Exercises for o-minimality Lectures I and II

MSRI Graduate Model Theory Workshop

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Real Closed Fields

Exercise 1

Let x and y be algebraically independent over $\mathbb{R}$.

(a) Show that $\mathbb{R}(x, y)$ is formally real and that we can find orders $<_1$ and $<_2$ of $\mathbb{R}(x, y)$ such that $x <_1 y$ and $y <_2 x$.

(b) Use (a) to show that the ordering $<$ is not quantifier-free definable in $\mathbb{R}$ in the language of rings.

Solution:

(a) Suppose $0 = \sum_{i=1}^{n} (\frac{f_i(x,y)}{g_i(x,y)})^2$ where $f_i, g_i \in \mathbb{R}[x, y]$, $g_i \neq 0$. So if we let $G_i(x, y) = \prod_{1 \leq j \leq n, i \neq j} g_i(x, y)$, then $0 = \sum_{i=1}^{n} (G_i(x, y) f_i(x, y))^2$, contradicting that x, y are algebraically independent.

Note that $(x - y)$ is not a sum of squares in $\mathbb{R}(x, y)$, since we would get $x - y = \sum_{i=1}^{n} (\frac{f_i(x,y)}{g_i(x,y)})^2$ where $f_i, g_i \in \mathbb{R}[x, y]$, $g_i \neq 0$. So if $G_i(x, y) = \prod_{1 \leq j \leq n, i \neq j} g_i(x, y)$, $0 = \sum_{i=1}^{n} G_i^2(x, y) f_i^2(x, y) - (x - y) \prod_{i=1}^{n} g_i^2(x, y)$, contradicting that x, y are algebraically independent. Similarly, $(y - x)$ is not a sum of squares in $\mathbb{R}(x, y)$.

Proposition 1. *If F is formally real and $a \in F$, $-a$ not a sum of squares in F, then there is an ordering of F where $a > 0$.*

Proof. Let $\sum F^2$ denote the sums of squares in F, and note that if $x \in \sum F^2$, then $x \geq 0$ in any ordering of F.

Also note that if $-a$ is not a sum of squares, $a \neq 0$.

If $\sqrt{a} \in F$, then $a \in F^2 \subset \sum F^2$, so in any ordering, $a > 0$.

We will show that $F(\sqrt{a})$ is formally real. Then, since in this field a is a square, any ordering will be such that $a > 0$, so we get the desired ordering of F by restricting this ordering to F.

Suppose $-1 = \sum(b_i + c_i\sqrt{a})^2$. Then $0 = \sum b_i^2 + 1 + \sum c_i^2 a + \sqrt{a}\sum 2b_ic_i$. So since $1, \sqrt{a}$ are a basis of $F(\sqrt{a})$, we must have $\sum b_i^2 + 1 + \sum c_i^2 a = 0$ and $\sum 2b_ic_i = 0$. In particular, $-a = \frac{\sum b_i^2 + 1}{\sum c_i^2} = \frac{\sum b_i^2 \sum c_i^2}{(\sum c_i^2)^2} + \frac{\sum c_i^2}{(\sum c_i^2)^2}$ which is the sum of squares $\Rightarrow\Leftarrow$.

Thus, $F(\sqrt{a})$ is formally real, as required.

$\square$

Hence, we get an ordering $<_1$ in which $y - x >_1 0$ so $x <_1 y$ and an ordering $<_2$ in which $x - y >_2 0$, so $y <_2 x$.

