# 1 Languages and structures

**Language: alphabet, terms, formulas.**
The **alphabet of a language** $L$ consists of, by definition, the following symbols:

(i) relation symbols $P_i$, $(i \in I)$, function symbols $f_j$, $(j \in J)$, and constant symbols $c_k$, $(k \in K)$ with some index sets $I, J, K$. Further, to each $i \in I$ and $j \in J$ is assigned a positive integer $\rho_i$, $\mu_j$, respectively, called the **arity** of the relation symbol $P_i$ or the function symbol $f_j$.
The symbols in (i) are called **non-logical** symbols and their choice determines $L$. In addition any language has the following symbols:
(ii) $\simeq$ - the equality symbol;
(iii) $v_1, \ldots, v_n, \ldots$ - the variables;
(iv) $\wedge$, $\neg$ - the connectives;
(v) $\exists$ - the existential quantifier;
(vi) (, ), , - parentheses and comma.

Words of the alphabet of $L$ constructed in a specific way are called $L$-terms and $L$-formulas:

$L$-**terms** are given by recursive definition as follows:
(i) $v_i$ is an $L$-term (any $i \geq 1$);
(ii) $c$ is an $L$-term (any constant symbol $c$ of $L$);
(iii) if $f$ is a function symbol of $L$ of arity $\mu$, and $\tau_1, \ldots \tau_\mu$ are $L$-terms, then $f(\tau_1, \ldots, \tau_\mu)$ is an $L$-term;
(iv) nothing else is an $L$-term.

We define the **complexity** of term $\tau$ to be just the length of $\tau$ as a word in the alphabet of $L$. It is obvious from the definition that any term of complexity $l > 1$ is obtained by an application of (iii) to terms of lower complexities.

We sometimes refer to a term $\tau$ as $\tau(v_{i_1}, \ldots, v_{i_n})$ to remark the fact that the variables occuring in $\tau$ are among $v_{i_1}, \ldots, v_{i_n}$. It may happen that there no variables occur in $\tau$, such terms are called **closed.**

**Atomic** $L$-**formulas** are the words of the form

(i) $\tau_1 \doteq \tau_2$ for any $L$-terms $\tau_1$ and $\tau_2$
or
(ii) $P(\tau_1, \ldots, \tau_\rho)$ for any relational $L$-symbol $P$ of arity $\rho$ and $L$-terms $\tau_1, \ldots, \tau_\rho$.

Notice, that (i) can be seen as a special case of (ii) if we view $\doteq$ as a relational symbol of arity 2.

We sometimes refer to an atomic formula $\varphi$ as $\varphi(v_{i_1}, \ldots, v_{i_n})$ to mark the fact that all the variables occuring in $\tau_1, \ldots, \tau_\rho$ are among $v_{i_1}, \ldots, v_{i_n}$.

An **$L$-formula** is defined by the following recursive definition:
(i) any atomic $L$-formula is an $L$-formula;
(ii) if $\varphi$ is an $L$-formula, so is $\neg\varphi$;
(iii) if $\varphi$, $\psi$ are $L$-formulas, so is $(\varphi \wedge \psi)$;
(iv) if $\varphi$ is an $L$-formula, so is $\exists v \varphi$ for any variable $v$;
(v) nothing else is an $L$-formula.

We define the **complexity of an $L$-formula** $\varphi$ to be just the number of occurences of $\wedge$, $\neg$ and $\exists$ in $\varphi$. It is obvious from the definition that an atomic formula is of complexity 0 and that any formula of complexity $l > 0$ is obtained by an application of (ii),(iii) or (iv) to formulas of lower complexities.

For an atomic formula $\varphi(v_{i_1}, \ldots, v_{i_n})$ the distinguished variables are said to be **free in** $\varphi$. The varibles which are free in $\varphi$ and $\psi$ in (ii) and (iii) are, by definition, also free in $\neg\varphi$ and $(\varphi \wedge \psi)$. The variable $v$ in (iv) is called **bounded** in $\exists v \varphi$ and the list of free variables for this formula is given by the free variables of $\varphi$ except $v$.
An $L$-formula with no free variables is called also an $L$-sentence.

We define a language $L$ to be the set of all $L$-formulas. Thus $|L|$ is the cardinality of the set.

**Exercise 1.1** *Show that*

$$|L| = \max\{\aleph_0, \text{card } (I), \text{card } (J), \text{card } (K)\}.$$

To give a meaning or interpretation of symbols of a language $L$ we introduce a notion of an $L$-**structure.** An $L$-structure $\mathfrak{A}$ consists of

(i) a non-empty set $A$, called a **domain** of the $L$-structure;
(ii) an assignment of an $r$-ary relation (subset) $P^{\mathfrak{A}} \subseteq A^r$ to any relation symbol $P$ of $L$ of arity $r$;
(iii) an assignment of an $m$-ary function $f^{\mathfrak{A}} : A^m \to A$ to any function symbol $f$ of $L$ of arity $m$;
(iv) an assignment of an element $c^{\mathfrak{A}} \in A$ to any constant symbol $c$ of $L$.

Thus an $L$-structure is an object of the form

$$\mathfrak{A} = \langle A; \{P_i^{\mathfrak{A}}\}_{i\in I}; \{f_j^{\mathfrak{A}}\}_{j\in J}; \{c_k^{\mathfrak{A}}\}_{k\in K}\rangle.$$

$\{P_i^{\mathfrak{A}}\}_{i\in I}$, $\{f_j^{\mathfrak{A}}\}_{j\in J}$ and $\{c_k^{\mathfrak{A}}\}_{k\in K}$ are called **the interpretations** of the predicate, function and constant symbols correspondingly.
We write $A = \text{dom }(\mathfrak{A})$.

**Example** Groups can be considered $L$-structures where $L$ is having one constant symbol $e$, one binary and one unary operation symbols $\cdot$ and $^{-1}$ and no relation symbols.

If $\mathfrak{A}$ and $\mathfrak{B}$ are both $L$-structures we say that $\mathfrak{A}$ is **isomorphic** to $\mathfrak{B}$, written $\mathfrak{A} \cong \mathfrak{B}$, if there is a bijection $\pi : \text{dom }(\mathfrak{A}) \to \text{dom }(\mathfrak{B})$ which **preserves** corresponding relation, function and constant symbols, i.e. for any $i \in I$, $j \in J$ and $k \in K$ :
(i) $\bar{a} \in P_i^{\mathfrak{A}}$ iff $\pi(\bar{a}) \in P_i^{\mathfrak{B}}$;
(ii) $\pi(f_j^{\mathfrak{A}}(\bar{a})) = f_j^{\mathfrak{B}}(\pi(\bar{a}))$;
(iii) $\pi(c_k^{\mathfrak{A}}) = c_k^{\mathfrak{B}}$.
The map $\pi$ is then called an **isomorphism**. If $\pi$ is only assumed being injective but still satisfies (i)-(iii), then it is called an **embedding** and can be wriiten as
$\pi : \mathfrak{A} \to \mathfrak{B}$ or $\mathfrak{A} \subseteq_\pi \mathfrak{B}$.

**Interpretation of terms in a structure**.
Given an $L$-structure $\mathfrak{A}$, we assign to each $L$-term $\tau(v_1, \ldots, v_n)$ a function

$$\tau^{\mathfrak{A}} : A^n \to A$$

by the following natural rule:
(i) if $\tau(v_1, \ldots, v_n)$ is just a variable $v_j$ then $\tau^{\mathfrak{A}}$ is the corresponding coordinate function $\langle a_1, \ldots a_n \rangle \mapsto a_j$;
(ii) if $\tau(v_1, \ldots, v_n)$ is a constant symbol $c$ then $\tau^{\mathfrak{A}}(a_1, \ldots, a_n) = c^{\mathfrak{A}}$;
(iii) if $\tau(v_1, \ldots, v_n)$ is $f(\tau_1(v_1, \ldots, v_n), \ldots, \tau(v_1, \ldots, v_n))$ then
$\tau^{\mathfrak{A}}(a_1, \ldots, a_n) = f^{\mathfrak{A}}(\tau_1^{\mathfrak{A}}(a_1, \ldots, a_n), \ldots, \tau_m^{\mathfrak{A}}(a_1, \ldots, a_n))$.

**Exercise 1.2** *Prove that if $\pi : \mathfrak{A} \to \mathfrak{B}$ is an embedding then $\pi$ preserves L-terms, i.e. for any term $\tau(\bar{v})$*

$$\pi(\tau^{\mathfrak{A}}(\bar{a})) = \tau^{\mathfrak{B}}(\pi(\bar{a})).$$

**Assigning truth values to $L$-formulas in an $L$-structure.**
Suppose $\mathfrak{A}$ is an $L$-structure with a domain $A$, $\varphi(v_1, \ldots, v_n)$ an $L$-formula with free variables $v_1, \ldots, v_n$ and $\bar{a} = \langle a_1, \ldots, a_n \rangle \in A^n$. Given these data we assign a truth value **true** (written $\mathfrak{A} \vDash \varphi(\bar{a})$ or **false** ($\mathfrak{A} \nvDash \varphi(\bar{a})$) by the following rules:
(i) $\mathfrak{A} \vDash \tau_1(\bar{a}) \doteq \tau_2(\bar{a})$ iff $\tau_1^{\mathfrak{A}}(\bar{a}) = \tau_2^{\mathfrak{A}}(\bar{a})$;
(ii) $\mathfrak{A} \vDash P(\tau_1(\bar{a}), \ldots, \tau_r(\bar{a}))$ iff $\langle \tau_1^{\mathfrak{A}}(\bar{a}), \ldots, \tau_r^{\mathfrak{A}}(\bar{a}) \rangle \in P_i^{\mathfrak{A}}$;
(iii) $\mathfrak{A} \vDash \varphi_1(\bar{a}) \wedge \varphi_2(\bar{a})$ iff $\mathfrak{A} \vDash \varphi_1(\bar{a})$ and $\mathfrak{A} \vDash \varphi_2(\bar{a})$;
(iv) $\mathfrak{A} \vDash \neg\varphi(\bar{a})$ iff $\mathfrak{A} \nvDash \varphi(\bar{a})$;
(v) $\mathfrak{A} \vDash \exists v_n \varphi(a_1, \ldots, a_{n-1}, v_n)$ iff there is $a_n \in A$ such that $\mathfrak{A} \vDash \varphi(a_1, \ldots, a_n)$.

