us that volume and spherical means coincide for uniformly continuous functionals, is due to P. Lévy, see [Lev1, p. 226]. The proof of this theorem is based on the fact that, for any $r \in (0,1)$,

$$\lim_{n\to\infty} \lambda_n(B_n(0,1) \setminus B_n(0,r))/\lambda_n(B_n(0,1)) = 1,$$

which can be expressed informally as that 'the volume of a ball in $\mathbf{R}^n$ concentrates outside of any ball with the same centre but less radius as $n \to \infty$'. Consequently, by Lemma 1, the volume mean of a bounded functional on a ball in $\mathbf{H}$ does not depend on its values on any ball with the same centre but less radius. The most striking example of this phenomenon called in geometry the *concentration of measure* is given in Fact 1, namely, that 'the mass of the unit sphere in $\mathbf{R}^n$ concentrates very close to any its subset of measure $(1/2)\sigma_n(S_n(0,1))$ as $n \to \infty$', see [Bal, p. 41–47] for details. It is just Fact 1 that the proof of Theorem 2 is based on. Theorem 2 is due to P. Lévy but his proof is not too clear (and the notion of spherical mean considered there is different from ours), see [Lev1, p. 226]. In [Bal, p. 43], Theorem 2 is shown for Lipschitz functionals. Corollaries 1 and 2 to Theorem 2, which are stated in [Pol2, p. 20 and 21], play a very important role in solving the Dirichlet and the Poisson problem as we have done in Theorems 3 and 4. In [Pol2, Theorem 5.1.1, p. 43] and [Pol2, Theorem 5.1.2, p. 47], the functionals H_f and P_f defined as in (13) and (17) are shown to be solutions of the Dirichlet and the Poisson problem on any bounded open convex set $G \subset \mathbf{H}$ satisfying (A), (B) and (C) for f being a Gâteaux functional, respectively. See also [Lev1, p. 320] and [Gat]. Results obtained in Remark 3 can be shown in a rather general context, see Theorem 7 and its proof in Chapter 4. The semigroup of type (22) on the class of Gâteaux functionals was investigated for the first time by E. M. Polishchuk, see [Pol2, p. 70].

The notion of Gâteaux functional goes back to [Gat], [Lev0] and [Lev1]. An up-to-date exposition of the theory of such functionals is presented in E. M. Polishchuk's book [Pol2]. Now a few results concerning Gâteaux functionals will be mentioned. In what follows let $m \in \mathbf{N}$ and a function $g : \mathbf{R}^m \times (0,1)^m \to \mathbf{R}$ satisfy the so-called *Carathéodory conditions*, namely, for almost all $t \in (0,1)^m$, the function $g(\cdot,t)$ is continuous on $\mathbf{R}^m$ and, for all $x \in \mathbf{R}^m$, the function $g(x,\cdot)$ is measurable on $(0,1)^m$. Moreover, assume that there are a function $h \in L^1((0,1)^m)$ and a $C > 0$ such that

$$|g(x,t)| \le h(t) + Cx_1^2 \cdots x_m^2, \ x \in \mathbf{R}^m \text{ and } t \in (0,1)^m. \tag{24}$$

For any $x \in L^2(0,1)$, put

$$F(x) = \int_{(0,1)^m} g(x(t_1),...,x(t_m),t_1,...,t_m)\,dt. \tag{25}$$

Such a real functional F on $L^2(0,1)$ is called a *Gâteaux functional*. Let us note that (24) is a necessary and sufficient condition for the continuity of F on $L^2(0,1)$. By the following assertion, the spherical mean of a Gâteaux functional can be expressed in an explicit form. Let E be a *uniformly dense* orthonormal basis in $L^2(0,1)$, see [Lev1, p. 249]. Then, for each $x \in L^2(0,1)$ and $r > 0$, the spherical mean $\mathcal{M}^E(F,x,r)$ exists and the so–called *Gâteaux formula* holds, i.e., $(2\pi)^{\frac{m}{2}}\mathcal{M}^E(F,x,r) =$

$$= \int_{(0,1)^m} \int_{\mathbf{R}^m} g(x(t_1)+ry_1,...,x(t_m)+ry_m,t_1,...,t_m) \exp(-\frac{y_1^2+\cdots+y_m^2}{2})\, dy\, dt,$$

see [Gat], [Lev1, p. 230] and [Pol2, p. 21]. Cf. also [KOS]. Moreover, F satisfies the condition (12), i.e.,

$$\mathcal{M}^E(\mathcal{M}^E(F,\cdot,r_1),x,r_2) = \mathcal{M}^E(F,x,\sqrt{(r_1^2+r_2^2)})$$

whenever $x \in L^2(0,1)$ and $r_1, r_2 > 0$, see [Pol2, p. 45 and p. 70].

Questions:

1. Does any uniformly continuous functional f having the spherical mean on any sphere in $\mathbf{H}$ (with respect to a given orthonormal basis E) satisfy the condition (12)? (This question seems to be a bit strange but it is not so difficult to prove that, for such a functional f, $x \in \mathbf{H}$ and $r_1, r_2 > 0$, the value of

$$\mathcal{M}^E(\mathcal{M}^E(f,\cdot,r_1),x,r_2) \tag{24}$$

depends only on the values of f on $S(x,\sqrt{(r_1^2+r_2^2)})$ if (24) makes sense.)

2. Theorems 3 and 4 give solutions of the Dirichlet and the Poisson problem. The uniqueness of these solutions should be cleared up. Cf. also Theorem 7 in Chapter 4.

2.4 The mean value property

Example 1. Let $E = \{e_n\}_{n=1}^\infty$ be an orthonormal basis of $\mathbf{H}$ and $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of real numbers. Then there is a functional $F \in \mathcal{C}^\infty(\mathbf{H})$ such that

(i) for each $n \in \mathbf{N}$, $\mathcal{A}_n(F,0,1) = a_n$ and $\mathcal{M}_n(F,0,1) = b_n$;

(ii) for each $x \in \mathbf{H}$ there is $r > 0$ such that F is equal to a simple functional on $B(x,r)$, see Example 2 in Section 2.2.

Now we are going to construct such a functional F. Let $n \in \mathbf{N}$. Put

$$\|x\|_n = \sqrt{\sum_{i=1}^n (x,e_i)^2},\ x \in \mathbf{H} \text{ and } T^n(x,r) = \{y \in \mathbf{H};\ \|y-x\|_n < r\}$$

for each $x \in \mathbf{H}$ and $r > 0$. Choose $h \in \mathcal{C}^\infty(\mathbf{R})$ such that $h > 0$ on $[0, 1/25)$, $h = 0$ on $[1/25, +\infty)$ and $0 \le h \le 1$. Put

$$\eta_n(x) = h(\|x - e_n\|_n^2),\ x \in \mathbf{H} \text{ and } \psi_n(x) = h(\|x - e_n/2\|_n^2),\ x \in \mathbf{H}. \tag{1}$$

Obviously, $\eta_n, \psi_n \in \mathcal{C}^\infty(\mathbf{H})$ are simple functionals, $\eta_n = 0$ on $\mathbf{H} \setminus T^n(e_n, 1/5)$ and $\psi_n = 0$ on $\mathbf{H} \setminus T^n(e_n/2, 1/5)$. By induction, we construct sequences $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ of real numbers and sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ of functionals on $\mathbf{H}$. First of all, there is $c_1 \in \mathbf{R}$ such that $b_1 = c_1 \mathcal{M}_1(\eta_1, 0, 1)$. Putting $g_1 = c_1 \eta_1$, there is $d_1 \in \mathbf{R}$ such that

$$a_1 = \mathcal{A}_1(g_1, 0, 1) + d_1 \mathcal{A}_1(\psi_1, 0, 1).$$

Denote $f_1 = d_1 \psi_1$. Now let $n \in \mathbf{N}$. Assume that we have defined $\{c_i\}_{i=1}^n$, $\{d_i\}_{i=1}^n$, $\{f_i\}_{i=1}^n$ and $\{g_i\}_{i=1}^n$. Then there is $c_{n+1} \in \mathbf{R}$ such that

$$b_{n+1} = \mathcal{M}_{n+1}(\sum_{i=1}^n (f_i + g_i), 0, 1) + c_{n+1} \mathcal{M}_{n+1}(\eta_{n+1}, 0, 1). \tag{2}$$

Putting $g_{n+1} = c_{n+1} \eta_{n+1}$, there is $d_{n+1} \in \mathbf{R}$ such that

$$a_{n+1} = \mathcal{A}_{n+1}(\sum_{i=1}^n (f_i + g_i) + g_{n+1}, 0, 1) + d_{n+1} \mathcal{A}_{n+1}(\psi_{n+1}, 0, 1). \tag{3}$$

Denote $f_{n+1} = d_{n+1} \psi_{n+1}$.

By means of sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ of functionals on $\mathbf{H}$ inductively constructed above define

$$F = \sum_{n=1}^\infty (f_n + g_n) \text{ on } \mathbf{H}. \tag{4}$$

The definition of F is correct, $F \in \mathcal{C}^\infty(\mathbf{H})$ and F satisfies (ii) because for each $x \in \mathbf{H}$ there are $n \in \mathbf{N}$ and $r > 0$ such that either $F = f_n$ or $F = g_n$ on $B(x, r)$, which follows easily from the obvious fact that sets of

$$\{T^n(e_n, 1/4)\}_{n \in \mathbf{N}} \cup \{T^n(e_n/2, 1/4)\}_{n \in \mathbf{N}}$$

are pairwise disjoint. Obviously, for each $n \in \mathbf{N}$,

$$F = \sum_{i=1}^n (f_i + g_i) \text{ on } P_n = \{\sum_{i=1}^n y_i e_i;\ (y_1, ..., y_n) \in \mathbf{R}^n\} \tag{5}$$

and

$$F = \sum_{i=1}^{n-1} (f_i + g_i) + g_n \text{ on } S(0, 1) \cap P_n. \tag{6}$$

By (2) and (3), F satisfies (i).

Of course, the functional $F \in \mathcal{C}^\infty(\mathbf{H})$ constructed above in Example 1 has the *local invariant mean value property* on $\mathbf{H}$, i.e., for each $x \in \mathbf{H}$ there is $r > 0$ such that $\mathcal{M}(f,x,s) = f(x)$ for each $0 < s < r$, and, as a result, $\Delta_L f = 0$ on $\mathbf{H}$. For such a functional F its volume and spherical means on the unit ball need not be equal nor be equal to its value at the origin, in general.

Let us state a well–known result from the classical potential theory. Let $n \in \mathbf{N}, n \geq 2$. Put $p_2(r) = -\log(r),\ r > 0$ and $p_n(r) = r^{2-n}/(n-2),\ r > 0$ if $n > 2$. The Laplace operator in $\mathbf{R}^n$ is defined by

$$\Delta_n = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Then the following fact can be derived from the theorem on three potentials, see [Hel, p. 9, Theorem 1.4].