(b) Let ϕ be a quantifier free formula in the language of rings such that for all $a, b \in \mathbb{R}$, $a < b \Leftrightarrow \mathbb{R} \vDash \phi(a, b)$. So $(\mathbb{R}, <) \vDash \forall a, b\phi(a, b) \leftrightarrow a < b$. Let $\mathcal{M}_i$ be the real algebraic closure of $(\mathbb{R}(x, y), <_i)$ for $i = 1, 2$ (since $(\mathbb{R}(x, y), <_i)$ is formally real by part (a), and thus, has a real algebraic closure whose order extends $<_i$). Since $(\mathbb{R}, <) \subset \mathcal{M}_i$, by model completeness of RCF, $(\mathbb{R}, <) \preceq \mathcal{M}_i$. So $\mathcal{M}_i \vDash \forall a, b\phi(a, b) \leftrightarrow a <_i b$. So $\mathcal{M}_1 \vDash \phi(x, y)$ and $\mathcal{M}_2 \vDash \neg\phi(x, y)$. Now consider $\mathcal{M}_1$ and $\mathcal{M}_2$ in the language of rings. $\mathbb{R}(x, y) \subset \mathcal{M}_1, \mathcal{M}_2$, so since ϕ is quantifier free and in the language of rings, $\mathbb{R}(x, y) \vDash \phi(x, y)$ and $\mathbb{R}(x, y) \vDash \neg\phi(x, y)$. $\Rightarrow\Leftarrow$

So $<$ is not quantifier free definable in $\mathbb{R}$ in the language of rings.

Exercise 2

Let F be a real closed field. We say that a function $g : F^n \to F$ is *algebraic* if there is a nonzero polynomial $p(X_1, \ldots, X_n, Y)$ over F such that for all $a \in F^n$, $p(a, g(a)) = 0$.

(a) Use quantifier elimination to show that every semialgebraic function is algebraic.

(b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is semialgebraic, then there are disjoint intervals $I_1, \ldots, I_m$ and a finite set X such that $\mathbb{R} = I_1 \cup \ldots \cup I_n \cup X$ and f is analytic on each I_j. (Hint: Use the Implicit Function Theorem for $\mathbb{R}$.)

Solution:

(a) Let $g : F^n \to F$ be semialgebraic. So $\{(\overline{x}, y) | g(\overline{x}) = y\} = \{(\overline{x}, y) | \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} p_{ij}(\overline{x}, y) = 0 \wedge q_{ij}(\overline{x}, y) > 0\}$ for some polynomials $p_{ij}, q_{ij} \in F[\overline{x}, y]$, $1 \leq i \leq m$, $1 \leq j \leq n$. Suppose for some i that $p_{ij} \equiv 0$ for all $j = 1..n$. Then $\{(\overline{x}, y) | \bigwedge_{j=1}^{n} q_{ij}(\overline{x}, y) > 0\} \subset$ *graph*(g). If this is empty, we can remove this whole disjunct for an equivalent defining formula. Otherwise, let $(\overline{x}, y)$ be such that $\bigwedge_{j=1}^{n} q_{ij}(\overline{x}, y) > 0$. Then, since the $q_{ij}'s$ are polynomials, they are continuous, so by choosing y' sufficiently close to but $\neq y$, we get

$q_{ij}(\overline{x},y')$ between 0 and $q_{ij}(\overline{x},y)$, so $\bigwedge\limits_{j=1}^{n} q_{ij}(\overline{x},y') > 0$. Thus, $(\overline{x},y),(\overline{x},y') \in graph(g)$, contradicting that g is a function. So, let $p(\overline{x},y) = \prod\limits_{i=1}^{m}\sum\limits_{j=1}^{n}(p_{ij}(\overline{x},y))^2$. Then for all $\overline{x}$, $p(\overline{x},g(\overline{x})) = 0$, since for some i, $\bigwedge_{j=1}^{n} p_{ij}(\overline{x},g(\overline{x})) = 0$, so $\sum\limits_{j=1}^{n}(p_{ij}(\overline{x},g(\overline{x})))^2 = 0$.

(b) By part (a), there is a polynomial p such that $p(x,f(x)) = 0$ for all x. $p = p_1 \cdot \ldots \cdot p_m$ where $p_1,\ldots,p_m$ are irreducible polynomials and wlog, are all distinct (since if we have a repeated root, if we divide by it, the result will still witness that f is algebraic). Thus, since they are irreducible, $p_i \nmid p_j$ for $1 \leq i \neq j \leq m$.

Let $X = \{a \in \mathbb{R} | \frac{\partial p}{\partial y}(a,f(a)) = 0\}$. We will show that X is finite.

Note, if $a \in X$, then $(a,f(a)) \in p \cap \frac{\partial p}{\partial y}$, so it will be enough to show that this is finite.