Given an $L$-structure $\mathfrak{A}$ and an $L$-formula $\varphi(v_1, \ldots, v_n)$ we can define the set

$$\varphi(\mathfrak{A}) = \{\bar{a} \in A^n : \mathfrak{A} \vDash \varphi(\bar{a})\}.$$

Sets of this form are called **definable**.

Since any subset of $A^n$ can be viewed as an $n$-ary relation, $\varphi(\bar{v})$ determines also an **$L$-definable relation.** If $\varphi(\mathfrak{A})$ coincides with a graph of a function $f : A^{n-1} \to A$, we say then that $f$ is an **$L$-definable function.**

**Proposition 1** *An isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ between L-structures preserves any L-formula $\varphi(v_1, \ldots, v_n)$ ($n \geq 0$), i.e. for any $\bar{a} \in A^n$*

$$(*) \qquad \mathfrak{A} \vDash \varphi(\bar{a}) \ \textit{ iff } \ \mathfrak{B} \vDash \varphi(\pi(\bar{a})).$$

**Proof** By the exercise 1.2 $\pi$ preserves terms. We use this to prove the proposition by induction on the complexity of $\varphi$.
Let first $\varphi$ be an atomic formula of the form $P(\tau_1(\bar{v}), \ldots, \tau_r(\bar{v})$, $P$ a relation symbol. Denote

$$\alpha_i = \tau_i(\bar{a}), \quad \beta_i = \tau_i(\pi(\bar{a})) \quad i = 1, \ldots, r.$$

Since terms are preserved, $\pi(\alpha_i) = \beta_i$. By the definition of an isomorphism

$$\langle \alpha_1, \ldots, \alpha_r \rangle \in P^{\mathfrak{A}} \text{ iff } \langle \beta_1, \ldots, \beta_r \rangle \in P^{\mathfrak{B}}.$$

This means that (*) holds for our formula.
Since $\doteq$ can be treated as a binary relation symbol, the above proves (*) for all atomic formulas.
Suppose now that the complexity of $\varphi$ $l > 0$, and for all formulas of complexity less than $l$ (*) holds. Then $\varphi$ is obtained by applying either (ii), (iii) or (iv) of the definition of formula to formulas of lower complexities. Consider e.g. case (iv). Then $\varphi(\bar{v}) = \exists v_{n+1} \psi(\bar{v}, v_{n+1})$. So
$\mathfrak{A} \vDash \varphi(\bar{a})$ iff $\mathfrak{A} \vDash \exists v_{n+1} \psi(\bar{a}, v_{n+1})$ iff there is $a_{n+1}$ $\mathfrak{A} \vDash \psi(\bar{a}, a_{n+1})$
the latter by the induction hypothesis is equivalent to $\mathfrak{B} \vDash \psi(\bar{b}, b_{n+1})$ where $\bar{b} = \pi(\bar{a})$ and $b_{n+1} = \pi(a_{n+1})$. Continuing in the reverse order we come to the equivalent statement $\mathfrak{B} \vDash \varphi(\bar{b})$, which proves (*) in this case. Similarly (*) holds in cases (ii) and (iii) and this completes the proof.□

**Corollary 1** *For definable subsets (relations)*

$$\pi(\varphi(\mathfrak{A})) = \varphi(\mathfrak{B}),$$

*in particular, when $\pi : \mathfrak{A} \to \mathfrak{A}$ is an automorphism (an isomorphism onto itself),*

$$\pi(\varphi(\mathfrak{A})) = \varphi(\mathfrak{A}).$$

The latter is very useful in checking non-definability of some subsets or relations.

**Exercise 1.3** *The multiplication is not definable in $\langle \mathbb{R}, + \rangle$.*

**Agreement about notations**. The proposition above about the properties of isomorphic structures says that there is no harm in identifying elements of $\mathfrak{A}$ with its images under an isomorphism. Correspondingly, when speaking about embedding $\pi : \mathfrak{A} \to \mathfrak{B}$ we identify $A = \text{dom } \mathfrak{A}$ with its image $\pi(A) \subseteq B = \text{dom } \mathfrak{B}$ elementwise. And so, by default, $\mathfrak{A} \subseteq \mathfrak{B}$ assumes $A \subseteq B$.

Given two $L$-structures $\mathfrak{A}$ and $\mathfrak{B}$ we say that $\mathfrak{A}$ is **elementary equivalent to** $\mathfrak{B}$, written $\mathfrak{A} \equiv \mathfrak{B}$, if for any $L$-sentence $\varphi$

$$\mathfrak{A} \vDash \varphi \text{ iff } \mathfrak{B} \vDash \varphi.$$

Two typical model-theoretic problems:
I. Given $\mathfrak{A}$, what are the definable subsets of $A$, or $A^n$?
II. Given $\mathfrak{A}$, what are the $\mathfrak{B}$ such that

$$\mathfrak{A} \equiv \mathfrak{B}?$$

**Some abbreviations**
Let $\phi$ and $\psi$ be $L$-formulas.
$(\phi \vee \psi)$ is an abbreviation for the formula $\neg(\neg\phi \wedge \neg\psi)$;
$(\phi \to \psi)$ is an abbreviation for the formula $\neg(\phi \wedge \neg\psi)$;
$(\phi \leftrightarrow \psi)$ is an abbreviation for the formula $((\phi \to \psi) \wedge (\psi \to \phi))$;
$\forall v\psi$ is an abbreviation for the formula $\neg\exists v\neg\psi$.

## 2 Compactness Theorem

Given an $L$-structure $\mathfrak{A}$ we define **the theory of** $\mathfrak{A}$

$$\mathrm{Th}(\mathfrak{A}) = \{\sigma : \ \sigma \text{ an } L\text{-sentence and } \mathfrak{A} \models \sigma\}$$

If $\Sigma$ is a set of $L$-sentences we also write $\mathfrak{A} \models \Sigma$ if $\Sigma \subseteq \mathrm{Th}(\mathfrak{A})$.

An $L$-sentence $\sigma$ is said to be **a logical consequence of** $\Sigma$, written $\Sigma \models \sigma$, if $\mathfrak{A} \models \Sigma$ implies $\mathfrak{A} \models \sigma$ for every $L$-structure $\mathfrak{A}$.
$\sigma$ is called **logically valid**, written $\models \sigma$, if $\mathfrak{A} \models \sigma$ for every Ł-structure $\mathfrak{A}$.

A set $\Sigma$ of $L$-sentences is said to be **satisfiable** if there is an $L$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma$. $\mathfrak{A}$ is then called **a model** of $\Sigma$.
$\Sigma$ is said to be **finitely satisfiable (f.s.)** if any finite subset of $\Sigma$ is satisfiable.

$\Sigma$ is said to be **complete** if for any $L$-sentence $\sigma$ $\sigma \in \Sigma$ or $\neg\sigma \in \Sigma$.

**Theorem 1 (Lindenbaum Theorem)** *For any f.s. set of $L$-sentences $\Sigma$ there is a complete f.s. set $\Sigma^\# \supseteq \Sigma$ of $L$-sentences.*

**Proof** Let

$$\mathcal{S} = \{\Sigma' : \Sigma \subseteq \Sigma' \text{ a f.s. set of } L\text{-sentences } \}.$$

Clearly $\mathcal{S}$ satisfies the hypothesis of Zorn's Lemma, so it contains a maximal element $\Sigma^\#$ say. This is complete for otherwise, say $\sigma \notin \Sigma^\#$ and $\neg\sigma \notin \Sigma^\#$. By maximality neither $\{\sigma\} \cup \Sigma^\#$ nor $\{\neg\sigma\} \cup \Sigma^\#$ is f.s.. Hence there exist finite $S_1 \subseteq \Sigma^\#$ and $S_2 \subseteq \Sigma^\#$ such that neither $\{\sigma\} \cup S_1$ nor $\{\neg\sigma\} \cup S_2$ is satisfiable. However, $S_1 \cup S_2 \subseteq \Sigma^\#$, finite, so has a model, $\mathfrak{A}$ say. But either $\mathfrak{A} \models \sigma$, so $\mathfrak{A} \models \{\sigma\} \cup S_1$, or $\mathfrak{A} \models \neg\sigma$, so $\mathfrak{A} \models \{\neg\sigma\} \cup S_2$, contradiction.$\square$

**Exercise 2.1** *(i) If $\Sigma' \subseteq \Sigma$ and $\Sigma' \models \sigma$ then $\Sigma \models \sigma$;*
*(ii) For complete f.s. $\Sigma$*

$$\Sigma \models \sigma \textit{ iff } \sigma \in \Sigma.$$

A set $\Sigma$ of $L$-sentences is said to be **full** if for any sentence in $\Sigma$ of the form $\exists v\varphi(v)$ there is a closed $L$-term $\lambda$ such that $\varphi(\lambda) \in \Sigma$.

An $L$-structure $\mathfrak{A}$ is called **canonical** if for every $a \in A$ there is a closed $L$-term $\lambda$ such that $\lambda^{\mathfrak{A}} = a$.