Fact 1. *If $G \subset \mathbf{R}^n$ is open, $f \in \mathcal{C}^2(G)$ and $\overline{B_n(x,r)} \subset G$, then*

$$\fint_{S_n(x,r)} f(y)\,\sigma_n(dy) - f(x) = \frac{1}{\beta_n} \int_{B_n(x,r)} (p_n(|y-x|_n) - p_n(r))\Delta_n f(y)\,\lambda_n(dy)$$

where $\beta_n = \sigma_n(S_n(0,1))$ and $|y|_n = \sqrt{\sum_{i=1}^n y_i^2}$, $y \in \mathbf{R}^n$; in particular,

$$\frac{1}{\beta_n} \int_{B_n(x,r)} (p_n(|y-x|_n) - p_n(r))\,\lambda_n(dy) = \frac{r^2}{2n}. \tag{7}$$

By Fact 1, if f is a harmonic function on an open set $G \subset \mathbf{R}^n$, i.e., $f \in \mathcal{C}^2(G)$ and $\Delta_n f = 0$ on G, then f has the spherical mean value property on G. For functionals on $\mathbf{H}$ we prove the following theorem.

Theorem 1. *Let $E = \{e_n\}_{n=1}^\infty$ be an orthonormal basis of $\mathbf{H}$, G be an open set in $\mathbf{H}$ and $f \in \mathcal{C}^2(G)$. If $B(x,\rho)$ is a ball contained with its closure in G and*

$$(\boldsymbol{L}^E f =) \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} D_{jj} f = 0$$

uniformly on $B(x,\rho)$, then $\mathcal{M}^E(f,x,\rho) = f(x) = \mathcal{A}^E(f,x,\rho)$.

Proof: Given $\varepsilon > 0$, take $m \in \mathbf{N}, m \geq 2$ such that, for each $n \geq m$,

$$\left| \frac{1}{n} \sum_{j=1}^{n} D_{jj} f \right| \leq \frac{2\varepsilon}{\rho^2} \text{ on } B(x,\rho). \tag{8}$$

By Fact 1, for each $n \geq m$ and $0 < r \leq \rho$, $\mathcal{M}_n(f,x,r) - f(x) =$

$$= \frac{1}{\beta_n} \int_{B_n(0,r)} (p_n(|y|_n) - p_n(r)) \sum_{j=1}^{n} D_{jj} f(x + \sum_{i=1}^{n} y_i e_i)\,\lambda_n(dy),$$

and hence, by (7) and (8),

$$(9)\qquad |\mathcal{M}_n(f,x,r)-f(x)| \le \frac{2n\varepsilon}{\rho^2\beta_n}\int_{B_n(0,r)} (p_n(|y|_n)-p_n(r))\,\lambda_n(dy) \le \varepsilon.$$

Thus $\mathcal{M}^E(f,x,\rho)=f(x)$. Since

$$\mathcal{A}_n(f,x,\rho)-f(x)=\frac{n}{\rho^n}\int_0^\rho r^{n-1}(\mathcal{M}_n(f,x,r)-f(x))\,dr,$$

by (9), we get

$$|\mathcal{A}_n(f,x,\rho)-f(x)|\le\varepsilon$$

for each $n\ge m$. Consequently, $\mathcal{A}^E(f,x,\rho)=f(x)$. □

Comments

In Example 1, we have constructed a functional $F\in\mathcal{C}^\infty(\mathbf{H})$ with $\Delta_L F=0$ on $\mathbf{H}$ which has not the mean value property on $\mathbf{H}$ but has still the local invariant mean value property on $\mathbf{H}$. It is quite natural to put the following questions.

Questions:

1. Is there any $F\in\mathcal{C}^\infty(\mathbf{H})$ with $\Delta_L F=0$ on $\mathbf{H}$ not having the local (invariant) mean value property on $\mathbf{H}$?
2. Under which conditions has a real functional F satisfying, for a given orthonormal basis E of $\mathbf{H}$, the equation $\Delta_L^E F=0$ on an open set $G\subset\mathbf{H}$ the mean value property on G?

Theorem 1, due to P. Lévy [Lev1, p. 316], does not provide too satisfactory answer to the second question. Cf. also [Lev1, footnote on p. 317]. In [Fel1], M. N. Feller referring to [Pol1] claims that if F is a uniformly continuous functional satisfying the equation $\Delta_L^E F=0$ on an open set $G\subset\mathbf{H}$ for a given orthonormal basis E and $B(x,r)$ is a ball contained with its closure in G, then $\mathcal{M}^E(F,x,r)=F(x)$ whenever $\mathcal{M}^E(F,x,r)$ exists. (I do not know whether the assertion is true or not. Besides, according to [Pol1], if it is assumed e.g. that F belongs to $\mathcal{C}^2(G)$ instead of being uniform continuous, then the corresponding assertion holds, which contrast with Example 1.)

2.5 Irregular differential operators

In this section, we show that the Lévy laplacian can be thought of as an example of irregular operators introduced in the article presented in Chapter 4 as follows. Recall that $\mathcal{N}_s^2=\mathcal{N}_s^2(\mathbf{H},\mathbf{R})$ is the Banach space of all symmetric bounded bilinear functionals on $\mathbf{H}$, $\mathcal{L}_s$ is the Banach space of all self-adjoint operators $S:\mathbf{H}\to\mathbf{H}$ and $\mathcal{L}_\infty$ is the Banach space of all compact linear operators on $\mathbf{H}$.

Since the mapping ϕ, defined for each $S \in \mathcal{L}_s$ by $\phi(S)(x,y) = (Sx, y),\ x, y \in \mathbf{H}$, is an isometry of $\mathcal{L}_s$ onto $\mathcal{N}_s^2$ we may identify elements of these Banach spaces.

A real bounded linear functional ψ on $\mathcal{L}_s$, i.e., $\psi \in \mathcal{L}_s^*$, is said to be *irregular* provided that $\psi = 0$ on $\mathcal{L}_\infty \cap \mathcal{L}_s$. Given $\psi \in \mathcal{L}_s^*$, the differential operator $\boldsymbol{D}^\psi$ (of 2-nd order corresponding to ψ) is defined by

$$\boldsymbol{D}^\psi f(x) = \psi(f''(x)) \tag{1}$$

if $x \in \mathbf{H}$ and f is a real functional having the second Fréchet derivative $f''(x)$ at x. Such a differential operator $\boldsymbol{D}^\psi$ is said to be *irregular* provided that ψ is irregular.

Let us recall the well-known Silverman–Toeplitz theorem.

Theorem 1. *(Silverman-Toeplitz) Given an infinite matrix $\{a_{n,k}\}_{n,k=1}^\infty$ with real entries, it holds that*

$$\lim_{n\to\infty} \sum_{k=1}^\infty a_{n,k} b_k = 0 \tag{2}$$

whenever $\{b_k\}_{k=1}^\infty \subset \mathbf{R}$ and $\lim_{k\to\infty} b_k = 0$, if and only if the following conditions are satisfied:

(a) there is $n_0 \in \mathbf{N}$ such that $\sup_{n\ge n_0} \sum_{k=1}^\infty |a_{n,k}| < +\infty$,

(b) for each $k \in \mathbf{N}$, $\lim_{n\to\infty} a_{n,k} = 0$.

A necessary and sufficient condition that

$$\lim_{n\to\infty} \sum_{k=1}^\infty a_{n,k} b_k = b \tag{3}$$

whenever $\{b_k\}_{k=1}^\infty \subset \mathbf{R}$ and $\lim_{k\to\infty} b_k = b$, is that (a) and (b) are satisfied together with

(c) $\lim_{n\to\infty} \sum_{k=1}^\infty a_{n,k} = 1$.

Proof: See [Coo, p. 64]. $\square$

A real infinite matrix $A = \{a_{n,k}\}_{n,k=1}^\infty$ is said to be a *T-matrix* and to *define a regular method of summability* provided that A satisfies the conditions (a), (b) and (c) and A has the property (3), respectively. By Theorem 1, A is a T-matrix if and only if A defines a regular method of summability. Furthermore, denote by ℓ^∞ the set of bounded sequences $b = \{b_n\}_{n=1}^\infty$ of real numbers endowed with the sup-norm

$$\|b\|_\infty = \sup_{n\in\mathbf{N}} |b_n|.$$

Since the limit is a linear functional on the linear space of convergent sequences of real numbers and the limes superior is a sublinear functional on ℓ^∞, by the

Hahn–Banach theorem, there is a linear functional $l : \ell^\infty \to \mathbf{R}$ (not uniquely determined) such that, for each $b = \{b_n\}_{n=1}^\infty \in \ell^\infty$, $l(b) \leq \limsup_{n\to\infty} b_n$ and $l(b) = \lim_{n\to\infty} b_n$ if $\{b_n\}_{n=1}^\infty$ is convergent. Easily, for each $b = \{b_n\}_{n=1}^\infty \in \ell^\infty$, $\liminf_{n\to\infty} b_n \leq l(b) \leq \limsup_{n\to\infty} b_n$ (and hence the value $l(b)$ does not depend on a finite number of terms of $\{b_n\}_{n=1}^\infty$) and $|l(b)| \leq \|b\|_\infty$, in particular, $l \in (\ell^\infty)^*$. Such an l is called here a *generalized limit* on ℓ^∞.

Example 1. Let $E = \{e_n\}_{n=1}^\infty$ be an orthonormal basis of $\mathbf{H}$ and l be a generalized limit on ℓ^∞. For $b = \{b_n\}_{n=1}^\infty \in \ell^\infty$ we shall write $\mathrm{LIM}_{n\to\infty} b_n$ instead of $l(b)$. Let $A = \{a_{n,k}\}_{n,k=1}^\infty$ be a real matrix satisfying the conditions (a) and (b) mentioned in Theorem 1 above. For each $S \in \mathcal{L}_s$ define the bounded sequence $b = \{b_n\}_{n=1}^\infty$ by

$$b_n = \sum_{k=1}^\infty a_{n,k}(Se_k, e_k),\ n \geq n_0,$$

and $b_n = 0$ otherwise, and put $\psi_A(S) = l(b)$, i.e.,

$$\psi_A(S) = \mathrm{LIM}_{n\to\infty} \sum_{k=1}^\infty a_{n,k}(Se_k, e_k). \tag{4}$$

Letting $\psi = \psi_A$, it is easy to see that $\psi \in \mathcal{L}_s^*$. If $S \in \mathcal{L}_\infty \cap \mathcal{L}_s$, then $\lim_{k\to\infty}(Se_k, e_k) = 0$ and, consequently, $\psi(S) = 0$, i.e., ψ is irregular. Moreover, if $x \in \mathbf{H}$ and a real functional f has the second Fréchet derivative $f''(x)$ at x, then

$$\boldsymbol{D}^\psi f(x) = \lim_{n\to\infty} \sum_{k=1}^\infty a_{n,k} D_{kk} f(x) \tag{5}$$

whenever the limit exists. If A is a T-matrix with nonnegative entries, i.e., $a_{n,k} \geq 0$ for all $n, k \in \mathbf{N}$, then

(P1) $\psi(I) > 0$ (even $\psi(I) = 1$) where I is the identity operator on $\mathbf{H}$ and

(P2) if $S \in \mathcal{L}_s$ and $S \geq 0$, i.e., $(Sh, h) \geq 0$ for each $h \in \mathbf{H}$, then $\psi(S) \geq 0$.