$\frac{\partial p}{\partial y} = \sum\limits_{i=1}^{m}(\prod\limits_{1\leq j\leq m, j\neq i} p_j)(\frac{\partial p_i}{\partial y})$. Since the degree of $\frac{\partial p_i}{\partial y}$ is strictly less than that of p_i, $p_i \nmid \frac{\partial p_i}{\partial y}$, and $\frac{\partial p_i}{\partial y} \nmid p_i$ since it is irreducible.

$p \cap \frac{\partial p}{\partial y} \subset \bigcup\limits_{i=1}^{m}(p_i \cap \frac{\partial p}{\partial y})$, so it is enough to show that each of these are finite. $\frac{\partial p}{\partial y} = \sum\limits_{j\neq i}(\prod\limits_{k\neq j} p_k)(\frac{\partial p_j}{\partial y}) + (\prod\limits_{k\neq i} p_k)(\frac{\partial p_i}{\partial y})$. p_i divides the first summand, but not the second, since $p_i \nmid p_k$ for $k \neq i$ and $p_i \nmid \frac{\partial p_i}{\partial y}$, and p_i is irreducible. $\frac{\partial p}{\partial y} \nmid p_i$, since then it would divide p, which would give us a repeated factor.

Thus, by Bezout's Theorem, $p_i \cap \frac{\partial p}{\partial y}$ is finite for each $1 \leq i \leq m$. Hence, X is finite.

So $X = \{a_1,\ldots,a_{m-1}\}$ with $a_1 < \ldots < a_m$. Let $a_0 = -\infty$, $a_m = \infty$, and $I_j = (a_{j-1},a_j)$ for $1 \leq j \leq m$. That is, $\mathbb{R} = I_1 \cup \ldots \cup I_m \cup X$.

We will show, using the implicit function theorem, that f is analytic on each I_j.

Theorem 2 (Implicit Function Theorem). *Suppose $F : W \to \mathbb{R}$ is analytic, $W \subset \mathbb{R}^2$ open, $(a,b) \in W$ such that $F(a,b) = 0 \neq \frac{\partial F}{\partial y}(a,b)$. Then there is a unique analytic function $\phi : \mathcal{U} \to \mathbb{R}$ for $\mathcal{U} \subset \Pi_1(W)$ open such that $F(x,\phi(x)) = 0$ and $\phi(a) = b$.*

Let $W = I_j \times \mathbb{R}$. So W is open in $\mathbb{R}^2$. Then $p : W \to \mathbb{R}$ is analytic since it is a polynomial. Then pick any $a \in I_j$, let $b = f(a)$, so $(a,b) \in W$, $p(a,b) = 0 \neq \frac{\partial p}{\partial y}(a,b)$. Then there is a unique analytic function $\phi : I_j \to \mathbb{R}$ such that $p(x,\phi(x)) = 0$ and $\phi(a) = b$. Thus, $f \equiv \phi$ on I_j, so f must be analytic on I_j, as required.

Exercise 3*

(Real Nullstellensatz) Let F be a real closed field and let $J \subset F[X_1,\ldots,X_n]$ be an ideal. We say that J is *real* if for any $p_1,\ldots,p_m \in F[X_1,\ldots,X_n]$ such that $\sum p_i^2 \in J$, then $p_i \in J$ for $1 \leq i \leq m$. Show that $I(V(J)) = J$ if and only if J is real.

Solution:

$\Rightarrow$: Let $\displaystyle\sum_{i=1}^{m} p_i^2 \in J$. If $V(J) = \emptyset$, then $J = I(V(J)) = F[X_1, \ldots, X_n]$, so J is real. Otherwise, let $x \in V(J)$.

Lemma 3. *A field F is formally real if and only if for all $a_1, \ldots, a_m \in F$, $\displaystyle\sum_{i=1}^{m} a_i^2 = 0 \Rightarrow a_i = 0\ \forall 1 \leq i \leq m$.*

Proof. $\Leftarrow$: If $m = 1$, then $a_1^2 = 0 \Rightarrow a_1 = 0$. Otherwise, suppose there is i such that $a_i^2 \neq 0$. Then $\sum 1 \leq j \leq m, j \neq i (\dfrac{a_j}{a_i})^2 = -1$. $\Rightarrow\Leftarrow$

$\Rightarrow$: Suppose $\displaystyle\sum_{i=1}^{m} a_i^2 = -1$ for some $a_1 \ldots a_n \in F$. Then $\displaystyle\sum_{i=1}^{m} a_i^2 + 1 = 0 \Rightarrow 1 = 0$. $\Rightarrow\Leftarrow$

So F is formally real.