**Theorem 2** *For any complete full f.s. set $\Sigma$ of $L$-sentences there is a canonical model $\mathfrak{A}$ of $\Sigma$.*

**Proof** Let $\Lambda$ be the set of closed terms of $L$. For $\alpha, \beta \in \Lambda$ define $\alpha \backsim \beta$ iff $\alpha \doteq \beta \in \Sigma$. This is an equibalence relation. Indeed, $\alpha \doteq \beta, \beta \doteq \gamma \vDash \alpha \doteq \gamma$ by definitions. Then by the exercise above $\alpha \backsim \beta, \beta \backsim \gamma$ implies $\alpha \backsim \gamma$. This proves the transitivity of $\backsim$, and similarly one gets the reflexivity and symmetry.
For $\alpha \in \Lambda$ let $\tilde{\alpha}$ denote the $\backsim$-equivalence class containing $\alpha$. Let

$$A = \{\tilde{\alpha} : \ \alpha \in \Lambda\}.$$

This will be the domain of our model $\mathfrak{A}$. We want to define relations, functions and constants of $L$ on $A$.
Let $P$ be an $n$-ary relation symbol of $L$ and $\alpha_1, \ldots, \alpha_n \in \Lambda$. Define

$$\langle \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n \rangle \in P^{\mathfrak{A}} \text{ iff } P(\alpha_1, \ldots, \alpha_n) \in \Sigma.$$

If $\beta_1, \ldots, \beta_n \in \Lambda$ and $\alpha_1 \backsim \beta_1, \ldots, \alpha_n \backsim \beta_n$ then $P(\beta_1, \ldots, \beta_n) \in \Sigma$ too, i.e. the definition does not depend on the choice of representatives in the $\backsim$-classes. Indeed, by definitions
$\alpha_1 \doteq \beta_1, \ldots, \alpha_n \doteq \beta_n, P(\alpha_1, \ldots, \alpha_n) \vDash P(\beta_1, \ldots, \beta_n)$ which implies the required.
For an unary function symbol $f$ of $L$ of arity $m$ and $\alpha_1, \ldots, \alpha_m \in \Lambda$ define

$$f^{\mathfrak{A}}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) = f(\widetilde{\alpha_1, \ldots}, \alpha_m).$$

By the same reason as above the definition does not depend on the choice of representatives in $\backsim$-classes.
Finally for a constant symbol $c^{\mathfrak{A}}$ is just $\tilde{c}$.
We now prove by induction on complexity of an $L$-formula $\varphi(v_1, \ldots, v_n)$ that

$$(*) \quad \mathfrak{A} \vDash \varphi(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \text{ iff } \varphi(\alpha_1, \ldots, \alpha_n) \in \Sigma.$$

For atomic formulas we have this by definition.
If $\varphi = (\varphi_1 \wedge \varphi_2)$ then
$\mathfrak{A} \models (\varphi_1(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \wedge \varphi_2(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n))$ iff $\mathfrak{A} \models \varphi_1(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ and $\mathfrak{A} \models \varphi_2(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ iff (by induction hypothesis) $\varphi_1(\alpha_1, \ldots, \alpha_n), \varphi_2(\alpha_1, \ldots, \alpha_n) \in \Sigma$ iff (by Exercise 2.1) $(\varphi_1(\alpha_1, \ldots, \alpha_n) \wedge \varphi_2(\alpha_1, \ldots, \alpha_n)) \in \Sigma$. Which proves (*) in this case.
The case $\varphi = \neg\psi$ is proved similarly.
In case $\varphi = \exists v \psi$
$\mathfrak{A} \models \exists v \psi(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n, v)$ iff there is $\beta \in \Lambda$ such that $\mathfrak{A} \models \psi(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n, \tilde{\beta})$ iff there is $\beta \in \Lambda$ such that $\psi(\alpha_1, \ldots, \alpha_n, \beta) \in \Sigma$. The latter implies by Exercise 2.1 that $\exists v \psi(\alpha_1, \ldots, \alpha_n, v) \in \Sigma$ and the converse holds because $\Sigma$ is full. This proves (*) for the formula and finishes the proof of (*) for all formulas. Finally notice that (*) implies that $\mathfrak{A} \models \Sigma$.$\square$

We will sometimes extend or reduce our language.
Let $L$ be a language with non-logical symbols $\{P_i\}_{i \in I} \cup \{f_j\}_{j \in J} \cup \{c_k\}_{k \in K}$ and $L' \subseteq L$ with non-logical symbols $\{P_i\}_{i \in I'} \cup \{f_j\}_{j \in J'} \cup \{c_k\}_{k \in K'}$ ($I' \subseteq I$, $J' \subseteq J$, $K' \subseteq K$). Let

$$\mathfrak{A} = \langle A; \{P_i^{\mathfrak{A}}\}_{i \in I}; \{f_j^{\mathfrak{A}}\}_{j \in J}; \{c_k^{\mathfrak{A}}\}_{k \in K} \rangle$$

and

$$\mathfrak{A}' = \langle A; \{P_i^{\mathfrak{A}}\}_{i \in I'}; \{f_j^{\mathfrak{A}}\}_{j \in J'}; \{c_k^{\mathfrak{A}}\}_{k \in K'} \rangle.$$

Under this conditions we call $\mathfrak{A}'$ **the $L'$-reduct of** $\mathfrak{A}$ and, correspondingly, $\mathfrak{A}$ is **an $L$-expansion of** $\mathfrak{A}'$.

**Remark** Obviously, under the notations above for an $L'$-formula $\varphi(v_1, \ldots, v_n)$ and $a_1, \ldots, a_n \in \operatorname{dom} A$

$$\mathfrak{A}' \models \varphi(a_1, \ldots, a_n) \text{ iff } \mathfrak{A} \models \varphi(a_1, \ldots, a_n).$$

**Theorem 3** *Any finitely satisfiable set $\Sigma$ has a model, i.e. is satisfiable.*

**Proof** We introduce new languages $L_i$ and complete set of $L_i$-sentences $\Sigma_i$ $(i = 0, 1, \ldots)$. Let $L_0 = L$. By the Lindenbaum Theorem there is $\Sigma_0 \supseteq \Sigma$ a complete set of $L_0$-sentences.

Given f.s. $\Sigma_i$ in language $L_i$ introduce new language

$$L_{i+1} = L_i \cup \{c_\phi : \ \phi \ \text{ a one variable } L_i\text{-formula}\}$$

and a new set of $L_{i+1}$ sentences

$$\Sigma_i^* = \Sigma_i \cup \{(\exists v \phi(v) \to \phi(c_\phi)) \ : \ \phi \text{ a one variable } L_i\text{-formula}\}.$$

Claim 1. $\Sigma_i^*$ is f.s. Indeed, for any finite $S \subseteq \Sigma_i^*$ let $S_1 = S \cap \Sigma_i$ and take a model $\mathfrak{A}$ of $S_1$ with domain $A$, which we assume well-ordered. Assign constants to symbols $c_\phi$ as follows:

$$c_\phi = \begin{cases} \text{the first element in } \phi(\mathfrak{A}) & \text{if } \phi(\mathfrak{A}) \neq \emptyset \\ \text{the first element in } A & \text{if } \phi(\mathfrak{A}) = \emptyset \end{cases}.$$

Denote the expanded structure $\mathfrak{A}^*$. By the definition for all $\phi(v)$ $\mathfrak{A}^* \models \exists v \phi(v) \to \phi(c_\phi)$. So $\mathfrak{A}^* \models S$. Which proves the claim.

Let $\Sigma_{i+1}$ be a complete f.s. set of $L_{i+1}$-sentences that contains $\Sigma_i^*$.
Take $\Sigma^* = \bigcup_{i \in \mathbb{N}} \Sigma_i$. This is a f.s., full and complete set of sentences in the language $\bigcup L_i = L+$ new constants. It has a canonical model $\mathfrak{A}^*$. The reduct of $\mathfrak{A}^*$ to language $L$ is a model of $\Sigma$. $\square$

**Remark** The cardinality of the model we constructed is less or equal to $|L|$ (see also Exercise 1.1). Thus any f.s. set of $L$-sentences has a model of cardinality not greater than $|L|$.

# 3 Applications of the compactness theorem

**Elementary embeddings**
An embedding of $L$ structures $\pi : \mathfrak{A} \to \mathfrak{B}$ is called **elementary** if $\pi$ preserves any $L$-formula $\varphi(v_1, \ldots, v_n)$, i.e. for any $a_1, \ldots, a_n \in \mathrm{dom}\, \mathfrak{A}$

$$\mathfrak{A} \vDash \varphi(a_1, \ldots, a_n) \text{ iff } \mathfrak{B} \vDash \varphi(\pi(a_1), \ldots, \pi(a_n)).$$

We denote the fact of elementary embedding

$$\mathfrak{A} \preccurlyeq \mathfrak{B}.$$

We usually identify $A = \mathrm{dom}\, \mathfrak{A}$ with the subset $\pi(A)$ of $B = \mathrm{dom}\, \mathfrak{B}$. Then $\pi(a) = a$ for all $a \in A$ and so for elementary embeddings

$$\mathfrak{A} \vDash \varphi(a_1, \ldots, a_n) \text{ iff } \mathfrak{B} \vDash \varphi(a_1, \ldots, a_n).$$

For an $L$-structure $\mathfrak{A}$ let $L_A = L \cup \{c_a : \ a \in A\}$ be the expansion of the language, $\mathfrak{A}^+$ the natural expansion of $\mathfrak{A}$ to $L_A$ assigning to $c_a$ the element $a$, and

$$\mathrm{Diag}(\mathfrak{A}) = \{\sigma : \ \sigma \ \text{ an atomic } L_A\text{-sentence or negation of an atomic } L_A\text{-sentence and } \mathfrak{A}^+ \vDash \sigma\}.$$

$$\mathrm{CDiag}(\mathfrak{A}) = \{\sigma : \ \sigma \ L_A\text{-sentence and } \quad \mathfrak{A}^+ \models \sigma\}.$$

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**Theorem 4 (Method of Diagrams)** *For an $L$ structure $\mathfrak{B}$ there is an expansion $\mathfrak{B}^+$ to language $L_{\mathfrak{A}}$ such that*
*(i)* $\mathfrak{B}^+ \models \mathrm{Diag}(\mathfrak{A})$ *iff* $\mathfrak{A} \subseteq \mathfrak{B}$.
*(i)* $\mathfrak{B}^+ \models \mathrm{CDiag}(\mathfrak{A})$ *iff* $\mathfrak{A} \preccurlyeq \mathfrak{B}$.