In what follows let A be a T-matrix such that, for each $n \in \mathbf{N}$, $a_{n,k} = \frac{1}{n}$ for $k = 1, ..., n$ and $a_{n,k} = 0$ otherwise. Such A defines the so-called *Cesàro method of summability.* Put $\boldsymbol{D} = \boldsymbol{D}^{\psi_A}$. Then ψ_A is an irregular functional satisfying (P1) and (P2) and the differential operator $\boldsymbol{D}$ is an extension of the Lévy laplacian $\boldsymbol{L}^E$. For a real functional f defined on a sphere $S(x, r)$ in $\mathbf{H}$ put

$$\overline{\mathcal{M}}^E(f, x, r) = \mathrm{LIM}_{n\to\infty}\mathcal{M}_n(f, x, r) \tag{6}$$

if the right-hand side exists. Similarly, for a real functional f defined on a ball $B(x, r)$ in $\mathbf{H}$ put

$$\overline{\mathcal{A}}^E(f, x, r) = \mathrm{LIM}_{n\to\infty}\mathcal{A}_n(f, x, r) \tag{7}$$

if the right-hand side exists. Then each bounded continuous functional f on a sphere $S(x,r)$ or on a ball $B(x,r)$ in $\mathbf{H}$ has $\overline{\mathcal{M}}^E(f,x,r)$ or $\overline{\mathcal{A}}^E(f,x,r)$, respectively. Hence, by Theorem 1 in Section 2.3, $\overline{\mathcal{M}}^E(f,x,r) = \overline{\mathcal{A}}^E(f,x,r)$ if f is a real uniformly continuous functional on $B(x,r)$ in $\mathbf{H}$. Compare the following theorem with Theorem 1 in Section 2.1.

Theorem 2. *If f is a real functional having the Fréchet derivative of 2-nd order at x, then*

$$\boldsymbol{D}f(x) = 2\lim_{r\to 0+}\frac{\overline{\mathcal{M}}^E(f,x,r)-f(x)}{r^2}. \tag{8}$$

PROOF: Since f has the second Fréchet derivative at x, it holds that

$$2\lim_{r\to 0+}\frac{\mathcal{M}_n(f,x,r)-f(x)}{r^2} = \frac{1}{n}\sum_{j=1}^{n} D_{jj}f(x) \tag{9}$$

uniformly for $n \in \mathbf{N}$, see the proof of Theorem 1 in Section 2.1. Futhermore, there is $R>0$ such that f is bounded and continuous on $B(x,R)$ and hence, for all $0<r<R$,

$$\mathrm{LIM}_{n\to\infty}\mathcal{M}_n(f,x,r) = \overline{\mathcal{M}}^E(f,x,r).$$

By the continuity of l on ℓ^∞ and by (9), $\boldsymbol{D}f(x) =$

$$= \mathrm{LIM}_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} D_{jj}f(x) = 2\ \mathrm{LIM}_{n\to\infty}\left(\lim_{r\to 0+}\frac{\mathcal{M}_n(f,x,r)-f(x)}{r^2}\right) =$$

$$= 2\lim_{r\to 0+}\left(\mathrm{LIM}_{n\to\infty}\frac{\mathcal{M}_n(f,x,r)-f(x)}{r^2}\right) = 2\lim_{r\to 0+}\frac{\overline{\mathcal{M}}^E(f,x,r)-f(x)}{r^2}.$$

□

Comments

In the article presented in Chapter 4, irregular operators were first explicitly defined. That the Lévy laplacian can be extended to a differential operator $\boldsymbol{D}^\psi$ with some $\psi \in \mathcal{L}_s^*$ and each such extension gives an irregular operator was first shown in Example 4 of this article, too. In Example 1 of this section, new irregular operators different from the Lévy laplacian are given, namely, for a real T-matrix $\{a_{n,k}\}_{n,k=1}^\infty$, the differential operator defined as in (5) is extended to an irregular operator. In particular, it is shown there that the Lévy laplacian has an irregular extension $\boldsymbol{D}^\psi$ with some ψ satisfying the conditions (P1) and (P2). In Chapter 4, for a general irregular operator $\boldsymbol{D}^\psi$ with ψ satisfying (P1) and (P2) the maximum principle is established and the related Dirichlet and Poisson problems are discussed. Considerable difficulties in the theory of P. Lévy are caused by non-existence of means for 'reasonable' functionals. Generalized volume and spherical means introduced in Example 1 are defined for a wide class of functionals, e.g., for bounded continuous ones, and are equal to volume and spherical means introduced by P. Lévy if the latter exist. Theorem 2 is a modification of Theorem 1 in Section 2.1 and illustrates usefulness of such generalized means.

Question: Are there any other irregular operators than those in Example 1?

2.6 Dependence on the choice of bases

Denote by $\mathcal{P}$ the set of all permutations of $\mathbf{N}$, i.e., the set of all one-to-one mappings ρ of $\mathbf{N}$ onto $\mathbf{N}$, and by $\mathcal{I}$ the set of all strictly increasing mappings $r : \mathbf{N} \to \mathbf{N}$. Then the following result holds.

Fact 1. *Let $D = \{d_{n,k}\}_{n,k=1}^{\infty}$ be a real T-matrix. If $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers, then the following statements are equivalent to each other:*

(i) the sequence $\{a_n\}_{n=1}^{\infty}$ has a limit;

(ii) for each $\rho \in \mathcal{P}$, $\lim_{n\to\infty} \sum_{k=1}^{\infty} d_{n,k} a_{\rho(k)}$ exists;

(iii) for each $r \in \mathcal{I}$, $\lim_{n\to\infty} \sum_{k=1}^{\infty} d_{n,k} a_{r(k)}$ exists.

In case D defines the Cesàro method of summability we can omit the hypothesis of boundedness of $\{a_n\}_{n=1}^{\infty}$.

Proof: First, let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers (not necessarily bounded) and $a \in [-\infty, +\infty]$ be such that $\lim_{n\to\infty} a_n = a$. Then, for each $\rho \in \mathcal{P}$ and $r \in \mathcal{I}$, $\lim_{n\to\infty} a_{\rho(n)} = a$ and $\lim_{n\to\infty} a_{r(n)} = a$ and, as a result,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} d_{n,k} a_{\rho(k)} = a \text{ and } \lim_{n\to\infty} \sum_{k=1}^{\infty} d_{n,k} a_{r(k)} = a.$$

Thus (i) implies (ii) and (iii).

On the other hand, let $\{a_n\}_{n=1}^{\infty}$ have no limit. We are going to show that there are $\rho \in \mathcal{P}$ and $r \in \mathcal{I}$ such that neither of the limits in (ii) and (iii) exists. Put $m = \liminf a_n$ and $M = \limsup a_n$. Let $\{m_n\}_{n=1}^{\infty}$ and $\{M_n\}_{n=1}^{\infty}$ be sequences of real numbers greater than m and less than M with $\lim_{n\to\infty} m_n = m$ and $\lim_{n\to\infty} M_n = M$, respectively. Moreover, there are $p, r \in \mathcal{I}$ such that $m = \lim_{n\to\infty} a_{p(n)}$ and $M = \lim_{n\to\infty} a_{r(n)}$. We can assume that $p(\mathbf{N}) \cap r(\mathbf{N}) = \emptyset$ and $\mathbf{N} \setminus (p(\mathbf{N}) \cup r(\mathbf{N}))$ is infinite. Take $s \in \mathcal{I}$ such that $s(\mathbf{N}) = \mathbf{N} \setminus (p(\mathbf{N}) \cup r(\mathbf{N}))$.

If $\{b_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $n \in \mathbf{N}$, then define here the n-th mean of $\{b_n\}_{n=1}^{\infty}$ (with respect to D) by

$$\sum_{k=1}^{\infty} d_{n,k} b_k$$

whenever the sum exists. In case D defines the Cesàro method of summability the n-th mean is the same as the n-th arithmetic mean. By induction we construct a sequence $\{b_n\}_{n=1}^{\infty}$ such that there is $\rho \in \mathcal{P}$ with $b_n = a_{\rho(n)}$, $n \in \mathbf{N}$, and, putting s_n equal to the n-th mean of $\{b_n\}_{n=1}^{\infty}$, $\liminf s_n = m$ and $\limsup s_n = M$. Before doing so, a lemma will be proved.

Lemma 1. *Let $c \in \mathbf{R}$, $\{c_n\}_{n=1}^{\infty}$ be a real sequence with $\lim_{n\to\infty} c_n < c$ and $K \in (0, +\infty]$ be such that $|c_n| \le K$, $n \in \mathbf{N}$. Assume that K is finite unless D defines the Cesàro method of summability. Then for each $n \in \mathbf{N}$ there is $N, k > n$ such that the N-th mean of any real sequence bounded by K whose first elements are $c_1, ..., c_k$ is less than c.*

Proof: Assume that $K < +\infty$. (In case D defines the Cesàro method of summability the proof is obvious.) By the Silverman-Toeplitz theorem, given $n \in \mathbf{N}$ and $\lim_{n\to\infty} c_n < c' < c$, there is $N > n$ such that the N-th mean t_N of $\{c_n\}_{n=1}^{\infty}$ is less than c' and

$$\sum_{j=1}^{\infty} |d_{N,j}| < +\infty.$$

Thus there is $k > n$ such that

$$\sum_{j=k+1}^{\infty} |d_{N,j}| < \frac{c - c'}{2K}. \tag{1}$$

For any real sequence $\{c'_n\}_{n=1}^{\infty}$ bounded by K whose first elements are $c_1, ..., c_k$ its N-th mean t'_N satisfies, by (1),

$$t'_N \le t_N + |t'_N - t_N| < c' + (c - c') = c,$$

which concludes the proof. □

Going back to the construction of $\{b_n\}_{n=1}^{\infty}$, put $K = \sup\{|a_n|;\ n \in \mathbf{N}\}$. Let $k_1 = 1$ and $l_1 = 1$. If we have defined $\{k_j\}_{j=1}^{n}$ and $\{l_j\}_{j=1}^{n}$ for some $n \in \mathbf{N}$, then let k_{n+1} be the least natural number greater than k_n for which there is $N > n+1$ such that the N-th mean of any real sequence bounded by K whose first elements are

$$a_{s(1)}, a_{p(k_1)}, a_{r(l_1)}, ..., a_{r(l_n)}, a_{s(n+1)}, a_{p(k_n+1)}, ..., a_{p(k_{n+1})}$$

is less than m_{n+1}. (By Lemma 1, such k_{n+1} exists because $\lim a_{p(n)} = m$ and $m < m_{n+1}$.) Similarly, let l_{n+1} be the least natural number greater than l_n for which there is $N > n+1$ such that the N-th mean of any real sequence bounded by K whose first elements are

$$a_{s(1)}, a_{p(1)}, a_{r(1)}, \ldots, a_{r(l_n)}, a_{s(n+1)}, a_{p(k_n+1)}, \ldots, a_{p(k_{n+1})}, a_{r(l_n+1)}, \ldots, a_{r(l_{n+1})}$$

is greater than M_{n+1}.