$\square$

Suppose for some $p_1, \ldots, p_m$, $\displaystyle\sum_{i=1}^{m} p_i^2 \in J$. Then $\displaystyle\sum_{i=1}^{m} (p_i(x))^2 = 0$, so by the Lemma, $p_i(x) = 0$ for all $1 \leq i \leq m$. Thus, $p_i \in I(V(J)) = J$.

$\Leftarrow$: We will need a few lemmas for the other direction.

Lemma 4. *If P is a real prime ideal of $F[X_1, \ldots, X_n]$, then if K is the field of fractions of $F[X_1, \ldots, X_n]/P$, K is formally real.*

Proof. Let $\displaystyle\sum_{i=1}^{m} (\frac{a_i + P}{b_i + P})^2 = P$ where $a_i, b_i \in F[X_1, \ldots, X_n]$, $b_i \notin P$, (so a sum of squares in K which is equal to 0). Let $c_i = \displaystyle\prod_{1 \leq j \leq m, j \neq i} b_j + P$. Then $\displaystyle\sum_{i=1}^{m} (c_i a_i + P)^2 = P$. So $\displaystyle\sum_{i=1}^{m} (c_i a_i)^2 \in P$, so $c_i a_i \in P$. We know that $c_i \notin P$, or else we would get some $b_j \in P$ since P is prime. So we must have $a_i \in P$. Thus, $a_i + P = P$, so $\frac{a_i+P}{b_i+P} = P$, that is, 0 in K.

Thus, by Lemma 3, K is formally real.

$\square$

Lemma 5. *If $J = \displaystyle\bigcap_{i=1}^{m} P_i$ where the P_i's are prime and J is real, then each P_i is real.*

Proof. Let $P_1, \ldots, P_m$ be such that $J = \displaystyle\bigcap_{i=1}^{m} P_i$, and no $P_i \subset P_j$ for $i \neq j$. If $m = 1$, then $P_1 = J$ is real. If not, consider P_i and let $c_j \in P_j \setminus P_i$ for all $j \neq i$, $c = \displaystyle\prod_{1 \leq j \leq m, i \neq j} c_j$. Let $\displaystyle\sum_{k=1}^{q} a_k^2 \in P_i$. Then $c^2 \displaystyle\sum_{k=1}^{q} a_k^2 = \sum_{k=1}^{q} (c a_k)^2$. Since, $\displaystyle\sum_{k=1}^{q} a_k^2 \in P_i$, this is in P_i, and since $c^2 \in P_j$ for all $j \neq i$, it is in P_j. Thus, $\displaystyle\sum_{k=1}^{q} (c a_k)^2 \in \bigcap_{i=1}^{m} P_i = J$. So since J is real, $c a_k \in J$, and thus, $c a_k \in P_i$ for each $1 \leq k \leq q$. So since P_i is prime, $c \in P_i$ or $a_k \in P_i$. But if $c \in P_i$ then

$c_j \in P_i$ for some $j \neq i$. Thus, we must have $a_k \in P_i$ for all $1 \leq k \leq q$. Thus, each P_i is real. $\square$

Since J is real, it is radical: let $f \in \sqrt{J}$, and n be such that $f^n \in J$. Let m be such that $m + n = 2^k$ for some k. then $f^m f^n \in J$, so $f^{2^k} \in J$. Thus, since J is real, $f \in J$.

So by the primary decomposition theorem, $J = \bigcup_{i=1}^{m} P_i$ for some prime ideals $P_1, \ldots, P_m \in F[X_1, \ldots, X_n]$. And by Lemma 5, each of these are real ideals.