**Proof** Indeed, by definitions $a \to c^{\mathfrak{B}^+}$ is an embedding iff $\mathfrak{B}^+ \models \mathrm{Diag}(\mathfrak{A})$. Correspondingly with the elementary embedding.□

**Corollary 2** *(i) For a theory $T$ the set $T \cup \mathrm{Diag}(\mathfrak{A})$ is f.s. iff there is a model $\mathfrak{B}$ of $T$ such that $\mathfrak{A} \subseteq \mathfrak{B}$.*
*(ii) For a theory $T$ the set $T \cup \mathrm{CDiag}(\mathfrak{A})$ is f.s. iff there is a model $\mathfrak{B}$ of $T$ such that $\mathfrak{A} \preccurlyeq \mathfrak{B}$.*

**Theorem 5 (Upward Lowenheim-Skolem Theorem)** *For any infinite $L$-structure $\mathfrak{A}$ and a cardinal $\kappa \geq \max\{|L|, ||\mathfrak{A}||\}$ there is an $L$-structure $\mathfrak{B}$ of cardinality $\kappa$ such that $\mathfrak{A} \preccurlyeq \mathfrak{B}$.*

**Proof** Let $M$ be a set of cardinality $\kappa$. Consider an extension $L_{A,M}$ of language $L$ obtained by adding to $L_A$ constant symbols $c_i$ for each $i \in M$. Consider now the set of $L_{A,M}$-sentences

$$\Sigma = \mathrm{CDiag}(\mathfrak{A}) \cup \{\neg c_i \doteq c_j : \ i \neq j \in M\}.$$

We claim that $\Sigma$ is f.s. Indeed, consider a finite subset $S \subseteq \Sigma$. We can represent

$$S \subseteq S_0 \cup \{\neg c_i \doteq c_j : \ i \neq j \in M_0\}$$

with $S_0 \subseteq \mathrm{CDiag}(\mathfrak{A})$ and $M_0$ a finite subset of $M$. By definition $\mathfrak{A}^+ \vDash S_0$. Now, since $A$ is infinite, we can expand $\mathfrak{A}^+$ to the model of $S$ by assigning to $c_i$ $(i \in M_0)$ distinct elements of $A$. Which proves the claim.
It follows from the claim and the compactness theorem with the consequent remark that $\Sigma$ has a model of cardinality $\kappa$, say $\mathfrak{B}^*$. The $L$-reduct $\mathfrak{B}$ of $\mathfrak{B}^*$, by the method of diagrams, satisfies the requrement of the theorem. □

L7

**Lemma 3.1 (Tarski Lemma)** *Suppose $\mathfrak{A} \subseteq \mathfrak{B}$ are $L$-structures with domains $A \subseteq B$. Then $\mathfrak{A} \preccurlyeq \mathfrak{B}$ iff the following condition holds:*
*for all $L$-formulas $\varphi(v_1, \ldots, v_n)$ and all $a_1, \ldots, a_{n-1} \in A$, $b \in B$ such that $\mathfrak{B} \vDash \varphi(a_1, \ldots, a_{n-1}, b)$ there is $a \in A$ with $\mathfrak{B} \vDash \varphi(a_1, \ldots, a_{n-1}, a)$*

**Proof** Obviously, given $\bar{a} = \langle a_1, \ldots, a_{n-1} \rangle$ the existence of $b \in B$ as above is equivalent to $\mathfrak{B} \vDash \exists v \varphi(\bar{a}, v)$.

Suppose $\mathfrak{A} \preccurlyeq \mathfrak{B}$. Then $\mathfrak{B} \vDash \exists v\varphi(\bar{a}, v)$ is equivalent to $\mathfrak{A} \vDash \exists v\varphi(\bar{a}, v)$ which is equivalent to existence of an $a \in A$ with $\mathfrak{A} \vDash \varphi(\bar{a}, a)$. The latter by $\mathfrak{A} \preccurlyeq \mathfrak{B}$ implies $\mathfrak{B} \vDash \varphi(\bar{a}, a)$.
For the converse, we assume

$$(*) \quad \mathfrak{B} \vDash \exists v\varphi(\bar{a}, v) \text{ implies for some } a \in A \quad \mathfrak{B} \vDash \varphi(\bar{a}, a)$$

and want to prove that

$$(**) \quad \mathfrak{A} \vDash \psi(\bar{a}) \text{ iff } \mathfrak{B} \vDash \psi(\bar{a})$$

for all $L$-formulas $\psi(v_1, \ldots, v_{n-1})$.
We use induction on the complexity of $\psi$. For $\psi$ atomic (**) is given by the definition of the embedding $\mathfrak{A} \subseteq \mathfrak{B}$. The cases of $\psi = \psi_1 \wedge \psi_2$ and $\psi = \neg\psi_1$ are easy. In the case $\psi = \exists v\varphi$ the $\Rightarrow$ side of (**) follows immediately from the induction hypothesis and the meaning of $\exists$, and
$\Leftarrow$:
$\mathfrak{B} \vDash \exists v\varphi(\bar{a}, v)$ implies $\mathfrak{B} \vDash \varphi(\bar{a}, b)$, some $b \in B$, implies $\mathfrak{B} \vDash \varphi(\bar{a}, a)$, some $a \in A$, implies by the induction hypothesis $\mathfrak{A} \vDash \varphi(\bar{a}, a)$ implies $\mathfrak{A} \vDash \exists v\varphi(\bar{a}, v)$.
$\square$

**Theorem 6 (Downward Lowenheim-Skolem Theorem)** *Let $\mathfrak{B}$ be an $L$-structure, $S$ a subset of $B = \text{dom }(\mathfrak{B})$. Then there exists $\mathfrak{A} \preccurlyeq \mathfrak{B}$ such that $S \subseteq A = \text{dom }(\mathfrak{A})$ and $||\mathfrak{A}|| \leq \max\{\text{card }(S), |L|\}$. In particular, given $\mathfrak{B}$ and a cardinal $||\mathfrak{B}|| \geq \kappa \geq |L|$ we can have $\mathfrak{A} \preccurlyeq \mathfrak{B}$ of cardinality $\kappa$.*

**Proof** Fix some $b_0 \in B$. For each $L$-formula $\phi(v_1, \ldots, v_n)$ define function $g_\phi : B^{n-1} \to B$ by

$$g_\phi(b_1, \ldots, b_{n-1}) = \begin{cases} \text{an element } b \in B : \mathfrak{B} \vDash \phi(b_1, \ldots, b_{n-1}, b) \\ \qquad \text{if such one exists} \\ b_0 \quad \text{if not} \end{cases}.$$

Notice that for $\phi$ of the form $\tau(v_1, \ldots, v_{n-1}) \doteq v_n$, for $\tau$ an $L$-term, $g_\phi$ coincides with the function $\tau^{\mathfrak{B}}$.
Let $A$ be the closure of $S$ under all $g_\phi$, i.e. $S = \bigcup_{i \in \mathbb{N}} S_i : S_0 = S$ and

$$S_{i+1} = \{g_\phi(b_1, \ldots, b_{n-1}) : b_1, \ldots, b_{n-1} \in S_i, \phi(v_1, \ldots, v_n) \ L-\text{formulas}\}.$$

Notice that card $A \leq$ card $S + |L|$.
Define an $L$-structure $\mathfrak{A}$ on domain $A$ interpreting the relation, function and constant symbols of $L$ on $A$ by restriction from $\mathfrak{B}$ :
(i) for an $n$-ary relation symbol $P$ or the equality symbol
$P^{\mathfrak{A}} = P^{\mathfrak{B}} \cap A^n$;
(ii) for an $m$-ary function symbol $f$ and $\bar{a} \in A^m$, $a \in A$ :
$f^{\mathfrak{A}}(\bar{a}) = a$ iff $f^{\mathfrak{B}}(\bar{a}) = a$;
(iii) for a constant symbol $c \quad c^{\mathfrak{A}} = c^{\mathfrak{B}}$.
(ii) and (iii) are possible since $A$ is closed under $L$-terms.

Clearly then $\mathfrak{A} \subseteq \mathfrak{B}$ and the condition of Tarski Lemma is satisfied, for if $\mathfrak{B} \vDash \exists v \phi(\bar{a}, v)$ then $\mathfrak{B} \vDash \phi(\bar{a}, g_\phi(\bar{a}))$.
Thus the lemma finishes the proof. $\square$

**Corollary 3** *Let $\Sigma$ be a set of $L$-sentences which has an infinite model. Then for any cardinal $\kappa \geq |L|$ there is a model of $\Sigma$ of cardinality $\kappa$.*

L8

**Axiomatizable classes and preservation theorems**

A formula of the form $\exists v_1 \ldots \exists v_m \psi$, where $\psi$ is a quantifier-free formula, is called **an existential formula**. The negation of an existential formula is called **a universal** formula.