Denote by $\{b_n\}_{n=1}^{\infty}$ the sequence inductively constructed above. Since $\{a_{p(n)}\}_{n=1}^{\infty}$, $\{a_{r(n)}\}_{n=1}^{\infty}$ and $\{a_{s(n)}\}_{n=1}^{\infty}$ are subsequences of $\{b_n\}_{n=1}^{\infty}$ such that each element of $\{b_n\}_{n=1}^{\infty}$ is in exactly one of these subsequences and $p(\mathbf{N}), r(\mathbf{N}), s(\mathbf{N})$ are pairwise disjoint subsets of $\mathbf{N}$ whose union is the whole set $\mathbf{N}$ there is $\rho \in \mathcal{P}$ such that $b_n = a_{\rho(n)},\ n \in \mathbf{N}$. Of course, $\{b_n\}_{n=1}^{\infty}$ is a desired sequence.

The proof will be concluded by showing that there is a subsequence $\{c_n\}_{n=1}^{\infty}$ of the sequence $\{a_n\}_{n=1}^{\infty}$ such that, putting t_n equal to the n-th mean of $\{c_n\}_{n=1}^{\infty}$, $\liminf t_n = m$ and $\limsup t_n = M$. Let $k_1^b = 1$, $k_1^e = 1$, l_1^b be the least natural number such that $r(l_1^b) > p(k_1^e)$ and $l_1^e = l_1^b$. (Let us note that b in l_1^b and e in l_1^e refer to the first letters of words *beginning* and *end*, respectively.) If we have defined $\{k_j^b\}_{j=1}^{n}$, $\{k_j^e\}_{j=1}^{n}$, $\{l_j^b\}_{j=1}^{n}$ and $\{l_j^e\}_{j=1}^{n}$ for some $n \in \mathbf{N}$, then let k_{n+1}^b be the least natural number greater than k_n^e such that $p(k_{n+1}^b) > r(l_n^e)$ and let k_{n+1}^e be the least natural number greater than k_{n+1}^b for which there is $N > n+1$ such that the N-th mean of any real sequence bounded by K whose first elements are

$$a_{p(1)}, a_{r(l_1^e)}, \ldots, a_{r(l_n^e)}, a_{p(k_{n+1}^b)}, \ldots, a_{p(k_{n+1}^e)}$$

is less than m_{n+1}. (By Lemma 1, such k_{n+1}^e exists because $\lim a_{p(n)} = m$ and $m < m_{n+1}$.) Similarly, let l_{n+1}^b be the least natural number greater than l_n^e such that $r(l_{n+1}^b) > p(k_{n+1}^e)$ and let l_{n+1}^e be the least natural number greater than l_{n+1}^b for which there is $N > n+1$ such that the N-th mean of any real sequence bounded by K whose first elements are

$$a_{p(1)}, a_{r(l_1^e)}, \ldots, a_{r(l_n^e)}, a_{p(k_{n+1}^b)}, \ldots, a_{p(k_{n+1}^e)}, a_{r(l_{n+1}^b)}, \ldots, a_{r(l_{n+1}^e)}$$

is greater than M_{n+1}.

Denote by $\{c_n\}_{n=1}^{\infty}$ the sequence inductively constructed above. Obviously, $\{c_n\}_{n=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ and has the desired properties. □

Denote by $\mathcal{F} = \mathcal{F}(\mathbf{H})$ the set of $T \in \mathcal{L}_s$ for which there is $p \in \mathbf{R}$ such that $T - pI \in \mathcal{L}_\infty$. The following proposition is a consequence of the previous assertion.

Proposition 1. *Let $T \in \mathcal{L}_s$ and $g(y) = (Ty, y),\ y \in \mathbf{H}$. Then the following statements are equivalent to each other:*

- *(i) g has the invariant spherical mean on any sphere in $\mathbf{H}$;*
- *(ii) there are $x \in \mathbf{H}$ and $r > 0$ such that $\mathcal{M}^E(g, x, r)$ exists for each orthonormal basis E;*

(iii) for each orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$, $\{g(e_n)\}_{n=1}^{\infty}$ is convergent;

(iv) $T \in \mathcal{F}$.

Furthermore, if $p \in \mathbf{R}$ is such that $T - pI \in \mathcal{L}_{\infty}$, then

$$\mathcal{M}(g, x, r) = g(x) + r^2 p$$

for each $x \in \mathbf{H}$ and $r > 0$.

PROOF: By the result from Example 2 in Section 2.1, (i) follows from (iv). Obviously, (i) implies (ii). Moreover, (ii) is equivalent to the fact that, for each orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$,

$$\frac{1}{n}\sum_{j=1}^{n} g(e_j),\ n \in \mathbf{N}$$

is convergent, see Example 2 in Section 2.1 again. In particular, (ii) implies that, given an orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$,

$$\frac{1}{n}\sum_{j=1}^{n} g(e_{\rho(j)}),\ n \in \mathbf{N}$$

is convergent for each $\rho \in \mathcal{P}$ and hence, by Fact 1, $\{g(e_n)\}_{n=1}^{\infty}$ is convergent. Thus (iii) follows from (ii). We are going to show that if (iii) holds, then there is $p \in \mathbf{R}$ such that $\lim_{n\to\infty} g(e_n) = p$ for each orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$, which is equivalent to (iv): Indeed, by Corollary to Theorem 1 in Section 1.2, $\lim_{n\to\infty}(Te_n, e_n) = p$ for each orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$ if and only if $T - pI \in \mathcal{L}_{\infty}$. Now it only remains to prove that if (iii) holds and $E = \{e_n\}_{n=1}^{\infty}$ and $F = \{f_n\}_{n=1}^{\infty}$ are orthonormal bases in $\mathbf{H}$ such that $\lim_{n\to\infty} g(e_n) = p$ and $\lim_{n\to\infty} g(f_n) = q$, then $p = q$. Indeed, let $h_1 \in \mathbf{H}$ and $h_2 \in \mathbf{H}$ be unit vectors with $(h_1, h_2) = 0$. If we have defined an orthonormal set $\{h_j\}_{j=1}^{2N}$ in $\mathbf{H}$ for some $N \in \mathbf{N}$ (i.e., for $i, j = 1, ..., 2N$, $\|h_j\| = 1$ and $(h_i, h_j) = 0$ if $i \neq j$), then it is easy to see that there is $m \in \mathbf{N}$ such that the vector

$$h_{2N+1} = (e_m - \sum_{j=1}^{2N}(e_m, h_j)h_j)/\|e_m - \sum_{j=1}^{2N}(e_m, h_j)h_j\|$$

satisfies $|g(h_{2N+1}) - p| = |(Th_{2N+1}, h_{2N+1}) - p| \leq 1/(N+1)$. Similarly, there is $n \in \mathbf{N}$ such that the vector

$$h_{2N+2} = (f_n - \sum_{j=1}^{2N+1}(f_n, h_j)h_j)/\|f_n - \sum_{j=1}^{2N+1}(f_n, h_j)h_j\|$$

satisfies $|g(h_{2N+2}) - q| = |(Th_{2N+2}, h_{2N+2}) - q| \leq 1/(N+1)$. Finally, there is an orthonormal basis containing the orthonormal set $\{h_n\}_{n=1}^{\infty}$ inductively constructed above and hence

$$p = \lim_{n\to\infty} g(h_{2n+1}) = \lim_{n\to\infty} g(h_n) = \lim_{n\to\infty} g(h_{2n}) = q,$$

which concludes the proof. □

Denote by $\mathcal{U}$ the set of unitary operators U on $\mathbf{H}$, i.e., U are linear and $(Uy, Uz) = (y, z)$, $y, z \in \mathbf{H}$. For $x \in \mathbf{H}$ and $U \in \mathcal{U}$ put $U_x(y) = x + U(y-x)$, $y \in \mathbf{H}$. Futhermore, let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathbf{H}$ and $\{d_{n,k}\}_{n,k=1}^{\infty}$ be a real T-matrix. Define

$$\boldsymbol{D}f(x) = \lim_{n\to\infty} \sum_{k=1}^{\infty} d_{n,k}(f''(x)e_k, e_k) \tag{2}$$

if a functional f has the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{H}$ and the limit in (2) exists. Given a real functional f having the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{H}$, it is easily seen that, for each $U \in \mathcal{U}$,

$$(f \circ U_x)''(x) = U^* f''(x) U \tag{3}$$

where U^* is the adjoint operator to U. Of course, a sequence $F = \{f_n\}_{n=1}^{\infty}$ of vectors of $\mathbf{H}$ is an orthonormal basis in $\mathbf{H}$ if and only if there is $U \in \mathcal{U}$ such that $f_n = Ue_n$, $n \in \mathbf{N}$. Given an orthonormal basis $F = \{f_n\}_{n=1}^{\infty}$ and $U \in \mathcal{U}$ with $f_n = Ue_n$, $n \in \mathbf{N}$, by (3),

$$\boldsymbol{D}(f \circ U_x)(x) = \lim_{n\to\infty} \sum_{k=1}^{\infty} d_{n,k}(f''(x)f_k, f_k) \tag{4}$$

whenever one side exists; in particular, $\boldsymbol{L}^E(f \circ U_x)(x) = \boldsymbol{L}^F f(x)$ whenever one side exists.

Proposition 2. *Let $\boldsymbol{D}$ be defined as in (2). If f is a real functional having the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{H}$, then the following statements are equivalent to each other:*

(i) $\boldsymbol{L}f(x)$ exists;

(ii) for each orthonormal basis F, $\boldsymbol{L}^F f(x)$ exists;

(iii) there is $p \in \mathbf{R}$ such that $\boldsymbol{D}(f \circ U_x)(x) = p$ for each $U \in \mathcal{U}$;

(iv) for each $U \in \mathcal{U}$, $\boldsymbol{D}(f \circ U_x)(x)$ exists;

(v) $f''(x) \in \mathcal{F}$.

Proof: By Proposition 1, in order to show that (i), (ii) and (v) are equivalent to each other it is sufficient to realize that, putting $g(y) = (f''(x)y, y),\ y \in \mathbf{H}$, it holds $\boldsymbol{L}^F f(x) = \mathcal{M}^F(g, 0, 1)$ if F is an orthonormal basis in $\mathbf{H}$ and one side exists, see Remark 3 in Section 2.1. Obviously, (iii) implies (iv).

By (4), it follows from (iv) that, for each orthonormal basis $F = \{f_n\}_{n=1}^{\infty}$,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} d_{n,k}\, g(f_k)$$

exists and hence

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} d_{n,k}\, g(f_{\rho(k)})$$

exists for any $\rho \in \mathcal{P}$. By Fact 1, for each orthonormal basis $F = \{f_n\}_{n=1}^{\infty}$, $\{g(f_n)\}_{n=1}^{\infty}$ is convergent, which is equivalent to (v) according to Proposition 1. By (4), (v) implies (iii). $\square$

The following assertions are counterparts of Fact 1 and Proposition 1.

Fact 2. *If $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers, $m = \liminf a_n$ and $M = \limsup a_n$, then for each $a \in [m, M]$ there is $\rho \in \mathcal{P}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} a_{\rho(j)} = a.$$

Proof: See Corollary to Fact 3 in the article from Chapter 5. $\square$

Proposition 3. *Let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathbf{H}$, $T \in \mathcal{L}_s$ and $g(x) = (Tx, x),\ x \in \mathbf{H}$. If $m^E = \liminf g(e_n)$ and $M^E = \limsup g(e_n)$, then for each $p \in [m^E, M^E]$ there is $\rho \in \mathcal{P}$ such that, putting $F = \{e_{\rho(n)}\}_{n=1}^{\infty}$,*

$$\mathcal{M}^F(g, x, r) = g(x) + r^2 p$$

for each $x \in \mathbf{H}$ and $r > 0$.

