Problem 1

Assume for the sake of contradiction that there is a model M s.t.:

- $M \models \Delta_0-PHP$
- $M \models \phi(0) \wedge \forall x\,(\phi(x) \to \phi(x+1)) \wedge \neg\phi(v)$ for some $\phi \in \Delta_0$ and $v \in M$

I consider the function f:

$$\forall y < v+1: \quad f(y) := \begin{cases} y & \text{if } M \models \phi(y) \\ y-1 & \text{if } M \models \neg\phi(y) \end{cases}$$

The graph of f may be defined by $A \in \Delta_0$:

$$A(y, z) \equiv (\phi(z) \wedge z = y) \vee (\neg\phi(z) \wedge z+1 = y)$$

I need to show:

$$M \models \forall y < v+1\ \exists! z < v\ A(y, z)$$

By the choice of M:

$$M \models A(0, 0)$$

$$M \models A(v, v-1)$$

The rest follows from (a) the general fact that it is always exactly one of the options $M \models \phi(a)$ and $M \models \neg\phi(a)$.

and (b) $PA^- \vdash \forall x \neq 0\, \exists y\ (y+1 = x)$.

(consequence of axiom 7 $(0<a \Rightarrow 1 \leq a)$ and axiom 4 $(a<b \Rightarrow (\exists x)\, a+x=b)$)

Using the axiom $PHP(A)$, I obtain:

$$M \models \exists y_1 < y_2 < v+1\ \exists z < v\ (A(y_1, z) \wedge A(y_2, z))$$

From the choice of A, it follows that:

$$M \models \exists y_1 < v+1\ (A(y_1, y_1) \wedge A(y_1+1, y_1))$$

Hence: $M \models \exists y_1 < v+1\ \phi(y_1) \wedge \neg\phi(y_1+1)$ $\lightning$