Clearly, $J \subset I(V(J))$, so let $f \in I(V(J))$. To show that $f \in J$, we will show that $f \in P_i$ for each i. Since F is a field, $F[X_1, \ldots, X_n]$ is Noetherian, so $P_i = \langle g_1, \ldots, g_k \rangle$. For any $\overline{v}$, if$g_1(\overline{v}) = \ldots = g_k(\overline{v}) = 0$, then $\overline{v} \in V(J)$. So since $f \in I(V(J))$, $f(\overline{v}) = 0$. Thus, $F \models \forall \overline{v}(\bigwedge g_i(\overline{v}) = 0 \to f(\overline{v}) = 0)$. Let K be the field of fractions of $F[X_1, \ldots, X_n]/P$. K is formally real by Lemma 4, so let L be its real algebraic closure. $F \subset L$, so by model completeness, $F \preceq L$, so $L \models \forall \overline{v}(\bigwedge g_i(\overline{v}) = 0 \to f(\overline{v}) = 0)$. $L \models \bigwedge g_i(X_1/P_i, \ldots, X_n/P_i) = 0$ since $g_i \in P_i$, so $L \models f(X_1/P_i, \ldots, X_n/P_i) = 0$. Thus, $f \in P_i$.

Note: the original exercise said: Let F be a real closed field, and let P be a prime ideal in $F[X_1, \ldots, X_n]$. Then, there is $x \in F^n$ with $f(x) = 0$ for all $f \in P$ if and only if whenever $p_1, \ldots, p_m \in F[X_1, \ldots, X_n]$ and $\sum p_i^2 \in P$, then all the $p_i \in P$.

This has a counterexample: Consider the ideal $(x^2 + y^2) \subset \mathbb{R}[x, y]$. This is prime since $x^2 + y^2$ is irreducible over $\mathbb{R}$. $(0, 0) \in V((x^2 + y^2))$. But $x, y \notin (x^2 + y^2)$.

Exercise 4

Prove that for all n and d there are M and D such that if $f(X_1, \ldots, X_n) = \frac{g}{h}$ where g and h are real polynomials of degree at most d and f is positive semidefinite, then there are polynomials $g_1, \ldots, g_M, h_1, \ldots, h_M$ of degree at most D such that

$$f = \sum_{i=1}^{M} \frac{g_i^2}{h_i^2}.$$

Solution:

Let $R \models RCF$ and suppose not. Then let n, d be such that for polynomials g, h in n variables of degree at most d such that $\frac{g}{h}$ is positive semidefinite, for all M, D, $g_1, h_1, \ldots, g_M, h_M$ polynomials in n variables of degree at most D, there is $\overline{x}$ such that $\frac{g(\overline{x})}{h(\overline{x})} \neq \sum_{i=1}^{M} \frac{(g_i(\overline{x}))^2}{(h_i(\overline{x}))^2}$.

For M, D, let $\Phi_{M,D}(\overline{a}, \overline{b})$ be a formula expressing the following: If $\overline{a}, \overline{b}$ are coefficients of polynomials $a(\overline{x})$ and $b(\overline{x})$ in n variables of degree at most d such that for all $\overline{x}$, $\frac{a(\overline{x})}{b(\overline{x})} \geq 0$ then for all $\overline{c_1}, \ldots, \overline{c_M}, \overline{d_1}, \ldots, \overline{d_M}$ coefficients of polynomials $c_1(\overline{x}), \ldots, c_M(\overline{x}), d_1(\overline{x}), \ldots, d_M(\overline{x})$ in n variables of degree at most D, there exists x such that $\frac{a(\overline{x})}{b(\overline{x})} \neq \sum_{i=1}^{M} \frac{(c_i(\overline{x}))^2}{(d_i(\overline{x}))^2}$.

Then, by assumption, $\{\Phi_{M,D}(\overline{a}, \overline{b}) | M, D \geq 0\}$ is finitely satisfiable.

So let $S \succeq R$ have realizations $\overline{a}, \overline{b}$ of this. Then, $S \models RCF$, and the corresponding $\frac{a(\overline{x})}{b(\overline{x})}$ is a positive semi-definite functions which cannot be expressed as a sum of squares. This contradicts Hilbert's 17$^{\text{th}}$ Problem.