Corollary. *If a real functional f has the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{H}$, $E = \{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathbf{H}$, $a_n^E = (f''(x)e_n, e_n)$ for each $n \in \mathbf{N}$, $m^E = \liminf a_n^E$ and $M^E = \limsup a_n^E$, then for each $p \in [m^E, M^E]$ there is $\rho \in \mathcal{P}$ such that, putting $F = \{e_{\rho(n)}\}_{n=1}^{\infty}$, $\boldsymbol{L}^F f(x) = p$.*

Proof: It follows directly from Fact 2 because

$$\mathcal{M}^E(g, x, r) = g(x) + r^2 \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(e_j)$$

if one side exists. $\square$

Comments

Fact 1 concerning sequences of real numbers is known at least in case D defines the Cesàro method of summability, see [Die2, Exercise 15, p. 116]. In [MW1], it is shown that (i) and (iv) from Proposition 1 are equivalent and so are (i) and (v) from Proposition 2. That (iii) and (v) from Proposition 2 are equivalent seems to be known because Theorem 2 in [NS] can be rephrased as follows: If $\psi \in \mathcal{L}_s^*$ is irregular, $\psi(I) > 0$ and a functional f has the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{H}$, then $f''(x) \in \mathcal{F}$ if and only if

$$\boldsymbol{D}^{\psi}(f \circ U_x)(x) = \boldsymbol{D}^{\psi} f(x), \ U \in \mathcal{U}. \tag{5}$$

On the other hand, in Theorem 4 of the article from Chapter 4, it is proved that, given a point $x \in \mathbf{H}$, $\psi \in \mathcal{L}_s^*$ is irregular if and only if each functional f that has the second Fréchet derivative $f''(x)$ belonging to $\mathcal{F}$ satisfies the condition (5).

Fact 2 is a consequence of Corollary to Fact 3 in the article from Chapter 5. Moreover, in this article, a more general result for sequences in Banach spaces is proved. Proposition 3 and its corollary follows directly from Fact 2. Of course, P. Lévy was aware of the fact that the definition of his laplacian depends on the choice of an orthonormal basis of $\mathbf{H}$. Let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathbf{H}$ and $x_n = (x, e_n)$, $x \in \mathbf{H}$ and $n \in \mathbf{N}$. In [Lev2, p. 63], P. Lévy says,'... les a_n étant tous égaux à 1 ou à 2, le laplacien de l'expression $\sum a_n x_n^2$ peut, si on change l'ordre des termes, prendre n'importe quelle valeur donnée entre 2 et 4.' Thus P. Lévy must have known the assertion of Fact 2.

Chapter 3

The Gross laplacian

In this chapter, our objective is not to give an exhaustive review of the potential theory on abstract Wiener spaces developed by L. Gross in his famous paper [Gro2] but especially to show main dissimilarities and loose connections with the Lévy theory presented in the previous chapter. Proofs of the majority of theorems contained in this chapter may be found in [Kuo1].

3.1 Abstract Wiener spaces

In what follows $(\mathbf{H}, (\cdot,\cdot))$ will be a real separable Hilbert space with the induced norm $|x| = \sqrt{(x,x)},\ x \in \mathbf{H}$. In case $\mathbf{H}$ is finite dimensional there is a unique probability measure μ on $\mathbf{H}$ (i.e., μ is a countably additive measure on the σ-algebra of Borel sets in $\mathbf{H}$ with $\mu(\mathbf{H}) = 1$) such that

$$(1) \qquad \int_{\mathbf{H}} e^{i(x,y)}\mu(dx) = e^{-\frac{|y|^2}{2}},\ y \in \mathbf{H}.$$

In fact, such a μ is called the *Gaussian measure* on $\mathbf{H}$ and if n is the dimension of $\mathbf{H}$, then

$$(2) \qquad \mu(A) = (2\pi)^{-n/2} \int_A e^{-\frac{|x|^2}{2}} \lambda_n(dx)$$

for each Borel set A in $\mathbf{H}$ where λ_n is the Lebesgue measure on $\mathbf{H}$. Following [Kuo2, p. 7], we show that in case $\mathbf{H}$ is infinite dimensional there is no probability measure μ on $\mathbf{H}$ satisfying (1). Indeed, suppose on the contrary that such a measure μ exists. Let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathbf{H}$. For each $n \in \mathbf{N}$ and $x \in \mathbf{H}$,

$$(3) \qquad \int_{\mathbf{H}} e^{i(x,e_n)}\mu(dx) = e^{-\frac{1}{2}}.$$

Note that, for every $x \in \mathbf{H}$, $\lim_{n\to\infty}(x, e_n) = 0$ and hence, by the Lebesgue dominated convergence theorem, we get a contradiction $1 = e^{-1/2}$.

Let $\mathbf{B}$ be a real Banach space and $\mathbf{B}^*$ be its dual space. Recall that a set $C \subset \mathbf{B}$ is said to be a *cylinder set* in $\mathbf{B}$ provided that there are an $n \in \mathbf{N}$, a Borel set D in $\mathbf{R}^n$ and $y_1, ..., y_n$ in $\mathbf{B}^*$ such that

$$C = \{x \in \mathbf{B};\ (< y_1, x >, ..., < y_n, x >) \in D\} \tag{4}$$

where $< \cdot, \cdot >$ is the natural bilinear pairing of $\mathbf{B}^*$ and $\mathbf{B}$. In addition, such a set C is said to be *based on* a finite dimensional linear subspace K of $\mathbf{B}^*$ provided that $y_1, ..., y_n \in K$. Obviously, the collection of cylinder sets in $\mathbf{B}$ is a algebra $\mathcal{R}$ and the collection of cylinder sets based on a given finite dimensional linear subspace K of $\mathbf{B}^*$ is a σ-algebra $\mathcal{R}_K$. Moreover, recall that a non-negative set function μ on $\mathcal{R}$ is said to be a cylinder set measure on $\mathbf{B}$ provided that $\mu(\mathbf{B}) = 1$ and μ is countably additive on $\mathcal{R}_K$ for any finite dimensional linear subspace K of $\mathbf{B}^*$. Easily, any cylinder set measure on $\mathbf{B}$ is additive on $\mathcal{R}$. In case $\mathbf{B}$ is separable one can prove that the σ-algebra of Borel sets in $\mathbf{B}$ is generated by $\mathcal{R}$ and hence any cylinder set measure on $\mathbf{B}$ which is countably additive on $\mathcal{R}$ can be uniquely extended to a probability measure on $\mathbf{B}$. It is easily seen that, for each cylinder set C in $\mathbf{H}$, there is a finite dimensional orthogonal projection P on $\mathbf{H}$ and a Borel set D in $P(\mathbf{H})$ such that

$$C = \{x \in \mathbf{H};\ Px \in D\}. \tag{4}$$

Now define the *Gaussian cylinder set measure* μ on $\mathbf{H}$ by

$$\mu(C) = \tilde{\mu}(D)$$

if a cylinder set C in $\mathbf{H}$ is as in (4) and $\tilde{\mu}$ is the Gaussian measure on $P(\mathbf{H})$. Obviously, in case $\mathbf{H}$ is infinite dimensional μ is not countably additive because μ satisfies (1). Futhermore, recall that a norm $\|\cdot\|$ on $\mathbf{H}$ is said to be *measurable* provided that for each $\varepsilon > 0$ there is a finite dimensional orthogonal projection P_0 on $\mathbf{H}$ such that, for any finite dimensional orthogonal projection P on $\mathbf{H}$ with $PP_0 = 0$,

$$\mu(\{x \in \mathbf{H};\ \|Px\| > \varepsilon\}) < \varepsilon.$$

Let $\|\cdot\|$ be a measurable norm on $\mathbf{H}$. Then it is known that there is a $c \in \mathbf{R}$ such that $\|x\| \leq c|x|$, $x \in \mathbf{H}$. Denoting by $\mathbf{B}$ the completion of $\mathbf{H}$ with respect to $\|\cdot\|$, the pair $(\mathbf{H}, \mathbf{B})$ is called an *abstract Wiener space*. Obviously, $\mathbf{B}$ is a separable Banach space and the natural injection i from $\mathbf{H}$ to $\mathbf{B}$ is continuous. On the other hand, for any real separable Banach space $\mathbf{B}$, there is a real separable Hilbert space $\mathbf{H}$ such that the pair $(\mathbf{H}, \mathbf{B})$ is an abstract Wiener space, see [Kuo1, Chapter I, Theorem 4.4].

Fact 1. *(Gross theorem) Let $(\mathbf{H}, \mathbf{B})$ be an abstract Wiener space, i be the natural injection from $\mathbf{H}$ to $\mathbf{B}$ and μ be the Gaussian cylinder set measure* *on* $\mathbf{H}$*. Then the cylinder set measure* $i(\mu)$ *on* $\mathbf{B}$ *is countably additive. Here* $i(\mu)(C) = \mu(i^{-1}(C))$ *for each cylinder set* C *in* $\mathbf{B}$.

PROOF: See [Kuo1, Chapter I, Theorem 4.1]. □

Assume that all hypotheses of Fact 1 are satisfied and denote by p the unique probability measure on $\mathbf{B}$ which is an extension of $i(\mu)$. For $t > 0$ put $p_t(E) = p(t^{-1/2}E)$ for any Borel set E in $\mathbf{B}$. The probability measure p_t is called a *Wiener measure* on $\mathbf{B}$ with variance parameter t.

Example 1. In case $\mathbf{H}$ is finite dimensional it is easily seen that $\mathbf{B} = \mathbf{H}$, μ is the Gaussian measure on $\mathbf{H}$ and $p = \mu$.

Example 2. Denote by $\mathcal{C}$ the Banach space of real continuous functions f on $[0,1]$ with $f(0) = 0$ endowed with the sup-norm and by $\mathcal{C}'$ the Hilbert space of absolutely continuous functions $f : [0,1] \to \mathbf{R}$ with f(0)=0 and $f' \in L^2(0,1)$ which is endowed with the norm

$$|f| = \left(\int_0^1 (f'(t))^2 \, dt \right)^{\frac{1}{2}}.$$

Then the pair $(\mathcal{C}', \mathcal{C})$ called the *classical Wiener space* is an example of an abstract Wiener space and in this case p is the so-called *classical Wiener measure*, see [Kuo1, Chapter I, Theorem 5.3].

Example 3. Assume that $\| \cdot \|$ is the norm on $\mathbf{H}$ induced by an inner product $((\cdot,\cdot))$ on $\mathbf{H}$. It is well-known that $\| \cdot \|$ is measurable on $\mathbf{H}$ if and only if there is an one-to-one operator $T \in \mathcal{L}_2(\mathbf{H})$ such that $\|x\| = |Tx|,\ x \in \mathbf{H}$. In other words, if $\mathbf{B}$ is a Hilbert space with the inner product $((\cdot,\cdot))$, then the pair $(\mathbf{H}, \mathbf{B})$ is an abstract Wiener space if and only if there is an one-to-one operator $T \in \mathcal{L}_2(\mathbf{B})$ such that $\mathbf{H} = T(\mathbf{B})$ and

$$((x,y)) = (Tx, Ty),\ x, y \in \mathbf{B}.$$

See [Kuo1, Chapter I, Example 18 and Theorem 4.3].