**Exercise 3.1** *Let $\phi_1, \ldots, \phi_n$ be existential formulas. Prove that*
*(i) $(\phi_1 \vee \ldots \vee \phi_n)$ and $(\phi_1 \wedge \ldots \wedge \phi_n)$ are logically equivalent to existential formulas;*
*(ii) $(\neg\phi_1 \wedge \ldots \wedge \neg\phi_n)$ and $(\neg\phi_1 \vee \ldots \vee \neg\phi_n)$ are logically equivalent to universal formulas.*

Given a set of sentences $\Sigma$ denote $\exists(\Sigma)$ its subset consisting of all existential formulas in $\Sigma$. Correspondingly, $\forall(\Sigma)$ are the universal formulas of $\Sigma$.
Thus $\exists\mathrm{Th}(\mathfrak{A})$ is the set of all existential $L$-sentences which hold in $\mathfrak{A}$.

**Lemma 3.2** $T \cup \mathrm{Diag}(\mathfrak{A})$ *is satisfiable iff* $T \cup \exists\mathrm{Th}(\mathfrak{A})$ *is satisfiable.*

**Proof** Any model of a finite part of $T \cup \mathrm{Diag}(\mathfrak{A})$ can be reduced to a model of a corresponding finite part of $T \cup \exists\mathrm{Th}(\mathfrak{A})$ and vice-versa, since for $\psi$ quantifier-free
$\exists v_1, \ldots v_n \psi(v_1, \ldots, v_n) \in \exists\mathrm{Th}(\mathfrak{A})$ iff $\mathfrak{A} \models \exists v_1, \ldots v_n \psi(v_1, \ldots, v_n)$ iff $\mathfrak{A} \models \psi(a_1, \ldots, a_n)$ for some $a_1, \ldots a_n$ iff $\psi(c_{a_1}, \ldots c_{a_n}) \in \mathrm{Diag}(\mathfrak{A})$.$\square$

A class $C$ of $L$-structures is called **axiomatizable** if there is a set $\Sigma$ of $L$-sentences such that

$$\mathfrak{A} \in C \text{ iff } \mathfrak{A} \vDash \Sigma.$$

We also write equivalently

$$C = \mathrm{Mod}\ (\Sigma).$$

$\Sigma$ is then called **a set of axioms for** $C$.
$C$ is called $\exists$**-axiomatizable** ($\forall$-axiomatizable) iff $\Sigma$ can be chosen to consists of only existential (universal) sentences.
Conversely, we call **the theory of class** $C$ the set

$$\mathrm{Th}(C) = \{\sigma : L - \text{sentence}, \mathfrak{A} \vDash \sigma \text{ for all } \mathfrak{A} \in C\}.$$

**Theorem 7** *Let $C$ be an axiomatizable class. Then the following conditions are equivalent:*
*(i) $C$ is $\exists$-axiomatizable;*
*(ii) If $\mathfrak{A} \in C$ and $\mathfrak{A} \subseteq \mathfrak{B}$ then $\mathfrak{B} \in C$.*

**Proof** (i) implies (ii) just immediately from the definitions.
To prove the converse consider $\mathrm{Th}(C)$, the theory of class $C$, and $\exists\mathrm{Th}(C)$. Let $\mathfrak{B} \models \exists\mathrm{Th}(C)$. We need to show that $\mathfrak{B} \models \mathrm{Th}(C)$ which would yield Mod $(\exists\mathrm{Th}(C)) =$ Mod $(\mathrm{Th}(C)) = C$.

Claim. $\mathrm{Th}(C) \cup \forall\mathrm{Th}(\mathfrak{B})$ is finitely satisfiable. For, otherwise, $\mathrm{Th}(C) \models \neg\sigma_1 \vee \ldots \vee \neg\sigma_n$ for some $\sigma_1, \ldots, \sigma_n \in \forall\mathrm{Th}(\mathfrak{B})$. But $\neg\sigma_1 \vee \ldots \vee \neg\sigma_n$ is equivalent to $\neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$ and $\mathfrak{B} \models \sigma_1 \wedge \ldots \wedge \sigma_n$. On the other hand $\neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$ is equivalent to an $\exists$-formula, and is a logical consequence of $\mathrm{Th}(C)$. So $\mathfrak{B} \models \neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$ The contradiction. Claim proved.

Let $\mathfrak{A}$ be a model of $\mathrm{Th}(C) \cup \forall\mathrm{Th}(\mathfrak{B})$. In particular, $\mathfrak{A} \in C$.
Consider $\mathrm{Th}(\mathfrak{B}) \cup \exists\mathrm{Th}(\mathfrak{A})$. This is satisfiable by $\mathfrak{B}$, since $\mathfrak{A} \models \forall\mathrm{Th}(\mathfrak{B})$. By Lemma 3.2 then $\mathrm{Th}(\mathfrak{B}) \cup \mathrm{Diag}(\mathfrak{A})$ is satisfiable. Let $\mathfrak{B}^*$ be a model of the latter. Then $\mathfrak{A} \subseteq \mathfrak{B}^*$ and $\mathfrak{B}^* \equiv \mathfrak{B}$. The first property implies by (ii) that $\mathfrak{B}^* \in C$, the second one implies $\mathfrak{B} \in C$.$\Box$

# 4 Categoricity in powers and $\aleph_0$-categoricity

We continue the study of axiomatizable classes by considering their theories. From now on **an $L$-theory** is a satisfiable set of $L$-sentences. An $L$-theory $T$ is said to be **complete** if for any $L$-sentence $\sigma$ either $T \vDash \sigma$ or $T \vDash \neg\sigma$. This slightly modifies the previous usage of the term.

A theory $T$ is said to be **categorical in power** $\kappa$ if there is a model $\mathfrak{A}$ of $T$ of cardinality $\kappa$ and any model of $T$ of the same cardinality is isomorphic to $\mathfrak{A}$.

**Theorem 8 (R.Vaught)** *Let $\kappa \geq |L|$ and $T$ be a $\kappa$-categorical $L$-theory without finite models. Then $T$ is complete.*

**Proof** Let $\sigma$ be an $L$-sentence and $\mathfrak{A}$ the unique, up to isomorphism, model of $T$ of cardinality $\kappa$. The either $\sigma$ or $\neg\sigma$ holds in $\mathfrak{A}$, let it be $\sigma$. Then $T \cup \{\neg\sigma\}$ does not have a model of cardinality $\kappa$, which by the Lowenheim-Skolem theorems means $T \cup \{\neg\sigma\}$ does not have an infinite model, which by our assumption means it is not satisfiable. It follows $T \vDash \sigma$. $\Box$

**Example** The trivial theory $T = \{\forall v\ v \doteq v\}$ in language $L$ with no non-logical symbols is categorical in every power. Indeed, any set $A$ determines a model $\mathfrak{A} = \langle A \rangle$ of $T$ and any other model of $T$ of the same cardinality is isomorphic to $\mathfrak{A}$ by a bijection.

**Example** Let $K$ be a field and $L_K$ be the language with alphabet $\{+, \lambda_k, 0\}_{k\in K}$ where $+$ is a symbol of a binary function and $\lambda_k$ symbols of unary functions, 0 constant symbol. Define $V_K$ to be the theory of vector spaces over $K$, i.e. $V_K$ consists of the standard axioms:
$\forall v_1 \forall v_2\ \ \lambda_k(v_1 + v_2) \doteq \lambda_k(v_1) + \lambda_k(v_2)$ an axiom for each $k \in K$;
$\forall v\ \ \lambda_1(v) \doteq v$;
$\forall v\ \ \lambda_0(v) \doteq 0$;
$\forall v\ \ \lambda_{k_1}(\lambda_{k_2}(v)) \doteq \lambda_{k_1 \cdot k_2}(v)$ an axiom for each $k_1, k_2 \in K$;

$\forall v \;\; \lambda_{k_1}(v) + \lambda_{k_2}(v) \doteq \lambda_{k_1+k_2}(v)$ an axiom for each $k_1, k_2 \in K$.

Mod $V_K$ is exactly the class of vector spaces over $K$. By linear algebra the isomorphism type of a vector space over a given field is determined by its dimension. Let $\mathfrak{A}$ be a model of $V_K$ of cardinality $\kappa > |L_K| = \max\{\aleph_0, \mathrm{card}\ K\}$. Then $||\mathfrak{A}|| = \dim \mathfrak{A}$, the dimension of the vector space (check it). It follows that, if $\mathfrak{B}$ is another model of $V_K$ of the same cardinality, $\mathfrak{A} \cong \mathfrak{B}$. Thus $V_K$ is categorical in any such power $\kappa$.

**Example** Let $L$ be the language with one binary symbol $<$ and DLO the theory of dense linear order with no end elements:
$\forall v_1 \forall v_2 \;\; (v_1 < v_2 \rightarrow \neg\, v_2 < v_1);$
$\forall v_1 \forall v_2 \;\; (v_1 < v_2 \vee v_1 \doteq v_2 \vee v_2 < v_1)$
$\forall v_1 \forall v_2 \forall v_3 \; (v_1 < v_2 \wedge v_2 < v_3) \rightarrow v_1 < v_3;$
$\forall v_1 \forall v_2 \;\; (v_1 < v_2 \rightarrow \exists v_3 \; (v_1 < v_3 \wedge v_3 < v_2));$
$\forall v_1 \exists v_2 \exists v_3 \;\; v_1 < v_2 \wedge v_3 < v_1.$

**The Cantor Theorem** *Any two countable models of* DLO *are isomorphic. In other words* DLO *is* $\aleph_0$*-categorical.*
The proof goes by numbering the two ordered sets and then applying the famous *back-and-forth construction* of a bijection preserving the orders.