Theorem 6 (Hilbert's 17^{th} Problem)**.** *If R is a real closed field, $\overline{x} \in R^n$, and $f \in R(\overline{x})$ is positive semidefinite, then there are $g_1, \ldots, g_m \in R(\overline{x})$ such that $f = \sum_{i=1}^{m} g_i^2$.*

Exercise 5

If K is a field, let $K[[t]]$ denote the field of formal power series over K in variable t, and let $K((t))$ denote its fraction field, the field of formal Laurent series over K. Let

$$K\langle\langle t\rangle\rangle = \bigcup_{n=1}^{\infty}((t^{\frac{1}{n}}))$$

be the field of formal *Puiseux series* over K. Series in $K\langle\langle t\rangle\rangle$ are of the form $\sum_{i=m}^{\infty} a_i t^{\frac{i}{n}}$ for some $m, n \in \mathbb{Z}$ with $n > 0$. An important theorem is that if K is algebraically closed, then $K\langle\langle t\rangle\rangle$ is also algebraically closed. It follows that if R is real closed then $R\langle\langle t\rangle\rangle$ is real closed.

(a) Show that $R \prec R\langle\langle t\rangle\rangle$, and t is a positive infinitesimal element of $R\langle\langle t\rangle\rangle$.

(b) Suppose that $r \in R$ and $f : (0, r) \to R$ is definable. Show that there is $\mu \in R\langle\langle t\rangle\rangle$ such that $R\langle\langle t\rangle\rangle \vDash f(t) = \mu$. Suppose that $\mu = at^q +$ higher-degree terms. Show that f is asymptotic to ax^q at 0. In other words, show that

$$R \vDash \forall\epsilon > 0 \exists\delta > 0 (0 < x < \delta \to |\frac{f(x)}{ax^q} - 1| < \epsilon).$$

.

Solution:

(a) Since $R, R\langle\langle t\rangle\rangle \vDash RCF$, by model completeness, since $R \subset R\langle\langle t\rangle\rangle$, $R \prec R\langle\langle t\rangle\rangle$.

We know that $t > 0$ in $R\langle\langle t\rangle\rangle$ since $t = (t^{\frac{1}{2}})^2$, and $t \neq 0$. Let $r \in R$, $r \neq 0$ be given. $r^2 - t = (\sum_{n=0}^{\infty}(-1)^n(r^{1-2n}\binom{\frac{1}{2}}{n})t^n)^2$ (by taking the Taylor expansion of $\sqrt{r^2 - t}$). So since their difference is a non-zero square in $R\langle\langle t\rangle\rangle$, $r^2 > t$.

(b) Let $f : (0, r) \to R$ be defined by $\phi(a, b)$. That is, $f(a) = b \Leftrightarrow R \vDash \phi(a, b)$. So $R \vDash \forall a(0 < a < r \to \exists! b(\phi(a, b)))$, and by elementarity, $R\langle\langle t\rangle\rangle \vDash \forall a(0 < a < r \to \exists! b(\phi(a, b)))$. Thus, since $R\langle\langle t\rangle\rangle \vDash 0 < t < r$ by part (a), $R\langle\langle t\rangle\rangle \vDash \exists! \mu(f(t) = \mu)$.

Then, note that $\frac{\mu}{ax^q} = 1 + g(x)$ where each power of x is strictly positive, since every term after ax^q in μ has degree strictly greater than q. Thus, as $x \to 0$, $g(x) \to 0$. So for any $\epsilon > 0$, we can choose $\delta > 0$ such that $0 < x < \delta \Rightarrow |\frac{\mu}{at^q} - 1| < \epsilon$. So, since $R\langle\langle t\rangle\rangle \vDash \mu = f(t)$, $R\langle\langle t\rangle\rangle \vDash |\frac{f(t)}{at^q} - 1| < \epsilon$. Thus, since $R \preceq R\langle\langle t\rangle\rangle$, and $\epsilon > 0$ was arbitrary, $R \vDash \forall\epsilon > 0 \exists\delta > 0 (0 < x < \delta \to |\frac{f(x)}{ax^q} - 1| < \epsilon)$.

