Introspective Semantics for Turing Machine Languages

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ABSTRACT: Define an axiom system α to be **self-verifying** iff

i) one of the theorems implied by α is the statement of its own consistency (by at least some reasonable definition of self-consistency), and

ii) the axiom system α is in fact consistent.

Kleene's Fixed Point Theorem implies that every recursively enumerable axiom system α can be easily extended into a broader system α^* which satisfies condition (i). The catch is that α^* can be inconsistent even when it formally proves its own consistency. For example, α can not simultaneously satisfy Conditions (i) and (ii) when it is an extension of Peano Arithmetic by Gödel's Incompleteness Theorem. However, the Incompleteness Theorem does not preclude the possibility that a weak axiom system can verify its own consistency, and our talk will give several examples of such systems.

For example, we will illustrate an axiom system α^* that can prove all the Π_1 theorems of Peano Arithmetic (when the Subtraction and Division primitives replace Addition and Multiplication in the formulation of Π_1 sentences), and which can verify its own consistency in that it can recognize that that for a fixed constant $\lambda > 0$ and an arbitrary variable x , *that an arbitrary Turing machine* running for x^λ units of time is unable to output a Hilbert proof of $0 = 1$ from α^* .

Let $\mathcal{BIN}$ denote the set of "binary sentences" of the form $\overline{2N} - \overline{N} = \overline{N}$. Since they are trivially Π_1, all the sentences in $\mathcal{BIN}$ are theorems of α^*. Yet oddly a system α^{**} will be inconsistent that differs from α^* only by

1. including the sentences of $\mathcal{BIN}$ as axioms rather than as theorems, and
2. including an added axiom stating that the Change 1 (above) does not cause the system α^{**} to have a different consistency property than α^* .

Thus, self-verifying systems have very tricky and almost paradoxical properties (which have no counterpart under more conventional logics). We will also present two conjectures about self-verifying systems, which if correct (?), would link the theory of self-verifying systems to the P =? NP open problem.