For $t > 0$ and a real bounded Borel function f on $\mathbf{B}$ put

$$(p_t f)(x) = \int_{\mathbf{B}} f(x+y) p_t(dy),\ x \in \mathbf{B} \tag{7}$$

and $p_0 f = f$ on $\mathbf{B}$.

Theorem 1. *The operators* $p_t,\ t \geq 0$ *form a strongly continuous contraction semigroup on the Banach space* $\mathcal{A}$ *consisting of bounded uniformly continuous functions on* $\mathbf{B}$ *endowed with the sup-norm on* $\mathbf{B}$.

PROOF: By the definition of Wiener measures, $p_t * p_s = p_{t+s},\ t, s > 0$. Here $p_t * p_s$ is the convolution of p_t and p_s. The remainder of the proof is straightforward. See [Kuo1, Chapter III, Theorem 4.1]. □

Now it is quite natural to ask about the infinitesimal generator of the semigroup considered above. Before discussing this point, a few preliminary remarks will be made. In what follows $(\mathbf{H},\mathbf{B})$ will be an abstract Wiener space. Let $T:\mathbf{B}\to\mathbf{B}^*$ be a bounded linear operator. Since the natural injection i^* from $\mathbf{B}^*$ to $\mathbf{H}^*$ is continuous and, by the Riesz representation theorem, $\mathbf{H}^*$ may be identified with $\mathbf{H}$, the restriction $T|_{\mathbf{H}}$ of T to $\mathbf{H}$ can be thought of as a bounded linear operator from $\mathbf{H}$ to $\mathbf{H}$. Moreover, one can prove that even $T|_{\mathbf{H}}$ is of trace class type, i.e., $T|_{\mathbf{H}}\in\mathcal{L}_1(\mathbf{H})$, and

$$\int_{\mathbf{B}} < Tx, x > p(dx) = tr(T|_{\mathbf{H}}), \tag{9}$$

see [Kuo1, Chapter I, Example 20 and Theorem 4.6]. The result just mentioned is sometimes referred to as the *Gross' trace class theorem*.

According to [Kur], [BF] and [Whi], in a general separable Banach space $\mathbf{B}$, the set $\mathcal{A}\cap\mathcal{C}^1(\mathbf{B})$ is not dense in $\mathcal{A}$ and even there is no non-zero Fréchet differentiable functional on $\mathbf{B}$ with bounded support. Thus the notion of Fréchet differentiability seems to be too strong in this context. Let f be a real functional defined on an open set G in $\mathbf{B}$. Given a point $x\in G$, there is a neighborhood U of the origin in $\mathbf{H}$ with $x+U\subset G$. Put $g(h)=f(x+h),\ h\in U$. If g has the n-th Fréchet derivative $g^{(n)}(0)$ at the origin, then f is said to be *n times $\mathbf{H}$-differentiable* at x and the n-linear form $g^{(n)}(0)$ on $\mathbf{H}$ is called the n-th $\mathbf{H}$-derivative of f at x and is denoted by $d^n f(x)$. Obviously, if f has the Fréchet derivative $f'(x)$ at x, then f is $\mathbf{H}$-differentiable at x and $(df(x),h) =< f'(x),h >$, $h\in\mathbf{H}$. By Theorem 1, for a given $f\in\mathcal{A}$, $\lim_{t\to 0+} p_t f = f$ uniformly on $\mathbf{B}$. Since one can prove that $p_t f$ is infinitely $\mathbf{H}$-differentiable on the whole space $\mathbf{B}$ for each $t>0$ we have that the set of infinitely $\mathbf{H}$-differentiable functionals of $\mathcal{A}$ is dense in $\mathcal{A}$, see [Kuo1, Chapter II, §6]. On the other hand, there is an infinitely $\mathbf{H}$-differentiable functional f on $\mathbf{B}$ which is continuous at none of points in $\mathbf{B}$ with respect to the norm $\|\cdot\|$ on $\mathbf{B}$. Indeed, consider $f=1$ on $\mathbf{H}$ and $f=0$ on $\mathbf{B}\setminus\mathbf{H}$.

Theorem 2. *Let $f\in\mathcal{A}\cap\mathcal{C}^2(\mathbf{B})$ be such that $f'':\mathbf{B}\to\mathcal{L}(\mathbf{B},\mathbf{B}^*)$ is bounded and uniformly continuous. Then f belongs to the domain of the infinitesimal generator A of the semigroup $p_t,\ t\geq 0$ on the Banach space $\mathcal{A}$ and*

$$(Af)(x) = \frac{1}{2} tr(d^2 f(x)),\ x\in\mathbf{B}. \tag{10}$$

Moreover, the set of functionals f of the domain of A satisfying that, for each $x\in\mathbf{B}$, $d^2 f(x)\in\mathcal{L}_1(\mathbf{H})$ and (10) holds is dense in the domain of A.

PROOF: The proof of the first part is standard as long as one notice that if f has the second Fréchet derivative $f''(x)$ at a point $x\in\mathbf{B}$, then, by the Gross' trace class theorem, $d^2 f(x)\in\mathcal{L}_1(\mathbf{H})$ and

$$\int_{\mathbf{B}} < f''(x)y, y > p(dy) = tr(d^2 f(x)),$$

see [Kuo1, Chapter III, Theorem 4.2].

The proof of the second part consists in the fact that, for any bounded Lipschitz functional f on $\mathbf{B}$, the function $v(t,x) = p_t f(x),\ (t,x) \in Q = [0,\infty) \times \mathbf{B}$ satisfies

$$\frac{\partial v}{\partial t} = \frac{1}{2} tr(d^2 v) \text{ on } Q \quad \text{and} \quad v(0,\cdot) = f \text{ on } \mathbf{B}.$$

For details, see [Kuo1, Chapter III, Theorem 4.6] and [Gro2, Corollary 3.2]. □

Of course, in case $\mathbf{B} = \mathbf{R}^n$ the infinitesimal generator A of the semigroup $p_t,\ t \geq 0$ on the Banach space $\mathcal{A}$ is equal to the Laplace operator in $\mathbf{R}^n$ divided by 2 for functions belonging to $\mathcal{A}$ together with their second partial derivatives.

Comments

The results presented in this section are mainly due to L. Gross, see [Gro1] and [Gro2]. Abstract Wiener spaces introduced by L. Gross in [Gro1] are named after N. Wiener who is the first person to show that the cylinder set measure called now the classical Wiener measure is countably additive, see e.g. [Wie]. The concept of abstract Wiener spaces was further generalized, see e.g. [Cat, p. 96] where *generalized Wiener spaces* are dealt with. Conditions under which cylinder set measures on linear topological spaces are countably additive and some related topics are also discussed in [DF] and [SF].

3.2 Potential theory and Brownian motion

In what follows the pair $(\mathbf{H}, \mathbf{B})$ will be an abstract Wiener space and $p_t,\ t > 0$ will be probability measures on $\mathbf{B}$ defined as in the previous section. Let Ω be the set of continuous functions $\omega : [0,\infty) \to \mathbf{B}$ with $\omega(0) = 0$. For $t \geq 0$ put $W(t)(\omega) = \omega(t),\ \omega \in \Omega$. Then it is well-known that there is a unique probability measure P on the σ-algebra in Ω generated by the functions $W(t),\ t \geq 0$ such that if $t > s$, then $W(t) - W(s)$ is distributed in $\mathbf{B}$ according to p_{t-s} and if $n \in \mathbf{N}$ and $0 = t_0 < t_1 < ... < t_n$, then $W(t_j) - W(t_{j-1}),\ j = 1, ..., n$ are independent, see [Gro2, Remark 2.3]. Of course, in case $\mathbf{B} = \mathbf{R}$ the measure P is the classical Wiener measure on Ω. Denote by $\mathcal{B}$ the σ-algebra of Borel sets in $\mathbf{B}$. Then $W(t),\ t \geq 0$ called a *Wiener process* in $\mathbf{B}$ is a Markov process with transition functions

$$p_t(x, A) = p_t(A - x),\ x \in \mathbf{B} \text{ and } A \in \mathcal{B}$$

for all $t > 0$. The corresponding *Brownian motion* starting at a point $x \in \mathbf{B}$ is $x + W(t),\ t \geq 0$. Moreover, denote by τ_x^V *the first exit time* from an open set $V \subset \mathbf{B}$ starting at a point $x \in \mathbf{B}$, i.e.,

(1) $$\tau_x^V(\omega) = \inf\{t \geq 0;\ x + W(t)(\omega) \notin V\},\ \omega \in \Omega.$$

In case V is a ball $B(x,r)$ in $\mathbf{B}$ we shall write $\tau_x^{(r)}$ instead of τ_x^V. Denoting by $E[\cdot]$ the expectation with respect to P, it is well-known that $c = E[\tau_0^{(1)}] \in (0,+\infty)$ and $E[\tau_x^{(r)}] = cr^2$, $x \in \mathbf{B}$ and $r > 0$, see [Gro2, Remark 3.3]. Moreover, define a probability measure π on the unit sphere $S(0,1)$ in $\mathbf{B}$ by

$$\pi(C) = P(W(\tau_0^{(1)}) \in C)(= P(\{\omega \in \Omega;\ W(\tau_0^{(1)}(\omega))(\omega) \in C\})\ )$$

for any Borel set C in $S(0,1)$. For a real Borel functional f defined on a sphere $S(x,r)$ in $\mathbf{B}$ define the *average* $L(f,x,r)$ of f on $S(x,r)$ by

$$L(f,x,r) = \int_{S(0,1)} f(x+ry)\,\pi(dy) \tag{2}$$

if the right-hand side exists.

Example 1. If $f \in \mathbf{B}^*$, then $L(f,x,r) = f(x)$ for each $x \in \mathbf{B}$ and $r > 0$. Indeed, it follows easily from the fact that $\pi(C) = \pi(-C)$ for any Borel set C of the unit sphere $S(0,1)$ in $\mathbf{B}$.

Example 2. If $T \in \mathcal{L}(\mathbf{B},\mathbf{B}^*)$ and $g(y) =< Ty, y >,\ y \in \mathbf{B}$, then

$$L(g,x,r) = g(x) + cr^2 \cdot tr(T|_{\mathbf{H}}),\ x \in \mathbf{B} \text{ and } r > 0,$$

see [Kuo1, the proof of Theorem 4.4].

Definition 1. *Let f be a real Borel functional defined on a ball in $\mathbf{B}$ with center at a point $x \in \mathbf{B}$. Then the Gross laplacian of f at x is defined by*

$$\Delta_G f(x) = 2c^{-1} \lim_{r \to 0+} \frac{L(f,x,r) - f(x)}{r^2} \tag{3}$$

whenever the limit exists.