**Exercise 4.1** *Show that* DLO *is not* $\kappa$*-categorical*
*(i) for* $\kappa = 2^{\aleph_0}$;
*(ii) for any* $\kappa > \aleph_0$.

Fix a *countable* language $L$. Henceforce $T$ denotes a complete $L$-theory having an infinite model, say $\mathfrak{A}$. By the Lowenheim-Skolem downward Theorem we may assume $\mathfrak{A}$ is countable. Also, by the definition, $T = \mathrm{Th}(\mathfrak{A})$.

Denote $F_n$ the set of all $L$-formulas with free variables $v_1 \dots v_n$ (abbreviated $\bar{v}$). Denote $E_n(T)$ the binary relation on $F_n$ defined by

$$\varphi(\bar{v}) \; E_n \; \psi(\bar{v}) \text{ iff } T \models \forall \bar{v}(\varphi(\bar{v}) \equiv \psi(\varphi)).$$

Equivalently, $\varphi(\mathfrak{A}) = \psi(\mathfrak{A})$.

Thus $E_n$ is an equivalence relation respecting Boolean operations $\wedge$, $\vee$ and $\neg$.

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**Theorem 9 (Ryll-Nardzewski)** *$T$ is $\aleph_0$-categorical iff $F_n/E_n(T)$ is finite for all $n \in\ N$.*

Proof of the theorem will follow from intermediate statements.

**Lemma 4.1** *For any $\bar{a} \in \mathfrak{A}^n$ there is $\varphi(\bar{v})$ such that*
*(i) $\mathfrak{A} \models \varphi(\bar{a})$*
*and*
*(ii) whenever $\psi(\bar{v})$ is such that $\mathfrak{A} \models \psi(\bar{a})$,*

$$T \models \forall \bar{v}(\varphi(\bar{v}) \to \psi(\bar{v})).$$

**Proof** By the finiteness assumption we can find the minimal definable subset of $A^n$ containing $\bar{a}$, say $\varphi(\mathfrak{A})$. Then for any $\psi(\mathfrak{A})$ containig $\bar{a}$ we have necessarily $\varphi(\mathfrak{A}) \subseteq \psi(\mathfrak{A})$. $\square$

Call $\varphi$ as above **principal** [for $\bar{a}$].

**Lemma 4.2 ($\Leftarrow$ of the Theorem)** *The finiteness of all $F_n/E_n(T)$ implies the $\aleph_0$-categoricity of $T$.*

**Proof** Suppose $\mathfrak{B}$ is another countable model of $T$. Enumerate

$$A = \{a_1, a_2, \ldots\}, \quad B = \{b_1, b_2, \ldots\}.$$

We will construct new enumerations $\{a'_1, a'_2, \ldots\}$ and $\{b'_1, b'_2, \ldots\}$ preserving $L$-formulas by the back-and-forth method:
Suppose $a'_1, \ldots, a'_{n-1} \in A$ and $b'_1, \ldots, b'_{n-1} \in B$ satisfy for all $\psi \in F_{n-1}$

$$\mathfrak{A} \models \psi(a'_1, \ldots, a'_{n-1}) \text{ iff } \mathfrak{B} \models \psi(b'_1, \ldots, b'_{n-1}). \tag{1}$$

Notice that (1) is true for $n = 1$ since $\mathfrak{A} \equiv \mathfrak{B}$. Let $a'_n$ be the first member in $A = \{a_1, a_2, \ldots\}$ not occuring among $a'_1, \ldots a'_{n-1}$. Let $\varphi$ be a principal

formula for $a'_1, \ldots, a'_n$. Then $\mathfrak{A} \models \varphi(a'_1, \ldots, a'_n)$ and $\mathfrak{A} \models \exists v \varphi(a'_1, \ldots, a'_{n-1}, v)$. By (1) $\mathfrak{B} \models \exists v \varphi(b'_1, \ldots, b'_{n-1}, v)$. Hence we may choose $b'_n \in B$ such that $\mathfrak{B} \models \varphi(b'_1, \ldots, b'_n)$.
Now suppose $\psi \in F_n$ and $\mathfrak{A} \models \psi(a'_1, \ldots, a'_n)$. Since $\varphi$ is principal

$$T \models \forall \bar{v}(\varphi(\bar{v}) \to \psi(\bar{v})).$$

Hence $\mathfrak{B} \models \psi(b'_1, \ldots, b'_n)$.
Thus (1) is satisfied for $a'_1, \ldots a'_n$ and $b'_1, \ldots b'_n$ too.
Similarly, if $b'_{n+1}$ is the first element in $B = \{a_1, a_2, \ldots\}$ not occuring among $b'_1, \ldots b'_n$ then we can find $a'_{n+1} \in A$ such that (1) is satisfied for $a'_1, \ldots a'_{n+1}$ and $b'_1, \ldots b'_{n+1}$.
Hence we may inductively construct in this way $A = \{a'_1, a'_2, \ldots a'_n \ldots\}$, $B = \{b'_1, b'_2, \ldots b'_n \ldots\}$ satisfying (1) for all $n$. Our construction guarantees that we get all of $A$ and all of $B$. Now it follows from (1) that $a'_i \to b'_i$ is an isomorphism. $\square$

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A subset $p \subset F_n$ is called an $n$-**type** (over $T$) if
(i) for all $\varphi \in p$ $\ T \models \exists \bar{v} \varphi(\bar{v})$;
(ii) if $\varphi, \psi \in p$ then $(\varphi \wedge \psi) \in p$.

Type $p$ is called **complete** if also the following is satisfied
(iii) for any $\varphi \in F_n$ either $\varphi \in p$ or $\neg\varphi \in p$.

Suppose $\bar{a} \in A^n$. Then we define **the $L$-type of $\bar{a}$ in $\mathfrak{A}$.**

$$\mathrm{tp}_{\mathfrak{A}}(\bar{a}) = \{\varphi \in F_n : \ \mathfrak{A} \models \varphi(\bar{a})\}.$$

Clearly, $\mathrm{tp}_{\mathfrak{A}}(\bar{a})$ is a complete $n$-type.

When $\mathfrak{A} \subseteq \mathfrak{B}$ then $\mathrm{tp}_{\mathfrak{A}}(a)$ and $\mathrm{tp}_{\mathfrak{B}}(a)$ might be different. But it follows immediately from the definitions that

$$\mathfrak{A} \preccurlyeq \mathfrak{B} \ \text{ implies } \ \mathrm{tp}_{\mathfrak{A}}(a) = \mathrm{tp}_{\mathfrak{B}}(a).$$

We say that an $n$-type $p$ is **realised** in $\mathfrak{A}$ if there is $\bar{a} \in A^n$ such that $p \subseteq \mathrm{tp}_{\mathfrak{A}}(\bar{a})$.
If there is no such $\bar{a}$ in $\mathfrak{A}$ we say that $p$ is **omitted** in $\mathfrak{A}$.

**Lemma 4.3** *Given a set $P = \{p^\alpha : \alpha < \kappa\}$ of $n$-types $p$, a model $\mathfrak{A}$ of $T$ and a cardinal $\kappa \geq |\mathfrak{A}|$, there is $\mathfrak{B} \succcurlyeq \mathfrak{A}$ of cardinality $\kappa$ such that all types from $P$ are realised in $\mathfrak{B}$. In particular, given one type $p$ there is a countable model $\mathfrak{B}$ of $T$ which realises $p$.*

**Proof** Consider the expansion $L^+$ of $L_{\mathfrak{A}}$ by new constants

$$\{c_1^\alpha, \ldots, c_n^\alpha : \alpha < \kappa\},$$

and the theory

$$T^+ = \mathrm{CDiag}(\mathfrak{A}) \cup \bigcup_{\varphi \in p^\alpha, \ \alpha < \kappa} \{\varphi(c_1^\alpha, \ldots, c_n^\alpha)\}$$

We claim that $T^+$ is f.s. in $\mathfrak{A}$. Indeed, any finite subset $S$ of $T^+$ contains only finitely many formulas $\varphi$ from the types. Since types are closed under conjuction, we may assume that there is at most one formula of the form $\varphi(c_1^\alpha, \dots, c_n^\alpha)$ in $S$ for a type $p^\alpha$. Since $\exists \bar{v} \varphi(\bar{v})$ holds in $\mathfrak{A}$, we can find in $\mathfrak{A}$ for $\varphi(c_1^\alpha, \dots, c_n^\alpha)$ an interpretation of $c_1^\alpha, \dots, c_n^\alpha$ which makes each such formula true in the corresponding expansion of $\mathfrak{A}$.
By the Compactness Theorem there is a model $\mathfrak{B}^+ \models T^+$ of cardinality $\kappa$. Since $\mathfrak{B}^+ \models \mathrm{CDiag}(\mathfrak{A})$ the $L$-reduct $\mathfrak{B}$ of $\mathfrak{B}^+$ is an elementary extension of $\mathfrak{A}$. Let for each $\alpha$ $a_1^\alpha, \dots, a_n^\alpha$ be the elements assigned to $c_1^\alpha, \dots, c_n^\alpha$ in $\mathfrak{B}^+$. By the construction $\langle a_1^\alpha, \dots, a_n^\alpha \rangle$ realize $p^\alpha$ in $\mathfrak{B}$.
If we start with a countable model $\mathfrak{A}$ of $T$ and $\kappa \leq \aleph_0$, then $\mathfrak{B}$ can be chosen countable. $\square$

**Corollary 4** *For any $n$-type there is $p' \supseteq p$ which is a complete $n$-type.*

Indeed, put $p' = \mathrm{tp}_{\mathfrak{B}}(\bar{a})$ for $\bar{a}$ in $\mathfrak{B}$ realising $p$.