Basic o-minimality

Exercise 15*

Suppose $(M, <, \ldots)$ is o-minimal, $a < b \in M$ and $f : (a, b) \to M$ is strictly increasing. Prove that $f|I$ is continuous for some interval $I \subset (a, b)$.

Solution:

Since f is injective and (a, b) is infinite, $f((a, b))$ is infinite, so let $(r, s) \subset f((a, b))$ be an interval. Again, since f is injective, there are unique $c, d \in (a, b)$ such that $f(c) = r$ and $f(d) = s$. So, since f is strictly increasing, $c < d$ and $f((c, d)) = (r, s)$. So let $(u, v) \subset (r, s)$ be an interval. Then there must be e, g with $c < e < g < d$ such that $f(e) = u$ and $f(g) = v$. For any $t \in (u, v)$, $f(e) < t < f(g)$, so for (the unique) $h \in (a, b)$ such that $f(h) = t$, $e < h < g$. So $f^{-1}((u, v)) = (e, g)$ is open. Thus, f is continuous on (r, s).

The problem originally said to prove that f is continuous on (a, b), which is not always true. This is Lemma 3 in Chapter 3 of van den Dries. It is also the exercise from the o-minimality lecture on Tuesday 7/24.

However, we can further prove that the number of points at which f is not continuous is finite.

Let $\phi(x)$ be $\forall c, d(c < f(x) < d \to \exists r, s(r < x < s \wedge \forall v(r < v < s \to c < f(v) < d)))$. Let $X = \{x \in (a, b) | f \text{ is not continuous}\} = \{x | \neg\phi(x)\}$. X is definable, so by o-minimality, if X is infinite, then there is some interval $J \subset X$. But then $f : J \to M$ is strictly increasing, so by the above argument, there is an interval $I \subset J$ on which f is continuous $\Rightarrow\Leftarrow$, since $I \subset X$. So X must be finite.

Definable Closure and Exchange

Exercise 18

[**Exchange**] Suppose $c \in dcl(A \cup \{b\})$. Then $c \in dcl(A)$ or $b \in dcl(A \cup \{c\})$.

Solution:

Let $c \in dcl(A \cup \{b\})$. Let $\phi(\overline{x}, y, z)$ be such that $\{c\} = \{x | \phi(\overline{a}, b, x)\}$ for some $\overline{a} \in A$. Let $B = \{y | \phi(\overline{a}, y, c) \wedge \exists! x \phi(\overline{a}, y, x)\}$.

Lemma 7. *If a set D is definable over some C and $x \in \partial D$ (the boundary of D), then $x \in dcl(C)$.*

Proof. Since D is definable, by o-minimality, $D = I_1 \cup \ldots \cup I_k \cup X$ where $I_1, \ldots, I_k$ are open intervals and X is finite. So $\partial D = X \cup$ the set of (non-infinite) endpoints of $I_1, \ldots, I_k$. Let $\partial D = \{b_1, \ldots, b_n\}$ with $b_1 < \ldots < b_n$. Let $x = b_i$, and $\psi(y) = \exists b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n(b_1 < b_2 < \ldots < b_{i-1} < y < b_{i+1} < \ldots < b_n \wedge \bigwedge_{1 \leq j \leq m, j \neq i} b_j \in \partial D \wedge y \in \partial D$. Then $\mathcal{M} \vDash \psi(y) \leftrightarrow y = x$. So $x \in dcl(C)$. □

So if $b \in \partial B$, then $b \in dcl(A \cup \{c\})$.

If b is not on the boundary of B, then $b \in I \subset B$ for some interval I. Let $\theta(y) = \exists! x \phi(\overline{a}, y, x)$, and let Y be the set defined by θ. Note that $I \subset B \subset Y$.

Define $f : Y \to M$ by $f(y) = x$ (the unique x guaranteed to exists by θ). f is an A-definable function on I (since $I \subset Y$), and $f \equiv c$ on I, since $I \subset B$, so for $y \in I$, $\phi(\overline{a}, y, c)$.

Let $\psi(x) = \exists u < v \forall u < y < v f(y) = x$. That is, there is an interval on which f has the constant value x.

