From Boolean values to Forcing

(1) Let $[\![\varphi]\!]$ be a valuation of sentences in our (or some other) model in a complete Boolean algebra B satisfying the conditions from the Scott paper. In particular:

$$[\![\exists x\, \varphi(x)]\!] = \bigvee_{\xi} [\![\varphi(\xi)]\!] .$$

(2) B is partially ordered by:

$$b_1 \leq b_2 \iff b_1 \wedge b_2 = b_1 .$$

Let P be the subordering with the universe $B \setminus \{0_B\}$; the ordering $\leq$ is the same.

(3) Elements of P will be called <u>forcing conditions</u> and are traditionally denoted by letters $p, q, r, \ldots$.

Let SENT be the set of all sentences in the language of our (or some other) world, including the names for reals, functions and functionals from $\mathbb{R}, \mathbb{R}^{\mathbb{R}}, \mathbb{R}^{\mathbb{R}^{\mathbb{R}}}$

(4) A <u>forcing relation</u> $\Vdash \subseteq \mathbb{P} \times \text{SENT}$ is defined by:

$$p \Vdash \varphi \underset{\text{df.}}{\iff} p \leq [\![\varphi]\!] .$$

<u>Intuition</u>: if we accept p as our "degree of truth" then φ is true.

(5) The relation satisfies three obvious properties:

(i) $(p \Vdash \varphi \;\&\; q \leq p) \Rightarrow q \Vdash \varphi$,

(ii) $p \Vdash \varphi \wedge \psi \iff (p \Vdash \varphi$ and $p \Vdash \psi)$,

(iii) $p \Vdash \forall x\, \varphi(x) \iff$ for all ξ, $p \Vdash \varphi(\xi)$), and three less obvious:

(iv) $p \Vdash \neg\varphi \iff \forall q \le p,\ q \not\Vdash \varphi$,

(v) $p \Vdash \varphi \vee \psi \iff \forall q \le p\ \exists r \le q\ \ r \Vdash \varphi$ or $r \Vdash \psi$,

(vi) $p \Vdash \exists x\, \psi(x) \iff \forall q \le p\ \exists r \le q\ \exists \xi,\ r \Vdash \psi(\xi)$.

As an example let us check (iv):

(⇒) Assume $p \Vdash \neg\varphi$, i.e. by the definition $\cancel{\forall q \le p,\ q \not\Vdash \varphi}$. $p \le [\![\neg\varphi]\!]$.

~~Any~~ Let $q \le p$ and assume $q \Vdash \varphi$, i.e. $q \le [\![\varphi]\!]$. But by (i) also $q \le [\![\neg\varphi]\!]$ and thus $q \le [\![\varphi]\!] \wedge [\![\neg\varphi]\!] = 0_B$. This is a contradiction as $0_B \notin P$.

(⇐) Assume $p \not\Vdash \neg\varphi$, i.e. $p \not\le [\![\neg\varphi]\!]$. Then $q := p \wedge [\![\varphi]\!]$ satisfies $q \le p$ and $q \le [\![\varphi]\!]$, i.e. $q \Vdash \varphi$.

It remains to check that $q \in P$, i.e. $q \ne 0_B$. But if $q = 0_B$ then we can calculate:

$$p \wedge [\![\neg\varphi]\!] = p \wedge (1_B - [\![\varphi]\!]) =$$

$$p - (p \wedge [\![\varphi]\!]) = p - 0_B = p.$$

So $p \Vdash \neg\varphi$ contradicting the hypothesis.

q.e.d.

(6) Now we turn the table around. Assume $(P, \leq)$ is any p. ordering and $\Vdash \subseteq P \times SENT$ is a relation satisfying the six conditions in (5).

First note that forcing of atomic sentences determines the whole relation.

<u>Claim</u>: Let $\Vdash$ and $\Vdash'$ be two forcing relations s.t. for all $p \in P$ and all atomic $\theta \in SENT$

$$p \Vdash \theta \iff p \Vdash' \theta.$$

Then $\Vdash = \Vdash'$.

$\square$

(7) One can thus think of forcing conditions as specifying a part of the (atomic) diagram of an unknown structure, and if $q \le p$ then q specifies more (by 5(ii)) than p, i.e. it is "stronger".

There is a way how to extract from $(P, \le)$ a structure using the notion of a <u>generic</u> set $G \subseteq P$. It has properties implying that

$$\exists p \in G, p \Vdash \varphi$$

satisfies Tarski's condition on model-th. satisfiability $\models$. The issue in constructing G in the language underlying SET: it has to be <u>countable</u> and have <u>enough names</u> for elements in order to be able to witness $\exists$ statements.

<u>Remark</u>: There are many interpretations in model-th. what forcing is (e.g. game-theoretic as in W. Hodges: Building models by games).

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