Remark 1. In case $\mathbf{B} = \mathbf{R}^n$ it is well-known that the average $L(f,x,r)$ of a real function f defined on a sphere $S(x,r)$ in $\mathbf{R}^n$ is the same as the spherical mean of f on $S(x,r)$, see [Chun, p. 149]. Moreover, if f is a real function having the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{R}^n$, then $\Delta_G f(x) = \Delta_n f(x)$ where Δ_n is the Laplace operator in $\mathbf{R}^n$. The proof of this assertion is quite analogous to that of the following theorem.

Theorem 1. *If f is a real functional having the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{B}$, then $\Delta_G f(x) = tr(d^2 f(x))$.*

PROOF: Since f has the second Fréchet derivative $f''(x)$ at x it holds that

$$f(x+y) - f(x) =< f'(x), y > + \frac{1}{2} < f''(x)y, y > + o(\|y\|^2)$$

as $y \to 0$ in $\mathbf{B}$. Consequently,

$$L(f,x,r) - f(x) = \frac{r^2}{2} \int_{S(0,1)} < f''(x)y, y > \pi(dy) + g(r)$$

with $\lim_{r\to 0+} g(r)/r^2 = 0$, see Example 1. By Definition 1,

$$\Delta_G f(x) = c^{-1} \int_{S(0,1)} < f''(x)y, y > \pi(dy)$$

and, finally, $\Delta_G f(x) = tr(d^2 f(x))$, see Example 2. □

The closed convex hull of the set consisting of a point $x \in \mathbf{B}$ and a ball in $\mathbf{B}$ not containing x is called a *cone* in $\mathbf{B}$ *with vertex* x. An open set $V \subset \mathbf{B}$ is called *strongly regular* at a point $x \in \partial V$ if there is a cone K in $\mathbf{B}$ with vertex x such that $V \cap K = \emptyset$. Finally, an open set $V \subset \mathbf{B}$ is said to be *strongly regular* provided that V is strongly regular at each of its boundary points. Recall that the *characteristic function* χ_M of a set M in Ω is defined by $\chi_M = 1$ on M and $\chi_M = 0$ on $\Omega \setminus M$.

Theorem 2. *Let V be a strongly regular open set in $\mathbf{B}$ and f be a bounded continuous function on ∂V. If $M_x = \{\omega \in \Omega;\ \tau_x^V(\omega) < +\infty\},\ x \in V$, then the function*

$$u(x) = E[f(x + W(\tau_x^V))\chi_{M_x}],\ x \in V \tag{4}$$

is continuous on V, $\Delta_G u = 0$ on V and

$$\lim_{x\in V, x\to a} u(x) = f(a),\ a \in \partial V.$$

Moreover, for each ball $B(x,r)$ contained with its closure in V, $L(f,x,r) = f(x)$.

Proof: See [Gro2, Corollary 1.1 and Remark 3.4]. □

Remark 2. The function u defined in Theorem 2 is infinitely $\mathbf{H}$-differentiable on V but we cannot expect in general that $u \in \mathcal{C}^2(V)$ as in finite dimensional case, see [Goo2].

The *Green's measure* on $\mathbf{B}$ is defined by

$$G(A) = \int_0^\infty p_t(A)\,dt,\ A \in \mathcal{B}. \tag{5}$$

The *potential* of a Borel functional f defined on $\mathbf{B}$ at a point $x \in \mathbf{B}$ is defined by

$$Gf(x) = \int_{\mathbf{B}} f(x+y)\,G(dy) \tag{6}$$

whenever the integral exists. If f is a bounded Borel functional on $\mathbf{B}$ with bounded support, then its potential Gf is defined at each point of $\mathbf{B}$ and is bounded on $\mathbf{B}$, see [Gro2, Remark 3.5].

Theorem 3. *Let f be a bounded Borel functional on $\mathbf{B}$ such that $G|f|$ is bounded on $\mathbf{B}$. Put $u = Gf$ on $\mathbf{B}$. If f is continuous at a point $x \in \mathbf{B}$,* *then $\Delta_G u(x) = -2f(x)$. If f is even a bounded Lipschitz functional on* $\mathbf{B}$ *with bounded support, then $d^2u(x) \in \mathcal{L}_1(\mathbf{H})$ and $tr(d^2u(x)) = -2f(x)$ for each $x \in \mathbf{B}$.*

PROOF: See [Gro2, Remark 3.6 and Theorem 3]. $\square$

Remark 3. Let $\mathbf{B}$ be a real separable Hilbert space with the inner product $((\cdot,\cdot))$. Recall that a differential operator $\boldsymbol{D}$ in $\mathbf{B}$ is regular provided that there is an $A \in \mathcal{L}_1(\mathbf{B}) \cap \mathcal{L}_s(\mathbf{B})$ such that

$$\boldsymbol{D}f(x) = tr(Af''(x)) \tag{7}$$

if $x \in \mathbf{B}$ and f is a real functional having the second Fréchet derivative $f''(x)$ at x. The differential operator $\boldsymbol{D}$ in $\mathbf{B}$ is called a *Daletzky's laplacian* if A is, in addition, one-to-one and positively definite on $\mathbf{B}$, i.e., $((Ab, b)) \geq 0,\ b \in \mathbf{B}$. Moreover, given an abstract Wiener space $(\mathbf{H}, \mathbf{B})$, there is an one-to-one operator $T \in \mathcal{L}_2(\mathbf{B})$ such that

$$\mathbf{H} = T(\mathbf{B}) \text{ and } (Tx, Ty) = ((x, y)),\ x, y \in \mathbf{B}, \tag{8}$$

see Example 3 in the previous section. Obviously, $A = TT^*$ is an one-to-one positively definite operator of $\mathcal{L}_1(\mathbf{B}) \cap \mathcal{L}_s(\mathbf{B})$. Let f be a real functional having the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{B}$. Then it holds that $\Delta_G f(x) = tr(Af''(x))$. Indeed, $(d^2f(x)h, h) = ((f''(x)h, h)),\ h \in \mathbf{H}$ and if $E = \{e_n\}_{n=1}^{\infty}$ is an orthonormal basis in $\mathbf{B}$, then $\{T(e_n)\}_{n=1}^{\infty}$ is an orthonormal basis in $\mathbf{H}$ and hence

$$\Delta_G f(x) = \sum_{n=1}^{\infty}((f''(x)Te_n, Te_n)) = \sum_{n=1}^{\infty}((T^*f''(x)Te_n, e_n)).$$

Since $tr(BC) = tr(CB)$ for each $B, C \in \mathcal{L}_2(\mathbf{B})$ we have

$$\Delta_G f(x) = tr(T^*f''(x)T) = tr(Af''(x)).$$

Consequently, the Gross laplacian on the abstract Wiener space $(\mathbf{H}, \mathbf{B})$ is an extension of the Daletzky's laplacian in $\mathbf{B}$ defined as in (7). On the other hand, if $A \in \mathcal{L}_1(\mathbf{B}) \cap \mathcal{L}_s(\mathbf{B})$ is one-to-one and positively definite, then $T = \sqrt{A}$ is an one-to-one operator of $\mathcal{L}_2(\mathbf{B})$ with $A = T^2$, see Section 1.2, and hence the Daletzky's laplacian defined as in (7) can be extended to the Gross laplacian on the abstract Wiener space $(\mathbf{H}, \mathbf{B})$ defined as in (8).

Comments

It is well-known that the classical potential theory in finite dimensional spaces is closely connected with probability theory, in particular, with the theory of Markov processes, see e.g. [Doo1], [Doo2], [Dyn] and [Chun]. In [Gro2], the probabilistic approach to potential theory was generalized for infinite dimensional spaces just as we have sketched in this section. See also [Kuo1, Chapter III, §4] and [Car], [Goo1], [Goo2]. In [Cat, p.108], the Gross laplacian is studied in a more general setting, namely, on a generalized Wiener space and it is shown there that the Green's measure up to a multiple can be thought of as a fundamental solution for the Gross laplacian. At the same time as L. Gross developed his theory, Yu. L. Daletzky dealt with parabolic equations arising from the differential operator called here Daletzky's laplacian, see [Dal] and also [DF]. For the theory of Daletzky's laplacian, see also [ASF] and [Bel1], [Bel2]. In Remark 3, we have shown that the notions of Gross laplacian and Daletzky's laplacian coincide on Hilbert spaces. Let us note that Daletzky's laplacian is an important example of regular differential operators, see Chapter 4. Easily, if $\boldsymbol{D}$ is a Daletzky's laplacian in a real separable Hilbert space $\mathbf{B}$, then there are an orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$ in $\mathbf{B}$ and a sequence $\{a_n\}_{n=1}^{\infty}$ of strictly positive numbers such that $\sum a_n < \infty$ and

$$\boldsymbol{D}f(x) = \sum_{n=1}^{\infty} a_n D_{nn} f(x)$$

whenever a real functional f has the second Fréchet derivative $f''(x)$ at a point $x \in \mathbf{B}$.

Chapter 4

The Lévy laplacian and differential operators of 2-nd order in Hilbert spaces

4.1 Comments

The article presented above is the same as that published in *Comment. Math. Univ. Carolinae* 39,1(1998), 115–135 except that the mistakes listed below in Errata were corrected. A reprint of the published article is enclosed at the end of this thesis. It is worth noting that, in the article, the transcription of Russian names is different from that used in the other part of the text. In the following 'equalities', the transcription of a Russian name adopted in the article stands on the left and that used in the other part of the text is on the right:
J. L. Daleckij = Yu. L. Daletzky,
M. G. Krejn = M. G. Krein,
A. S. Nemirovskij = A. S. Nemirovsky,
O. G. Smoljanov = O. G. Smolyanov,
V. Ja. Sikirjavyj = V. Ya. Sikiryavy,
G. E. Šilov = G. E. Shilov.

In what follows $\mathbf{H}$ will be a real separable Hilbert space unless otherwise specified.

1. Regular and irregular operators

In [ASF], for a given $\varphi \in \mathcal{L}^* = (\mathcal{L}(\mathbf{H}, \mathbf{H}))^*$ the differential operator $\boldsymbol{D}$ (of 2-nd order corresponding to φ) is defined by

$$\boldsymbol{D}f(x) = \varphi(f''(x)) \tag{1}$$

if $x \in \mathbf{H}$ and a functional f has the second Fréchet derivative $f''(x)$ at x. Moreover, a functional $\varphi \in \mathcal{L}^*$ is said to be *regular* provided $\|\varphi\| = \|\varphi|_{\mathcal{L}_\infty}\|$, i.e.,

$$\sup\{|\varphi(f)|;\ f \in \mathcal{L}, \|f\| = 1\} = \sup\{|\varphi(f)|;\ f \in \mathcal{L}_\infty, \|f\| = 1\},$$

and in this case $\boldsymbol{D}$ is called a *regular operator.* It is shown there that $\varphi \in \mathcal{L}^*$ is regular if and only if there is $A \in \mathcal{L}_1$ such that

$$\varphi(B) = tr(AB),\ B \in \mathcal{L}. \tag{2}$$

Futhermore, it is proved that for each $\varphi \in \mathcal{L}^*$ there is a unique regular functional $\varphi_r \in \mathcal{L}^*$ such that $\varphi = \varphi_r$ on $\mathcal{L}_\infty$. The proof of uniqueness of such a φ_r is rather complicated, see [ASF, 7.31]. Finally, it is mentioned there that quite a few works, including [Lev1, Shi3] and articles of E. M. Polishchuk, deal with *irregular* operators without defining explicitly the notion of irregularity. These results can be interpreted so that each $\varphi \in \mathcal{L}^*$ can be uniquely written as the sum of a regular functional $\varphi_r \in \mathcal{L}^*$ (for which there is $A \in \mathcal{L}_1$ such that $\varphi_r(B) = tr(AB),\ B \in \mathcal{L}$) and a functional $\varphi_i \in \mathcal{L}^*$ vanishing on $\mathcal{L}_\infty$ and hence, for $\boldsymbol{D}$ defined as in (1),

$$\boldsymbol{D}f(x) = tr(Af''(x)) + \varphi_i(f''(x)) \tag{3}$$

if $x \in \mathbf{H}$ and f has the second Fréchet derivative $f''(x)$ at x. Put $A_s = (A + A^*)/2$ and $C = (A - A^*)/2$. Obviously, $A_s \in \mathcal{L}_1 \cap \mathcal{L}_s$. Since $C^* = -C$ we have that $tr(CD) = 0,\ D \in \mathcal{L}_s$. (By using this fact, the proof of Theorem 1 shortens.) Consequently, $tr(AD) = tr(A_s D),\ D \in \mathcal{L}_s$ and, as a result, our notion of regular operator is the same as that introduced in [ASF]. The uniqueness of the decomposition of φ into φ_r and φ_i does not, however, imply directly the uniqueness of the decomposition of $\boldsymbol{D}$ into a regular and an irregular part. (Unfortunately, I am wrong in the last sentence of Remark following Theorem 2.)

