Getting an ordinary 2-valued structures

(1) General situation:

$\mathcal{L}$: a countable lang.

$\mathbb{B}$: a complete Bool. alg.

$\mathbb{A}$: a $\mathbb{B}$-valued $\mathcal{L}$-structure

H : a $\mathcal{L}$-sentence of special interest (e.g. the CH in ours case).

W.l.o.g. we assume that all parameters from $\mathbb{A}$ occuring in H have names in $\mathcal{L}$.

(2) For each $\mathcal{L}$-fla $A(x_1, \ldots, x_n)$ introduce a f. symbol $val_A(x_1, \ldots, x_n)$ and interpret it by the map:

$$val_A : \vec{\xi} \in \mathbb{A}^n \longrightarrow [\![A(\vec{\xi})]\!] \in \mathbb{B}$$

Note that there are countably many of these new symbols.

(3) Define a "combined structure" $\mathbb{C}$:

$$(\mathbb{A}, \mathbb{B}, val_A, \ldots\ldots)$$

with one sort for $\mathbb{A}$, another for $\mathbb{B}$. $\mathbb{C}$ is a first-order ordinary (i.e. 2-valued) structure.

Note that the lang. of $\mathbb{C}$ (i.e. L, the lang. of $\mathbb{B}$ plus all val_A) is countable.

(4) Apply to $\mathbb{C}$ the Löwenheim-Skolem thm to get its countable elementary substructure $\mathbb{C}' \preccurlyeq \mathbb{C}$:

$$\mathbb{C}' : (\mathbb{A}', \mathbb{B}', \mathrm{val}_A, \ldots)$$

Claim: The truth evaluation of sentences over $\mathbb{A}'$ with values in $\mathbb{B}'$ defined by:

$$[\![A(\xi)]\!]' := \mathrm{val}_A ([\![A(\xi)]\!])$$

for $\xi_i \in \mathbb{A}'$, satisfies all conditions required.

Proof: Let us check e.g. the condition about $\exists$: we want:

$$[\![\exists y\, B(\xi, y)]\!]' = \sup_{\eta \in \mathbb{A}'} [\![B(\xi, \eta)]\!]'.$$

This can be expressed using the language with val-symbols by the sentences:

$$\forall x \forall y,\ val_{\exists y B(x,y)}(x) \geq val_{B(x,y)}(x,y)$$

and

$$\forall b \forall x \left(val_{\exists y B(x,y)}(x) > b \rightarrow \exists y\ val_{B(x,y)}(x,y) \geq b \right)$$

where x, y range over $\mathbb{A}$'-sort and b over $\mathbb{B}$'-sort.

Both these sentences are true in $\mathcal{C}$ and hence also in $\mathcal{C}'$.

$\square$ claim.

(5) We shall need the (corollary of the) Rasiowa-Sikorski thm: Let $\mathbb{B}$ be a complete Bool. algebra, $\mathcal{U} \subseteq \mathcal{P}(\mathbb{B})$ a countable collection of its subsets. Then there exists a homomorphism

$$h : \mathbb{B} \longrightarrow \{0,1\} \ (= \text{the 2-element alg.})$$

s.t. for all $U \in \mathcal{U}$:

$$(*) \quad h(\sup U) = \sup \{h(u) \mid u \in U\}.$$

(6) We shall apply it to $\mathbb{B}'$ and to the following collection $\mathcal{U} \subseteq \mathcal{P}(\mathbb{B}')$:

– for each L-fla $B(\bar{x}, y)$ and each n-tuple $\bar{\xi}$ of parameters from $\mathbb{A}'$ include in $\mathcal{U}$ the set

$$\mathcal{U}_{B(\bar{\xi}, y)} := \{ [\![B(\bar{\xi}, \eta)]\!]' \mid \eta \in \mathbb{A}' \}.$$

Note that ~~(...) for (...) for $\mathcal{U}_{B(\bar{\xi},y)}$~~

$$\sup \mathcal{U}_{B(\bar{\xi}, y)} = [\![\exists y\, B(\bar{\xi}, y)]\!]'.$$

Let $h : \mathbb{B}' \to \{0, 1\}$ be a homomorphism guaranteed to exist by the R.-S. thm.

(7) Define an ordinary (2-valued) L-structure $\mathbb{A}^*$ by:

– factor $\mathbb{A}'$ by the equiv. rel. $\approx$:

$$\xi_1 \approx \xi_2 \overset{df}{\iff} h([\![\xi_1 = \xi_2]\!]') = 1$$

– define for each atomic $\mathcal{L}$-sentence A with parameters from $\mathbb{A}^*$:

$$\mathbb{A}^* \models A \underset{\text{df.}}{\iff} h(\llbracket A \rrbracket') = 1$$

Remark: Elmt's of $\mathbb{A}^*$ are blocks of $\approx$ and we represent them in the right-hand side by any representant of the respective blocks.

(8) Claim: For any $\mathcal{L}$-sentence B with parameters from $\mathbb{A}^*$: $\mathbb{A}^* \models B \iff h(\llbracket B \rrbracket') = 1$.

In particular,

(i) If $\llbracket B \rrbracket = 1_{\mathbb{B}}$ in $\mathbb{A}$ then $\mathbb{A}^* \models B$.

(Hence any theory $\mathbb{B}$-valid in $\mathbb{A}$ is true in $\mathbb{A}^*$.)

(ii) $\mathbb{A}^* \models \neg H$.

(In particular, CH fails in $\mathbb{A}^*$ in our case.)

Prf: (i) is proved by induction on the logical complexity of B. The cases of $\exists/\forall$ need (*) of (5) for all $u_{B(\xi,y)}$ from (6).

(ii): By (1), $\llbracket H \rrbracket = 0_{\mathbb{B}}$, so (i) implies $\mathbb{A}^* \models \neg H$.

Qed.

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