**Remark** If $\pi : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism, $\bar{a} \in A^n$, $\bar{b} \in B^n$, and $\pi : \bar{a} \to \bar{b}$ then $\mathrm{tp}_{\mathfrak{A}}(\bar{a}) = \mathrm{tp}_{\mathfrak{B}}(\bar{b})$.

**Example** There is a countable elementary extension of the group of integers $\mathbb{Z} = (\mathbb{Z}; +; 0)$ which is not isomorphic to $\mathbb{Z}$.
Given $n > 0$ denote $n|v$ the formula $\exists w (v = w + \ldots + w)$ ($n$ summands).
Let

$$p = \{1|v \& \ldots \& n|v : \ n \in \mathbb{N}\}.$$

$p$ clearly is a type, thus it is realised in some countable elementary extension. But $p$ is obviously omitted in $\mathbb{Z}$.

A type $p$ is called **principal** if there is $\varphi \in p$ such that for any $\psi \in p$ $T \models \forall \bar{v}(\varphi(\bar{v}) \to \psi(\bar{v}))$.
$\varphi$ is called then a principal formula of type $p$.
A type which is not principal is called **non-principal**.

**Remark** If $p$ is a complete type then a principal formula for $p$ is a principal formula.

**Exercise 4.2** *A principal type $p$ is realised in any model $\mathfrak{A}$ of $T$.*

**Lemma 4.4** *If $F_n/E_n(T)$ is infinite then there exists a non-principal $n$-type in $T$.*

**Proof** Take $p = \{\neg\varphi_1 \wedge \ldots \wedge \neg\varphi_k \in F_n : \varphi_i$ principal formulae $\}$. We claim that $p$ is an $n$-type.
Suppose not. Then

$$T \models \forall\bar{v}(\varphi_1(\bar{v}) \vee \ldots \vee \varphi_k(\bar{v}))$$

for some principal formulas $\varphi_1, \ldots, \varphi_k \in F_n$.
Define for $\psi \in F_n$

$$W_\psi = \{i \in \{1, \ldots, k\} : \ T \models \exists\bar{v}(\varphi_i(\bar{v}) \wedge \psi(\bar{v})\}$$

Notice that since $\varphi_i$'s are principal formulas
$T \models \exists\bar{v}(\varphi_i(\bar{v}) \wedge \psi(\bar{v})\}$ iff $T \models \forall\bar{v}(\varphi_i(\bar{v}) \to \psi(\bar{v})\}$.
It follows that for any $\psi, \chi \in F_n$ $\psi E_n \chi$ iff $W_\psi = W$. Thus card $F_n/E_n(T) = 2^k$ which contradicts the assumtions and proves the claim.
Take now a complete $n$-type extending $p$. It is non-principal by the construction.□

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**Theorem 10 (Omitting Type Theorem)** *Let $p$ be a non-principal $n$-type in a complete theory $T$ of a countable language $L$. Then there is a countable model of $T$ which omits $p$.*

**Proof** Let $L' = L \cup C$, $C$ a set of countably many new constant symbols.
Let $\bar{c}_1, \ldots, \bar{c}_k, \ldots$ be an enumeration of all $n$-tuples of constant symbols of $L'$ and $\phi_1, \ldots, \phi_l, \ldots$ an enumeration of all sentences in $L'$.
We construct a chain of finite sets of $L'$-sentences

$$S_0 \subseteq \ldots S_m \subseteq \ldots$$

by induction on $m \geq 1$ so that
(i) $T \cup S_m$ are satisfiable,
(ii) for $m \geq 1$ either $\phi_m$ or $\neg\phi_m$ is in $S_m$, ,
(iii) if $\phi_m$ is in $S_m$ and has the form $\exists v\ \varphi(v)$, for some 1-variable $L'$-formula $\varphi(v)$, then $\varphi(c) \in S_m$ for some $c \in C$

(iv) for $m \geq 1$ there is a formula $\psi \in p$ such that $\neg\psi(\bar{c}_m) \in S_m$.

Start with $S_0 = \emptyset$.
Suppose $S_0 \subseteq \ldots S_{m-1}$ are constructed.
If $T \cup S_{m-1} \cup \{\phi_m\}$ is satisfiable then put $S'_m = S_{m-1} \cup \{\phi_m\}$. Otherwise $S'_m = S_{m-1} \cup \{\neg\phi_m\}$. It is easy to see that $T \cup S'_m$ is satisfiable.

Claim. There exists $\psi \in p$ such that $T \cup S'_m \cup \{\neg\psi(\bar{c}_m)\}$ is satisfiable.
Proof of Claim. Suppose for all $\psi \in p$ the converse holds. Let $\Phi = \bigwedge S'_m$. We can represent $\Phi$ as $\varphi(c_{m,1}, \ldots, c_{m,n}, d_1, \ldots, d_p)$, where $\varphi(v_1, \ldots v_n, u_1, \ldots, u_p)$ is an $L$-formula with free variables $v_1, \ldots v_n, u_1, \ldots, u_p$ and $\langle c_{m,1} \ldots, c_{m,n}\rangle = \bar{c}_m$, $d_1, \ldots, d_p$ constant symbols not in $L$ and different from $c_{m,i}$'s. We write corresponding formulas in the short form $\varphi(\bar{c}_m, \bar{d})$ and $\varphi(\bar{v}, \bar{u})$.
Then, by our assumption, for any $\psi \in p$

$$T \models (\varphi(\bar{c}_m, \bar{d}) \to \psi(\bar{c}_m)).$$

Since no component of $\bar{c}_m$ and $\bar{d}$ do occur in $T$, it follows

$$T \models \forall\bar{v}\forall\bar{u}(\varphi(\bar{v}, \bar{u}) \to \psi(\bar{v})).$$

The formula can be equivalently rewritten as $\forall\bar{v}(\exists\bar{u}\varphi(\bar{v}, \bar{u}) \to \psi(\bar{v}))$, so

$$T \models \forall\bar{v}(\exists\bar{u}\varphi(\bar{v}, \bar{u}) \to \psi(\bar{v}))$$

for every $\psi \in p$. This means that $\exists\bar{u}\varphi(\bar{v}, \bar{u})$ is a principal formula for $p$. The contradiction, which proves the claim.

Now take $S''_m = S'_m \cup \{\neg\psi(\bar{c}_m)\}$.
Suppose $\phi_m$ is in $S''_m$ and has the form $\exists v\ \varphi(v)$. Choose $c \in C$ which does not occur in $S''_m$. Then $T \cup S''_m \cup \{\varphi(c)\}$ has a model: any model $\mathfrak{A}$ of $T \cup S''_m$ in the language $L \cup \{$ constants of $S''_m\}$ can be expanded by assigning to $c$ the values of $v$ for $\exists v\varphi(v)$.
Denote $S_m = S''_m \cup \{\varphi(c)\}$. If $\phi_m$ does not have this form then put $S_m = S''_m$.
This $S_m$ satisfies (i)-(iv) by the construction.
To finish the proof of the theorem consider now

$$T^* = T \cup \bigcup_{m \in \mathbb{N}} S_m.$$

By the properties (i)-(iii) $T^*$ is satisfiable, complete and full set of sentences. By Theorem 2 $T^*$ has a canonical model $\mathfrak{A}$. Notice that by (iii) for any closed term $\lambda$ $T^*$ says $\lambda = c$ for some $c \in C$. Thus all elements of the canonical model $\mathfrak{A}$ are named by symbols from $C$. Consequently, (iv) says that no $n$-tuple in $\mathfrak{A}$ realises type $p$.$\square$

**End of the proof of the Ryll-Nardzewski Theorem**: *If $F_n/E_n(T)$ is infinite then $T$ is not $\aleph_0$-categorical.*
Indeed, from Lemma 4.4 it follows, under the assumption, that there is a non-principal $n$-type in $T$. By the Omitting Type Theorem there is a countable model $\mathfrak{A}$ that omits $p$. On the other hand, by Lemma 4.3, there is a countable model $\mathfrak{B}$ which realises $p$. It follows $\mathfrak{A}$ is non-isomorphic to $\mathfrak{B}$ and thus $T$ is not $\aleph_0$ categorical. $\square$

**Remark** Slight changes in the proof of the Omitting Type Theorem yield

**Theorem 11 (Omitting Types Theorem of R.Vaught)** *Let $P$ be a countable set of non-principal $n$-types in a complete theory $T$ of a countable language $L$. Then there is a countable model of $T$ which omits every type in $P$.*

# 5 Theories with few types

Let $T$ be a complete theory of a countable language $L$.
We denote $\mathrm{S}_n(T)$ the set of all complete $n$-types in $T$, **the Stone space of** $T$.

For $\varphi \in F_n$ denote
$$U_\varphi = \{p \in \mathrm{S}_n(T) : \varphi \in p\}.$$

**Exercise 5.1** *Prove that*
$U_{\varphi_1} \cup U_{\varphi_2} = U_{\varphi_1 \vee \varphi_2}$
$U_{\varphi_1} \cap U_{\varphi_2} = U_{\varphi_1 \wedge \varphi_2}$
$U_{v_1 = v_1} = \mathrm{S}_n(T)$
$U_{\neg\varphi} = \mathrm{S}_n(T) \setminus U_\varphi$.

**Remark** The Stone space of $T$ can be made a topological space by letting $U_\varphi$ be the basic open subsets. With this topology $\mathrm{S}_n(T)$ is Hausdorff and compact.

For $T$ as before, $T$ is called **0-stable** if card $\mathrm{S}_n(T) \leq \aleph_0$ for all $n \in \mathbb{N}$.