2. Characteristic properties of irregular operator

The fact that the Lévy laplacian satisfies (2) (and hence (1)) in Theorem 3 is due to P. Lévy, see [Lev1, p. 300, (2)]. In Theorem 3, it is shown that a differential operator $\boldsymbol{D}$ satisfies (1) or (2) if and only if $\boldsymbol{D}$ is irregular. This result suggests a distinctive feature of irregular operators: In spite of being differential operators of the second order their characteristic properties (1) and (2) are typical of a 'derivation', i.e., they behave like differential operators of the first order. Since the set $\mathcal{L}_f \cap \mathcal{L}_s$ is dense in $\mathcal{L}_\infty \cap \mathcal{L}_s$ it is easily seen that a differential operator $\boldsymbol{D}$ in $\mathbf{H}$ is irregular if and only if $\boldsymbol{D}f = 0$ on $\mathbf{H}$ for any twice Fréchet differentiable simple functional f on $\mathbf{H}$.

Theorem 4 due to the author gives another characterization of irregular operators, namely, $\psi \in \mathcal{L}_s^*$ is irregular if and only if $\boldsymbol{D}^\psi$ is independent of the choice of orthonormal bases in the sense of (iii).

3. Differential operators satisfying the maximum principle

In [MW2], it is shown that the Lévy laplacian satisfies the maximum principle on $\mathcal{C}_b^2(G)$ if G is a bounded open set in $\mathbf{H}$. A modification of their proof gives that it holds even for general differential operators (not necessarily irregular) satisfying the conditions (P1) and (P2), see Theorem 5. In this theorem, it is proved that the conditions (P1) and (P2) are also necessary for differential operators to satisfy the maximum principle. Extending differential operators satisfying the maximum principle is quite analogous to extending the Lévy laplacian in [MW2], cf. also [Shi1], [NS] and [Sik]. See also [Fel1]. It is easily seen that any irregular differential operator in $\mathbf{H}$, including the Lévy laplacian, fails to have the strong maximum principle. Indeed, consider a simple functional $f(x) = -(x,a)^2,\ x \in B(0,1)$ if $a \in \mathbf{H}$ and $a \neq 0$.

4. The Dirichlet and Poisson problems

This paragraph deals with the Dirichlet and Poisson problems for $\bar{\boldsymbol{D}}$, the extension of a differential operator $\boldsymbol{D}$ satisfying the maximum principle produced in the previous paragraph. In Theorem 7, the Dirichlet and Poisson problems are shown to be uniquely solvable on the class of fundamental functionals defined on a fundamental set G in $\mathbf{H}$ by generalizing results obtained for the Lévy laplacian in [MW2], see also [Shi1] and [Shi3]. Moreover, it is easily seen that even for a functional f defined as in (2) with F being only continuous, the unique solutions of (DP) and (PP) may be obtained in the same way as in the proof of Theorem 7. The notions of fundamental functional and fundamental set go back to [Shi1], [Shi3] and [Sik]. In Lemma, it is shown that $\bar{\boldsymbol{D}}$ satisfies a stronger condition than (2) in Theorem 3 and, as a result, we were able to generalize the notion of a fundamental functional in comparison with [MW2]. Simple functionals and sets called here s-fundamental were studied in [Shi1]. As to Theorem 8, the fact that weakly continuous functionals are harmonic with respect to the Lévy laplacian is known, see Comments in Section 2.2.

Errata

The following errata in the article (*Comment. Math. Univ. Carolinea* 39,1(1998), 115-135) are to be noted:
p. 116, line 1: read 'isometrical' for 'isometric'.
p. 116, line 6: insert 'the set of' after '$\mathcal{G}_x$'.
p. 117, line 8: transfer '$\{\mu_n\}$' after 'sequence'.
p. 117, line -4: insert 'real' before 'symmetric'.
p. 118, line -11: transfer '$\{\lambda_n\}$' to the beginning of the line.
p. 119, line 1: insert 'an' after 'or'.
p. 119, line 4: read 'operator' for 'operators'.
p. 119, line 7: read ', as given in [1]' for 'in [1]'.
p. 119, line 11: read 'there is' for 'there was'.
p. 121, line -11: replace the line by 'on the choice of'.
p. 123, line -15: read '$\boldsymbol{D}_a^{\psi}$' for '$\boldsymbol{D}^{\psi}$.
p. 123, line -12: read 'Ω_a is the set' for 'Ω_a is a set'.
p. 123, line -3: read '$\mathcal{C}_b(G) \cap \mathcal{C}^2(G)$' for '$\mathcal{C}_b(G)$'.
p. 124, line -6: read 'has' for 'have'.
p. 124, line -5: read 'the case' for 'this case'.
p. 134, line -7: read '$\boldsymbol{D}_a^{\psi}$' for '$\boldsymbol{D}$'.
p. 135, line 9: insert 'Non-self-adjoint' after 'Linear'.

Part II

Sequences in Banach spaces

Chapter 5

Limit points of arithmetic means of sequences in Banach spaces

5.1 Comments

The text presented above is an article prepared for publication. Propositions 1 and 2 are known and so are the assertions (a) and (b) from Remark 2, see Comments in Section 2.6 of Part I for details. On the other hand, Theorem is due to the author. In Fact 2, a new characterization of reflexive spaces in terms of limit points and cores of bounded sequences is given. In Fact 3, 'infinite' limit points of real sequences are dealt with in the way due to the author. Examples 1, 2, 3 and 4 are also new. An easy modification of Example 2 is

Example 5. The sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, defined by $a_{3n-2} = 0$, $a_{3n-1} = n$ and $a_{3n} = -n$ for all $n \in \mathbf{N}$, satisfies $\mathcal{K}(\{a_n\}_{n=1}^{\infty}) = \{0\}$ and $\mathcal{N}(\{a_n\}_{n=1}^{\infty}) = \mathbf{R}$. The proof is quite analogous as in Example 2.

It may be interesting to answer the following questions.

Questions:

1. Is it possible to find a bounded sequence $\{a_n\}_{n=1}^{\infty}$ in a Banach space $\mathbf{X}$ with the weak Banach-Saks property such that $\mathcal{K}(\{a_n\}_{n=1}^{\infty}) \neq \mathcal{N}(\{a_n\}_{n=1}^{\infty})$ or $\mathcal{N}(\{a_n\}_{n=1}^{\infty}) \neq \mathcal{C}(\{a_n\}_{n=1}^{\infty})$?
2. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence in a Banach space $\mathbf{X}$. Are the sets $\mathcal{K}(\{a_n\}_{n=1}^{\infty})$ and $\mathcal{N}(\{a_n\}_{n=1}^{\infty})$ closed and convex? (By Theorem, in case $\mathbf{X}$ is (BS) the answer is of course 'yes'.)

Following [Die2], we give a brief history of the theory of Banach spaces with the so-called Banach-Saks property. This theory goes back to the note [BaS] published in 1930 where S. Banach and S. Saks showed that, for $p > 1$, the spaces $L^p(0,1)$ and ℓ^p enjoy the property now named after them and, on the other hand, claimed to produce a weakly null sequence in $L^1(0,1)$ having no subsequences with norm convergent arithmetic means. However, in 1965, W. Szlenk showed that they were mistaken, namely, that $L^1(0,1)$ has the weak Banach-Saks property, see [Szl]. That uniformly convex spaces have the Banach-Saks property was first shown by S. Kakutani in 1938, see [Kak] (unfortunately, this article is not available to the author). In [NW], it was first proved that any Banach space with the Banach-Saks property is reflexive and, in [Bae], A. Baernstein II gave the first example of a reflexive Banach space without the Banach-Saks property. Futhermore, c_0 and ℓ^1 are (WBS), whereas $C[0,1]$ and ℓ^∞ are not (WBS). An account of the theory may be found in J. Diestel's books [Die1] and [Die2], see also [Par1].

That a Banach space $\mathbf{X}$ is (WBS) if and only if each weakly convergent sequence in $\mathbf{X}$ admits even a stable subsequence was first shown in [FS] by using an infinite version of the so-called Ramsey theorem, cf. also [EM]. Ramsey theorems play an important role in the Banach space theory, see e.g. [Die2], [Bea] and [Par2]. The notion of stable sequence in a Banach space $\mathbf{X}$ was introduced in [BrS]. It is easily seen that any norm convergent sequence in $\mathbf{X}$ is stable and, on the other hand, any orthonormal basis $E = \{e_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathbf{H}$ is stable but not norm convergent. The following assertion follows immediately from Fact 1 in Section 2.6 of Part I.

Fact 4. *Let* $D = \{d_{n,k}\}_{n,k=1}^{\infty}$ *be a real T-matrix and* $\{a_n\}_{n=1}^{\infty}$ *be a bounded sequence in a Banach space* $\mathbf{X}$. *Then the following statements are equivalent to each other:*

(i) *the sequence* $\{a_n\}_{n=1}^{\infty}$ *is weakly convergent;*

(ii) *for each* $\rho \in \mathcal{P}$, $w\text{-}\lim_{n\to\infty} \sum_{k=1}^{\infty} d_{n,k} a_{\rho(k)}$ *exists;*

(iii) *for each* $r \in \mathcal{I}$, $w\text{-}\lim_{n\to\infty} \sum_{k=1}^{\infty} d_{n,k} a_{r(k)}$ *exists.*

In case D *defines the Cesàro method of summability we can omit the hypothesis of boundedness of* $\{a_n\}_{n=1}^{\infty}$.

By Fact 4, it is easily seen that any stable sequence in $\mathbf{X}$ is weakly convergent and in case $\mathbf{X}$ is finite dimensional a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbf{X}$ is (norm) convergent if and only if $\{a_n\}_{n=1}^{\infty}$ is stable.

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