Claim 1. *$\psi(Y)$ is finite.*

Proof. By the Monotonicity theorem, $Y = I_1 \cup \ldots \cup I_m \cup X$ where X is finite, I_i's are intervals, and f is either strictly monotone or constant on each interval. So if x is such that f is constantly x on some interval J, it must be on the whole I_i in which J is contained. Thus, there can only be m many such x's, so $\psi(Y)$ is finite. □

So let $\psi(Y) = \{a_1, \ldots, a_m\}$. So since $c \in \psi(Y)$, $c = a_i$ for some $1 \leq i \leq m$.

By Lemma 7, since $c \in \partial(\psi(Y))$ which is definable over A, $c \in dcl(A)$

Consequences of Cell Decomposition

Exercise 21

Suppose $\mathcal{M}$ is o-minimal and $\mathcal{N}$ is elementarily equivalent to $\mathcal{M}$. Prove that $\mathcal{N}$ is o-minimal.
Solution:

Let $S \subset \mathcal{N}$ be definable and let $\phi(\overline{x}, y)$ and $\overline{a} \in N^m$ be such that $S = \{y \in N | \mathcal{N} \models \phi(\overline{a}, y)\}$. We want to show that S is a finite union of intervals and points.

Let $A = \{(\overline{r}, y) | \phi(\overline{r}, y)\}$. So for each $\overline{r} \in M^m$, $A_{\overline{r}}$ is definable.

Let $B = \{(\overline{r}, c, d) | (c, d) \subset A_{\overline{r}} \wedge \forall (e, f) \supset (c, d), (e, f) \not\subset A_{\overline{r}}\}$. (We can express this in a first order way.)

So $B_{\overline{r}}$ is the set of (c, d) which are disjoint intervals contained in $A_{\overline{r}}$. By o-minimality, $B_{\overline{r}}$ is finite for all $\overline{r} \in M^m$. So by Uniform Bounding, there is N such that $|B_{\overline{r}}| < N$ for all $\overline{r} \in M^m$.

Similarly, let $C = \{(\overline{r}, c) | c \in A_{\overline{r}} \wedge \forall (e, f) \ni c, (e, f) \not\subset A_{\overline{r}}\}$. Again, by o-minimality, since $C_{\overline{r}}$ is the set of isolated points in $A_{\overline{r}}$, $C_{\overline{r}}$ is finite, so by Uniform Bounding, there is M such that $|C_{\overline{r}}| < M$ for all $\overline{r} \in M^m$.

Let $\theta_1(\overline{r}, c)$ be $(-\infty, c) \subset A_{\overline{r}} \wedge \forall d > c (-\infty, d) \not\subset A_{\overline{r}}$ and $\theta_2(\overline{r}, d)$ be $(d, \infty) \subset A_{\overline{r}} \wedge \forall c < d (c, \infty) \not\subset A_{\overline{r}}$. So

$\mathcal{M} \models \forall \overline{r} \exists c_1 < d_1 < \ldots < c_N < d_N \exists x_1, \ldots, x_M$
$(A_{\overline{r}} = (c_1, d_1) \cup \ldots \cup (c_N, d_N) \cup \{x_1, \ldots, x_M\})$
$\vee (\exists c \theta_1(c) \wedge A_{\overline{r}} = (-\infty, c) \cup (c_1, d_1) \cup \ldots \cup (c_N, d_N) \cup \{x_1, \ldots, x_M\})$
$\vee (\exists d \theta_2(d) \wedge A_{\overline{r}} = (c_1, d_1) \cup \ldots \cup (c_N, d_N) \cup (d, \infty) \cup \{x_1, \ldots, x_M\})$
$\vee (\exists c, d \theta_1(c) \wedge \theta_2(d) \wedge A_{\overline{r}} = (-\infty, c) \cup (c_1, d_1) \cup \ldots \cup (c_N, d_N) \cup (d, \infty) \cup \{x_1, \ldots, x_M\})$.

Since $\mathcal{N} \equiv \mathcal{M}$, $\mathcal{N}$ satisfies this as well. Thus, since $S = A_{\overline{a}}$, S is a finite union of points and intervals (c, d) with $c, d \in N \cup \{\pm\infty\}$.