A model $\mathfrak{A}$ of $T$ is called **atomic** if for all $n \in \mathbb{N}$, every complete $n$-type realised in $\mathfrak{A}$ is principal.

**Remark** We can equivalenly say in the definition: 'every $n$-tuple in $\mathfrak{A}$ satisfies a principal formula'.

**Warning** 'Atomic' here is connected to the notion of atoms of Boolean algebra $F_n/E_n(T)$. Nothing to do with atomic formulas.

A model $\mathfrak{A}$ of $T$ is called **prime** if for any model $\mathfrak{B}$ of $T$ there is an elementary embedding $\pi : \mathfrak{A} \to \mathfrak{B}$.

**Theorem 12** *Let $T$ be 0-stable. Then*
*(i) $T$ has a countable atomic model.*

*(ii) Any countable atomic model is prime.*
*(iii) Any two countable atomic models of $T$ are isomorphic.*

**Proof** Since $T$ is 0-stable, there are only countably many non-principal types in $\bigcup_n \mathrm{S}_n(T)$. By The Omitting Types Theorem of Vaught there is a countable model $\mathfrak{A}$ of $T$ that omits all the non-principal types. This $\mathfrak{A}$ is atomic by definition.
Proofs of the two other statements are left to the reader as exercises (Problems 2 and 3, sheet 6). Use the inductive construction similar to the one in the proof of the first part of The Ryll-Nardzewski Theorem.$\square$

A model $\mathfrak{A}$ of $T$ is called **countable saturated** if $\mathfrak{A}$ is countable and for any expansion $\mathfrak{A}_{c_1,\dots,c_m}$ of $\mathfrak{A}$ by finitely many contant symbols $c_1,\dots,c_m$, every 1-type in $\mathrm{Th}(\mathfrak{A}_{c_1,\dots,c_m})$ is realised in $\mathfrak{A}c_1,\dots,c_m$ .

A model $\mathfrak{A}$ of $T$ is called **$\aleph_0$-universal** if for any countable model $\mathfrak{B}$ of $T$ there is an elementary embedding $\pi : \mathfrak{B} \to \mathfrak{A}$.

**Theorem 13** *Let $T$ be 0-stable. Then*
*(i) $T$ has a countable saturated model.*
*(ii) Any countable saturated model is $\aleph_0$-universal.*
*(iii) Any two countable saturated models of $T$ are isomorphic.*

**Proof** We start with

**Lemma 5.1** *Let $T' = T(c_1,\dots,c_m)$ be a complete theory extending $T$ in language $L(c_1,\dots,c_m)$, the extension of $L$ by finitely many extra constants symbols $c_1,\dots,c_m$, and suppose $T$ is 0-stable. Then $T'$ is 0-stable too.*

**Proof** Fix $n$. For each $p \in \mathrm{S}_n(T')$ define

$$p^* = \{\phi(v_1,\dots,v_{n+m}) \in F_{n+m} : \ \phi(v_1,\dots,v_n,c_1,\dots,c_m) \in p\}.$$

It follows from the definition that $p^* \in \mathrm{S}_{n+m}(T)$, and if $p_1 \neq p_2$ then $p_1^* \neq p_2^*$. Hence we have mapping $\mathrm{S}_n(T') \to \mathrm{S}_{m+n}(T)$ which is injective. Since card $\mathrm{S}_{m+n} \leq \aleph_0$, by the hypothesis, we have $\mathrm{S}_n \leq \aleph_0$. $\square$

Proof of the theorem. Let $\mathfrak{A}$ be a countable model of $T$. Enumerate $\{a_1, \ldots, a_n, \ldots\}$ elements of $\mathfrak{A}$. Let $C = \{c_1, \ldots, c_n, \ldots\}$ be a set of new constant symbols, $\mathfrak{A}_C$ the structure in the language $L(C)$ obtained by assigning $a_i$ to $c_i$, $T(C)$ the theory of the structure, and $T(c_1, \ldots c_m)$ the fragment of the theory containing formulas with only first $m$ constants symbols of $C$.
By Lemma above the set $\bigcup_m \mathrm{S}_1(T(c_1, \ldots c_m))$ is countable. By Lemma 4.3 we can construct a countable $\mathfrak{B} \succ \mathfrak{A}$ with the property that any 1-type in an expanded theory $\mathrm{Th}(\mathfrak{A}_{\{c_1,\ldots,c_m\}})$ is realised in $\mathfrak{B}$.
Repeating this construction we get an elementary chain

$$\mathfrak{A}^{(0)} \preccurlyeq \mathfrak{A}^{(1)} \preccurlyeq \ldots \preccurlyeq \mathfrak{A}^{(n)} \ldots$$

of countable models of $T$ with $\mathfrak{A}^{(0)} = \mathfrak{A}$ and the property that any 1-type in $\mathrm{Th}(\mathfrak{A}^{(n)}_{c_1,\ldots,c_m})$ is realised in $\mathfrak{A}^{(n+1)}_{c_1,\ldots,c_m}$ for any assignement of constant symbols $c_1, \ldots, c_m$, any $m$. It follows that $\mathfrak{A}^* = \bigcup_n \mathfrak{A}^{(n)}$ is a countable saturated model of $T$ which also elementarily extends $\mathfrak{A}$. This proves (i).

The proofs of (ii) and (iii) are exercises: see problem sheet 7.
**Remark** It is an easy set-theoretic calculation that shows that the number of countable structures in a countable language (up to isomorphism) is not greater than $2^{\aleph_0}$. But it remains

**Open Problem (R.Vaught)** Is it true that the number of countable models of $T$ is either $\leq \aleph_0$ or equal to $2^{\aleph_0}$?

# 6 Theories with many types

**Theorem 14** *Suppose* card $\mathrm{S}_n(T) = \kappa > \aleph_0$. *Then $T$ has at least $\kappa$ non-isomorphic countable models.*

**Proof** For any $n$-type there is a countable model that realises the type, and in a countable model at most countably many complete types can be realized.□

**Theorem 15** *Suppose* $\mathrm{S}_n$ *is uncountable. Then* card $\mathrm{S}_n(T) = 2^{\aleph_0}$.

We start the proof of the theorem by introducing a new notion and proving an intermediate lemma.
A formula $\varphi \in F_n$ is called **fat** (in $T$) if $U_\varphi$ is uncountable.

**Lemma 6.1** *For any fat $\varphi$ there are fat $\varphi_0$ and $\varphi_1$ such that $U_\varphi = U_{\varphi_0} \dot{\cup} U_{\varphi_1}$ (the intersection is empty).*

**Proof** Suppose not. Define

$$q = \{\psi \in F_n : \psi \wedge \varphi \text{ is fat }\}.$$

This is a complete type. Indeed, (i) of the definition of type follows from the fact that every $\psi$ in $q$ belongs to a type, since $\psi$ is fat.
(ii) follows from the assumption that $U_\varphi$ can not be decomposed into two fat parts: $\psi_1 \wedge \psi_2 \wedge \varphi$ is fat, if $\psi_1, \psi_2 \in q$.
(iii) is immediate from the assumption.
Now notice that

$$U_\varphi = \{q\} \cup \bigcup \{U_{\neg\psi\wedge\varphi} : \psi \in q\}.$$

Finally, since by the assumption $U_{\neg\psi\wedge\varphi}$ is at most countable for $\psi \in q$, we came to a contradiction. Lemma proved.
**Proof** of the theorem. Notice first that the number of $n$-types is not greater than $2^{\aleph_0}$ since each type is just a subset of the countable set $F^n$. So we want to show that the number is not less than $2^{\aleph_0}$.

Let $\mathcal{M} = \{\mu : \mathbb{N} \to \{0,1\}\}$ be the set of all $\{0,1\}$-sequences. For each $\mu$ and $n \in \mathbb{N}$ define $\mu_{|n}$, the initial $n$-cut of $\mu$, to be the reduction of $\mu$ to $\{1,\ldots,n\}$. Define a fat formula $\varphi_{\mu,n}$ by induction on $n$ : For $n = 0$ let it be the formula $v_1 = v_1$. If $\varphi_{\mu,n}$ is defined then $\varphi_{\mu,n+1}$ is either one of the two fat formulas that divide $\varphi_{\mu,n+1}$, as given by the lemma above, depending on whether $\mu(n+1)$ is 0 or 1. So if $\mu_{|n}{=}\nu_{|n}$ and $\mu_{|n+1} \neq \nu_{|n+1}$, then $\varphi_{\mu,n} = \varphi_{\nu,n}$, and $\varphi_{\mu,n+1}$ and $\varphi_{\nu,n+1}$ can not belong to a common type. Also $T \vDash \forall\bar{v}(\varphi_{\mu,n+1} \to \varphi_{\mu,n})$.
Let now for each $\mu$

$$q_\mu = \{\varphi_{\mu,i_1} \wedge \ldots \wedge \varphi_{\mu,i_n} : i_1,\ldots,i_n \in \mathbb{N}\}.$$

This, by definition, is a type. So, there is an extension $p_\mu \supseteq q_\mu$ which is a complete type. If $\mu \neq \nu$, say $n$ is the first number such that $\mu(n) \neq \nu(n)$, then $\varphi_{\mu,n} \in p_\mu$, $\varphi_{\nu,n} \in p_\nu$ are the two mutually inconsistent formulas dividing $\varphi_{\mu,n}$, and so $p_\mu \neq p_\nu$.
Thus the number of complete types is not less then the number of infinite $\{0,1\}$-sequences, which is $2^{\aleph_0}$. $\square$

**Corollary 5** *Suppose for some $n$, $\mathrm{S}_n(T)$ is uncountable. Then $T$ has exactly $2^{\aleph_0}$ non-isomorphic models.*

$\square$